Composition of functions and modeling

Order of composition: the T-shirt problem

Every time I buy a T-shirt from WalMart, it is TOO LONG (it comes down to my knees!!). I know of two ways to fix this: have my mother hem the T-shirt, or wash and dry the shirt so that it shrinks. The final length of the T-shirt will depend on how long it was to begin with and which process(es) I use to shorten it.

The initial length of the shirt will be called "x".

1. Let's start by making a "hemming function" called $h(x)$. If x is the length of the shirt before hemming, then $h(x)$ will represent the length after hemming.

   Suppose my mother is a creature of habit, and always hems a T-shirt by 5 inches. Write a formula for $h(x)$.

2. Now we need a shrinking function, $s(x)$, which tells us the length of the T-shirt after shrinking (depending on x, the length of the T-shirt before shrinking).

   Suppose that the T-shirt shrinks by 10% when it is washed and dried. Write a formula for $s(x)$.

3. Suppose we start with a shirt that is 35 inches long, and we decide to shrink it first and then get it hemmed. After the first step (shrinking) how long is the shirt?

4. Now we start with the shrunken shirt from the last problem and ask mom to hem it. How long is the shirt after hemming?

5. Which composition of functions represents this process of shrink-then-hem, $h(s(x))$ or $s(h(x))$? (Think: the output of which process is input for the next?)

6. Write out and simplify a formula $f(x)$ which gives the final length of the shirt after shrinking and then hemming. (CHECK: does it agree with your answer above when $x = 35$?)

7. Suppose we did the two processes in the reverse order: hem the shirt first and then shrink it. Write out a composition of s and h and simplify it to get a formula $g(x)$ for the hem-then-shrink process.

8. Is the shrink-then-hem function f the same as the hem-then-shrink function g? Why or why not?

9. What process does the composition $h(h(x))$ represent?

10. Find a formula for $h(h(x))$. It should make sense!

Blowing up a balloon

Suppose we are blowing up a (more-or-less) spherical balloon. As we blow the air in at a constant rate, the volume and radius of the balloon are increasing.

We'll use v for volume (measured in cm^3), r for radius (measured in cm) and t for time (measured in seconds).

11. The volume of a sphere with radius r is given by the formula $v = \dfrac{4}{3}\pi r^3$. Solve this equation to get r as a function of v. (If you are too lazy to write out "$\sqrt[3]{\frac{3}{4\pi}}$", you can replace it with the decimal approximation "0.62".)

$$r(v) = \underline{\hspace{4cm}}$$

12. Let's investigate the function $v(t)$, the volume of the balloon at time t. If we start measuring time ($t = 0$) when the balloon has 5 cm^3 of air in it, this means that

$$v(\underline{\hspace{1cm}}) = \underline{\hspace{1cm}}.$$

(Fill in the blanks.)

13. If we add air at the rate of 10 cm^3 per second, what is the volume when $t = 1$? $t = 2$? $t = 3$? $t = 4$?

14. Find a formula for the volume $v(t)$. Make sure it agrees with the numbers from the last two problems.

$$v(t) = \underline{\hspace{4cm}}$$

15. What kind of function is $v(t)$? Explain the meaning of the two constants in the function, in terms of the balloon.

16. How can you use composition of functions to find a formula for the radius r at any time t? Write a formula for r in terms of t.

17. You can check your formula for r in terms of t using the volumes computed in part 13. According to your formula, what is the radius at time $t = 2$? Using this value for the radius, compute the volume using $v = \frac{4}{3}\pi r^3$. (This should agree with part 13.)

Challenge:

18. Sketch a graph of $r(t)$ at right.
Can you explain the shape of the graph?

19. Find the surface area as a function of time.

20. Suppose you blow the air twice as fast.
How does this change $v(t)$? $r(t)$?
How does it change the graphs?

Constructing a closed box

An open box with a square base is required to have a volume of 10 cubic feet. (An "open box" has a bottom and four sides, but no top.) Express the amount A of material used to make such a box as a function of the length of a side of the square base.

1. Draw a picture of the box here, labeling the length, width, and height as l, w, and h, respectively.

2. The bottom is square. What does this tell you about the relationship between l, w, and h?

   (Go back to your picture now and relabel w so that you only have two variables, l and h.)

3. What do you want to find? Use your picture to write this (the amount of material, A) as a function of the variables in your picture.

4. You're almost done ... except that A looks like a function of **two** variables instead of one. How are these two variables related (if they are)?

5. The box is "required to have a volume of 10 cubic feet". Write an equation here which says "volume equals 10." (Assume l and h are in feet, of course.)

6. This last equation tells you how l and h are related. Solve this equation for h as a function of l.

7. Combine your answers from parts 3 and 6 to get a formula for A as a function of l alone.

CHECK your answer:

8. Logically (don't use your formulas!), if the base is a square with side length of 1 foot, how high must it be to have the correct volume? (Sketch a picture.) What is the area of the base? What is the area of each of the four sides? Add these areas up and compare with your formula for $A(l)$ when $l = 1$.

USE your answer:

9. How much material is required for a base that is 2 feet by 2 feet?

10. Graph $A = A(l)$ on your calculator. For what value of l is A smallest?

For Instructors:

SUGGESTIONS on how to USE worksheet #5

- Be sure you have discussed composition of functions before using this worksheet.
  - Inputs/Outputs: the notation $f(g(x))$ means that x is input to g, and then the output from g is input to f ...
  - Algebra: $f(\smiley) = \smiley^2$ means the same thing as $f(x) = x^2$.
- Remind students that the questions will walk them through the problems step-by-step. They will have to read the questions and decide exactly what each is asking. (YES, some questions may seem silly and the answers may be obvious! That's life! The important thing is to be able to tell the difference between hard questions and easy ones, and to be able to answer whichever question you were asked.)
- Make sure the students have their graphing calculators.
- Emphasize that the answers should make *logical* sense to the students, and that talking through this worksheet is just as important as doing the symbolic manipulations.