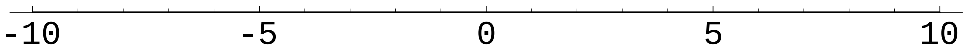
Inequalities and Graphs

Inequalities and graphs on the number line: absolute values

This page will help you investigate inequalities graphically, algebraically, and verbally by analyzing the statement

$$|a - b| \text{ means "the distance between } a \text{ and } b\text{."}$$

1. On the number line below, graph the set of numbers which are less than 3 units from the point 2.



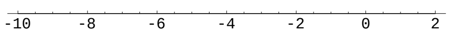
2. Write an algebraic statement which says "The distance between x and 2 is less than three units." (Note that the solutions to this inequality are the points described in the previous question.)

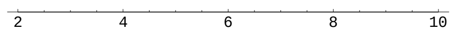
3. Now let's solve this inequality *algebraically*. Your equation should involve $|x-2|$ (or $|2-x|$, which is the same thing – why?). Remember that the meaning of the absolute value depends on the sign (positive or negative) of what's inside.

   Split this inequality into two cases, remembering that if $(x-2)$ is negative, then $|x-2|$ can be replaced by $-(x-2)$ (why?). (Note: $x = 2$ works in both cases.)

$(x-2)$ is negative	$(x-2)$ is positive

   Sketch a quick graph of the solutions to the two cases. (Remember that that first should only have x's that are less than 2 and the second only x's greater than 2. (Why?))





4. Combine your two answers and write the solution as ?? $< x <$???.

5. Does your answer agree with the description in the first question? Does it agree with the graph? Plug in some points to the inequality you wrote in part 2 to check your solution.

Now we will concentrate on more complicated algebraic inequalities (but without absolute values), like

$$4x \leq \frac{2x^2}{x-2}.$$

The moral for this story is:

> To solve an **in**equality, first solve the equality.
> The solutions to the equality, plus any points at which the expression is undefined, are possible candidates for where your inequality could change from true to not true, or vice-versa. Then plug in x's to get TRUE or FALSE.

To use this knowledge, you need to solve the equality correctly (Don't divide by something that might be zero! Check your answers before moving on to the inequality!). Then you'll want to plot these points on the number line and decide what whether the inequality holds for the x-values in the intervals between these points.

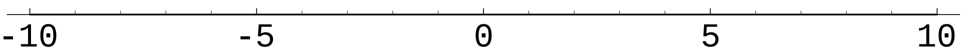
5. The following computation is part of the process of solving the inequality $4x \leq \frac{2x^2}{x-2}$.

$$\begin{aligned} 4x &= \frac{2x^2}{x-2} \\ 2x &= \frac{x^2}{x-2} \\ 2 &= \frac{x}{x-2} \\ 2(x-2) &= x \\ 2x-4 &= x \\ x-4 &= 0 \\ x &= 4 \end{aligned}$$

   Check the answer(s) using your calculator (graph $4x - \frac{2x^2}{x-2}$ to see where it is zero). You should see that the computation is wrong. Where is the error?

6. Do the computation correctly (hint: skip the illegal step, and then factor). Write your correct computation next to the wrong one above.

7. Are there any x-values at which $4x$ or $\frac{2x^2}{x-2}$ are not defined? Plot that value on the number line below along with all the solutions to the equality. If these points do not satisfy the inequality, then draw them as open circles.



8. Now that you know all the points where the truth of $4x \leq \frac{2x^2}{x-2}$ might change, how can you figure out where it's true? Of course, this is the solution set. Find it now and graph it on the axis above.

9. For a quick graphical check, use your calculator to graph $f(x) = 4x - \frac{2x^2}{x-2}$.
   (Note: Of course a plot cannot **guarantee** that you are correct because you don't know what's happening at x-values you didn't plot, and you can't tell exactly what the relevant x-values are.)

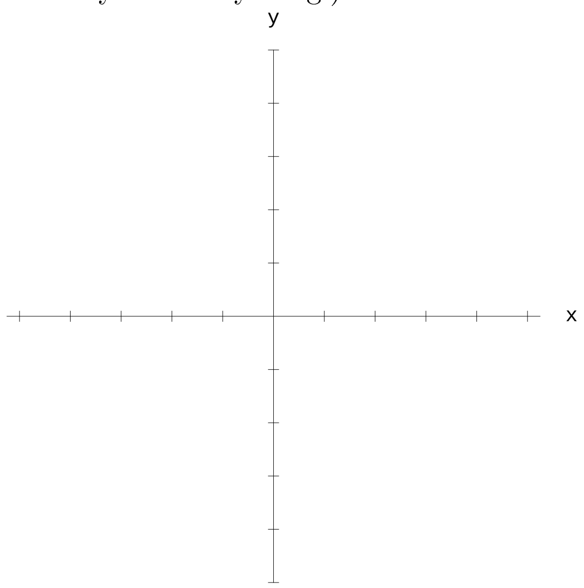
Graphs in the plane

11. Use the distance formula to write an expression for the distance between $(-2,-1)$ and $(1,3)$ in the x-y plane. Put the numbers in, but DON'T simplify it ... leave the subtractions and square root and all the junk.

12. Write an expression for the distance between an arbitrary point (x, y) in the plane and the point $(1,3)$.

13. Write an equation which says "The distance between (x, y) and $(1,3)$ is 5 units."

14. Describe the graph which consists of all points (x, y) which are 5 units from the point $(1,3)$ (i.e., what is the graph of this equation?). Sketch it here, labeling the axes so the complete graph fits on the axes. (You'll probably want to determine the leftmost, rightmost, topmost, and bottommost points before you actually draw anything.)



(Did you bring your compass?)

15. The equation you wrote in part 13 probably doesn't look like the usual equation for a circle. Square both sides to get the "standard equation" for this circle. Do you see why there's an "r^2" in the equation?

16. Is $(1,3)$ a point on the circle? Is $(-2,-1)$ a point on the circle? You should be able to check this both graphically (part 14) and algebraically (part 13).

17. Are there any x-intercepts? y-intercepts? Find them from the equation above and visually verify your answers based on your graph.

18. Draw the line segment from $(1,3)$ to $(-2,-1)$, and find the midpoint of this segment by averaging the x-coordinates and then averaging the y-coordinates.

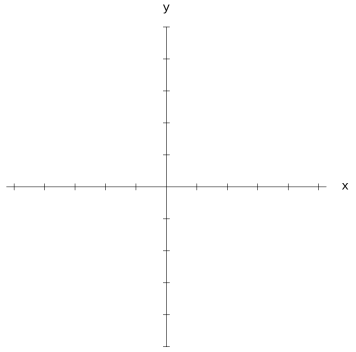
19. This point (the midpoint) should be on a circle of radius ______ with center at ______. (Please fill in the blanks.)
Challenge: Find the equation for this circle (with radius half as big as the original) and verify algebraically that your midpoint is actually on that circle.

The remaining questions should get you thinking about **symmetries**.

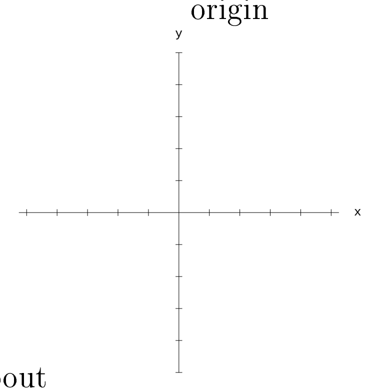
(Recall that a "symmetry" is present when you can take a graph and move it so that it coincides with itself. Symmetry about the y-axis is seen by folding along the y-axis (similarly for symmetry about the x-axis), and symmetry about the origin is seen by sticking a pin through the origin and rotating the picture by 180^o about the pin.)

19. For the this problem, draw a graph in the first quadrant of the first axes (anything! make it interesting!). Then copy it into the first quadrant of the other 3 axes.

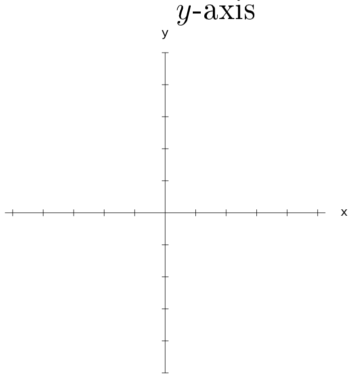
Symmetric about x-axis



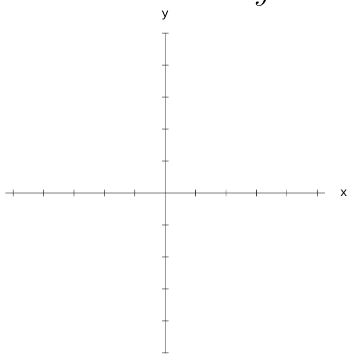
Symmetric about origin



Symmetric about y-axis



Symmetric about both x and y axes



Next, pass this along to the the person on your right (in your group). They must draw the rest of each graph so that it has the indicated symmetry(ies).

When they're done drawing, take your own paper back and correct the mistakes!

20. Match the statements below with the type of symmetry:

about origin **about x-axis** **about y-axis** **none of these**

(a) "When I replace x with $-x$, the equation doesn't change."

(b) "It doesn't matter whether I put in a positive y or a negative y, I still get the same value for x."

(c) "When I turn the paper upside down, the graph looks the same."

(d) "When I plug in $x = 2$, I get $y = 5$, but when I plug in $x = -2$ then I get $y = -5$. The same thing happens for any value of x: plugging in *minus* x gives me *minus* y."

(e) "When I switch x with y, the equation doesn't change."

(f) "The y-value at any x is the same as the y-value at $-x$."

Compare each of these statements to the graphs at the top of the page.

For Instructors:

SUGGESTIONS on how to USE worksheet #3

This is a long worksheet: some students will not be able to finish it in one class period. To account for this you could try one of the following:

- Instruct your students to do only two or three pages during class (let them know which pages they must complete!). If you had time to do an example on one of these problems during lecture, then tell them to skip that problem and come back to it only if they have time and/or interest.
- Have the students begin the worksheet on Wednesday (one day earlier). Use the worksheet over two days: each day give a 10-15 minute introduction and then have the students work through two pages of the sheet during the remainder of class time. (This way, the students are working through examples rather that watching you work at the board.)