Polynomial and rational functions

Quadratic functions
In order to get a feeling for the shape of an arbitrary parabola, we will start with the function $g_1(x) = x^2$ and transform it into the function $f(x) = 4(x-3)^2 - 9$. We'll follow the following steps:

$$g_1(x) = x^2 \quad \xrightarrow{1} \quad g_2(x) = 4x^2 \quad \xrightarrow{2} \quad g_3(x) = 4(x-3)^2 \quad \xrightarrow{3} \quad f(x) = 4(x-3)^2 - 9.$$

Fill in the blanks and answer the questions below.

1. The graph of $g_1(x)$ is a parabola opening upwards. The function $g_2(x)$ is obtained (algebraically) from $g_1(x)$ by replacing __x__ with _____ (be careful!). How is the graph of $g_2(x)$ obtained from the graph of $g_1(x)$?

2. The graph of $g_3(x)$ comes from $g_2(x)$ by replacing _____ with _____. This corresponds to shifting the graph of g_2 by _____ units in the _____ direction.

3. We finally get to $f(x) = 4(x-3)^2 - 9$ by subtracting 9 from $g_3(x)$. What effect does this have on the graph?

4. Summarize here all the steps (in order) that it takes to change the graph of $y = x^2$ into the graph of $y = 4(x-3)^2 - 9$.

5. Where is the vertex of the graph of $y = x^2$? Which steps change the x-value of the vertex? Where is the vertex in the graph of $y = 4(x-3)^2 - 9$?

6. Which step alters the width of the parabola? Does it narrow or widen it?

7. Here's another way to find the vertex of the parabola $y = 4(x-3)^2 - 9$: the key is noticing that the size of y is determined by how big $4(x-3)^2$ is.

   (a) Notice that the quantity $4(x-3)^2$ can never be negative. What is the *smallest* value it could be?

   (b) What is the x-value at which $4(x-3)^2$ is smallest?

   (c) What is the smallest value that $y = 4(x-3)^2 - 9$ could be?

   (d) What is the x-value at which y is smallest?

   (e) Explain why this tells you the vertex of the graph of $y = 4(x-3)^2 - 9$.

If you have time:

8. How would your answers change if the 4 was a -2 instead?

9. How would your answers change if the -3 was a 2 instead?

10. How would your answers change if the -9 was a $\frac{1}{2}$ instead?

11. Repeat the problems on this page with the parabola $y = 3(x-1)^2 + 2$ or a generic parabola of the form $y = a(x-h)^2 + k$ (where a, h, and k are (unknown) constants).

Completing the square

You are probably used to dealing with quadratic functions in the form $f(x) = ax^2 + bx + c$. But the first page should have shown you that it is sometimes easier to deal with functions in another form, like $f(x) = a(x - h)^2 + k$. Now you'll get a chance to see how to get from one form to the other.

In the column below, expand the expression $f(x) = 4(x - 3)^2 - 9$ so it looks like $f(x) = ax^2 + bx + c$. Write out every step explicitly, since we'll be interested in reversing these steps.

$$f(x) = 4(x - 3)^2 - 9$$

Write your expression (in the form $f(x) = ax^2 + bx + c$) below and fold the page in half to cover up what you just did in the left column. Then try to reverse your steps, attempting to rewrite the expression as $f(x) = a(x - h)^2 + k$.

The process you just performed (undoing the expansion) is known as "completing the square." It is useful here because seeing the function in the form $a(x-h)^2+k$ says a lot more than in the form ax^2+bx+c. You will come across other situations in which grouping things together this way will make your work a lot easier, so it is worth your time to understand it.

- Which steps are hardest?
- What clues do you look for to figure out what to do at these steps?
- Why is it important to "pull out the 4"?
- How could you figure out that you need the "3" in $(x - 3)$?
- How can you check that you "completed the square" correctly?

Factored form

Of course, it is also useful to look at functions (quadratic or otherwise) in FACTORED form. Take a minute now to write our function $f(x) = 4x^2 - 24x + 27$ as a product of two factors.

(Notice that this problem has been carefully rigged so that there are no square roots involved. This is NOT a typical case for a random polynomial!!)

Record your three forms here:

completed square: $f(x) =$ ________________

expanded: $f(x) =$ ________________

factored: $f(x) =$ ________________

For each of the following, decide which form of the equation is best for finding out the information. Then decide exactly how to get that information.

- x-intercepts of $f(x)$
- y-intercept of $f(x)$
- bowl-shaped or dome-shaped?
- x-value of the vertex
- y-value of the vertex
- axis of symmetry
- is the parabola wide or narrow?

polynomial functions

1. Examine graphs of polynomials of degree 3, which look like $ax^3 + bx^2 + cx + d$ (with constants a, b, c, and d). Have each person in the group graph at least two different third-degree polynomials (so you have examples of at least 8 different functions), and notice the number of <u>x-intercepts</u> and <u>turning points</u> in each. Do you notice any patterns/similarities?

2. Repeat the previous exercise with polynomials of degree 4 (i.e., ones of the form $ax^4 + bx^3 + cx^2 + dx + e$, where a, b, c, d, and e are arbitrary constants).

3. Repeat the previous exercise with polynomials of degree 5.

4. Any guess about polynomials with higher degrees?

Rational functions

Compare the following functions, making note of how the differences in the graph can be seen from the symbolic formula.

Be sure to note the **domains** of the functions and the behaviors near any **intercepts** (both x and y) and/or **asymptotes** (horizontal and vertical).

$$Y_1(x) = \frac{1}{x-2} \qquad Y_2(x) = \frac{-2}{x-2} \qquad Y_3(x) = \frac{x^2}{x-2}$$

$$Y_4(x) = \frac{1}{(x-2)^2} \qquad Y_5(x) = \frac{1}{(x-2)^3} \qquad Y_6(x) = \frac{1}{x^2-2}$$

$$Y_7(x) = \frac{x^2-3x+2}{x-2} \qquad Y_8(x) = x-1 \qquad Y_9(x) = \frac{x^2}{(x-2)(x-3)^2}$$

For Instructors:

SUGGESTIONS on how to USE worksheet #6

The worksheet is pretty heavy on quadratic functions, with very little on rational functions and polynomials in general. You might ask the students to do only 2 of the first three pages, assign one for homework, or use one(some) for part of Wednesday's class.

The students NEED to have GRAPHING calculators for some of the explorations on this worksheet.

The students should know a little bit about

- **shifting and stretching** graphs
- the definition of the **vertex** of a parabola
- how to find x and y **intercepts**
- what an **axis of symmetry** is
- what a "**turning point**" is (where a graph goes from increasing to decreasing or vice-verse)
- how to use their calculators for **graphing** (factoring might also be useful)
- what "**multiplicity**" means (number of times a factor appears in a completely factored expression)