Exponential and logarithmic functions

Exponential functions are used to describe any quantity that increases or decreases by a certain proportion every year. Two common examples are population growth and exponential decay, which you'll explore in this worksheet.

Before you start in on the algebraic formulas, take a minute to fill in the following data for future reference.

Population growth

Suppose we have a population $P(t)$ which is growing in such a way that the population increases by 20% every year. If t is measured in years since 1/1/60 and the population was 200,000 on that date, fill in the following table. (You don't need to find a formula to complete this table!)

t	0	1	2	3	4	5
P	200,000					

Use the table to estimate how long it will take for the population to double in size (this is the "doubling time" of the population).

Radioactive decay

Suppose we have a quantity Q of a radioactive substance which is decaying in such a way that the quantity decreases by 30% every year. If t is measured in years since 1/1/60 and the quantity was 300 tons on that date, fill in the following table. (You don't need to find a formula to complete this table!)

t	0	1	2	3	4	5
Q	300					

Use the table to estimate how long it will take for the quantity to decay to half its original amount (this is the "half-life" of the substance).

An exponential function for population growth can be written in either of the following forms:

$$P(t) = P_0 a^t \qquad \text{OR} \qquad P(t) = Ce^{kt},$$

where e is the special constant $e = 2.71828\ldots$, and P_0, a, C, and k are constants which determine the shape of the graph.

1. Explain why the constants P_0 and C must be the same number by describing their meaning graphically or physically. What is the value of P_0 (and C) for our example? (Hint: What is the population at time $t = 0$?)

2. Find the value of a so that the first formula above describes the population. Record your formula here. (Hint: What must P be when $t = 1$?)

   (Check that your answer agrees with the numbers in the table!)

3. Find the value of k so that the second formula above also describes the population. Record your formula here. (Hint: What must P be when $t = 1$?)

   (Check that your answer agrees with the numbers in the table!)

4. How are a and k related? (Hint: What is $\ln a$?) (Why?)

5. What property of a tells you that this is exponential GROWTH (and not decay)?

6. What property of k tells you that this is exponential GROWTH (and not decay)?

Now we will look at the same questions for exponential decay.

An exponential function for radioactive decay can also be written in two different forms:

$$Q(t) = Q_0 b^t \qquad \text{OR} \qquad Q(t) = C_0 e^{\alpha t},$$

where e is the special constant $e = 2.71828\ldots$, and Q_0, b, C_0, and α are constants which determine the shape of the graph.

7. Explain why the constants Q_0 and C_0 must be the same number by describing their meaning graphically or physically. What is the value of Q_0 (and C_0) for our example? (Hint: What is the quantity at time $t = 0$?)

8. Find the value of b so that the first formula above describes the population. Record your formula here. (Hint: What must Q be when $t = 1$?)

   (Check that your answer agrees with the numbers in the table!)

9. How are b and α related? (Why?)

10. What property of b tells you that this is exponential DECAY (and not growth)?

11. What property of α tells you that this is exponential DECAY (and not growth)?

12. The pH of a chemical solution is given by the formula

$$\mathrm{pH} = -\log_{10}(H^+)$$

where H^+ is the concentration of hydrogen ions in moles per liter. Values of pH range from 0 (acidic) to 14 (alkaline).

(a) If we increase the hydrogen concentration, will the pH increase or decrease?

(b) Find the pH of a 1-liter container of water with 0.0000001 mole of hydrogen ion.

(c) Suppose we had a 2-liter container of solution with 0.0000001 mole of hydrogen ion (i.e., a concentration *half* of that in the previous problem). What would the pH be? Is it double the pH of the 1-liter water solution? Why or why not?

(d) How much solution would we need to have (with 0.0000001 mole of hydrogen ion) in order to double the pH from the 1-liter solution in part b?

(e) Find the hydrogen ion concentration of a mildly acidic solution with a pH of 4.2.

If you have more time:

13. Use logarithms to compute (by hand) either the doubling time of the population or the half-life of the radioactive substance. Does your answer agree with your estimate based on the table?

14. Suppose you have \$100 in a savings account which earns interest at an annual rate of 6%, compounded yearly. Fill in the following table which gives the balance B in your account as a function of the number of years t since you deposited the \$100. (Assume that $t = 1$ is immediately after the first year's interest has been paid.)

t	0	1	2	3	4	5	6	7
B								

(a) Estimate how many years it will take to double your money at this interest rate.

(b) Find an equation for B of the form $B = B_0 a^t$.

For Instructors:

Suggestions on how to use worksheet #8

Page 1 could be done as a class (recording the answers on the board). Then students could work in groups on pages 2-5.

You might consider having the students skip page 3 on their first go-through, since it is really the same as page 2 (only with decay instead of growth). Have them come back to page 3 if they have extra time.

Before using the worksheet, students should have seen

1. Definition of logarithm; how to write exponential equalities as logarithmic equalities and vice-versa.
2. Exponential and logarithmic functions:
   - domain and range
   - these are inverse functions
   - basic shapes of graphs
   - increasing and decreasing
3. Basic properties of exponents and logarithms (have the students refer to section 4.4 for formulas, if needed).