Exponential and logarithmic equations

The purpose of this worksheet is to give you experience in manipulating expressions involving exponentials and logarithms. Although computer algebra systems can do most of these problems fairly quickly (and sometimes even correctly), it is still a useful skill to be able to write equations in different forms (in fact, many computer algorithms behave differently depending on the form of the equation).

For this worksheet, you are asked to solve several equations which arise in common applications. In each case, you should write an equation and try to obtain the **exact** solution **by hand**. In most cases this will be possible, but in at least one example you will have to settle for an approximate solution, obtained numerically.

Restricted population growth

The growth of a population with competition for limited resources is often modeled by a *logistic equation*. For example, suppose we have a population of bacteria growing in a petri dish according to

$$P(t) = \frac{C}{1+3e^{-\frac{1}{2}t}}.$$

P = number of bacteria (in millions)

t = hours since the culture was isolated

C = a constant

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1. The value of the constant C depends on the initial population. If the initial population was 20 million bacteria, determine C.

2. How long will it take the population to reach 40 million?

3. Challenge: How long will it take the population to reach 80 million? 100million?

Loudness and intensity of sound

When physicists measure noise levels, they generally measure the *intensity*, which is the quantity of energy passing through a fixed area. The perceived loudness of sound (i.e., how loud it sounds to your ears) depends on the intensity according to

$$L(x) = 10 \log \frac{x}{I_0}.$$

$$\begin{aligned} L &= \text{loudness (in decibels)} \\ x &= \text{intensity (in watts per square meter)} \\ I_0 &= 10^{-12} \text{ watts per square meter (a constant)} \end{aligned}$$

(This "log" means "$\log_{10}$".)

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1. In game 5 of the NBA finals between the Chicago Bulls and the Utah Jazz at the Delta Center, the crowd noise was 110 decibels. What was the intensity of the noise?

2. NBA guidelines say sound levels are not to exceed 95 decibels. What intensity corresponds to a noise level of 95 decibels?

3. By what factor did the intensity exceed the maximum stated in the guidelines?

4. At a science fair with your little brother, you measure the intensity of his screaming to be 10^{-3} watts per square meter, and you estimate that your mother can scream 10% louder. What would be the intensity of your mother's screaming? (Compare the energy output of the two.)

Compounded interest

When you open an interest-bearing bank account, the quantity that you deposit is called the "principle" (P)and the amount in your account at any later time is called the "balance" (B). The bank pays you interest n times a year, at a certain annual interest rate r. If you do not make any more deposits yourself, the balance in the account will vary over time according to the formula

$$B(t) = P\left(1 + \frac{r}{n}\right)^{nt}.$$

$$\begin{aligned} B &= \text{balance (in dollars)} \\ t &= \text{time since deposit (in years)} \\ r &= \text{interest rate (fraction)} \\ n &= \text{number of time compounded per year} \\ P &= \text{principle (in dollars)} \end{aligned}$$

(Technically, this formula is only valid for certain values of t, but we shall ignore this restriction here and pretend that it is valid for all values of t.)

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1. Suppose that your initial deposit is \$1000 dollars and interest is being compounded bi-monthly ($n = 6$). If it takes you 20 years to double your money, what is the annual interest rate, r?

2. Suppose again that your initial deposit is \$1000 dollars and interest is being compounded bi-monthly ($n = 6$). If the annual interest rate is 4% ($r = 0.04$), how long will it take for your money to double?

3. Start again with your deposit of \$1000 dollars and an annual interest rate of 4%. If your balance is \$1820.30 after 15 years, how many times per year was the interest being compounded?

If you have more time:

Newton's "law" of cooling

When a warm object is placed in a cooler environment, its temperature u can be modeled by Newton's law of cooling (which is really just an approximation, not a law):

$$u(t) = u_0 e^{kt} + T(1 - e^{kt})$$

u = temperature of the object (in ^oF)
t = time (in hours)
u_0 = initial temperature of the object (in ^oF)
k = a negative constant (depends on the cooling rate)
T = Constant temperature of the environment (in ^oF)

The questions below will walk you through the solution to the following problem.

> The hotel Bora-Bora is having a pig roast. At noon, the chef put the pig in a large earthen oven. The pig's original temperature was 75^oF. When the chef checked the pig's temperature at 2:00 pm, he was upset because it had reached only 100^oF. If the oven's temperature remains a constant 325^oF, at what time may the hotel serve its guests, assuming that pork is done when it reaches 175^oF?

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1. You know the initial temperature and the temperature of the environment (the oven). Plug these two numbers into Newton's law of cooling and write it here (simplify it if you'd like).

2. You know the temperature after two hours. Plug in this value for u (and the appropriate value for t) and solve the resulting equation to compute the value of k.

3. Now you have a formula for $u(t)$ without extra constants. Use this equation to find how long it will take for the temperature of the pig to reach 175^oF.

For Instructors:

Suggestions on how to introduce worksheet #9

Students need to be reminded of the rules for exponents and logarithms (especially logarithms of products and exponentials).

Also they need to be reminded that, when solving for a variable, they first need to **gather** all terms involving the variable and then to **isolate** that variable (usually by factoring).

Try doing the following (WRONG!) examples at the front board. Ask the students to check the answers by plugging in. Then ask them to identify which step is wrong and why.

$$\begin{aligned} e^{5x} &= 2e^{2x} \\ \ln(e^{5x}) &= \ln(2e^{2x}) \\ 5x &= 2(2x) \\ 5x - 4x &= 0 \\ x &= 0. \end{aligned}$$

$$\begin{aligned} \ln(20x) &= 3\ln(x) \\ e^{\ln(20x)} &= e^{3\ln(x)} \\ 20x &= 3x \\ 17x &= 0 \\ x &= 0. \end{aligned}$$

Students should know the basic "legal moves" to use when solving equations:

1. add the same number to both sides
2. multiply both sides by the same (non-zero) number
3. take the logarithm of both sides
4. change from log to exponent form, or vice-versa (take the logarithm of both sides or exponentiate both sides)

Note: The examples in this worksheet aren't nearly as convoluted as the problems in the text/homework (although the students may find the worksheet problems "messy" because the answers are not all integers). I can't imagine any way for the text's convoluted problems to ever occur in applications.