Angles and trigonometric functions on the unit circle

For the purposes of this worksheet, assume that the radius of a bicycle wheel is $33cm = .33m$ (from the center of the wheel to the point where the tire contacts the road).

If no units are specified, all angles are measured in RADIANS.

The first few questions deal with a stationary bicycle wheel turning at a rate of 260 revolutions per minute.

1. Express the rotation rate of the bicycle wheel in revolutions per second.

2. What is the circumference of the wheel?
   (in meters)

3. What is the angular speed of the wheel?
   (in radians per second)

4. How many revolutions does the wheel make in $\frac{1}{2}$ second?
   In 2 seconds? In 25 seconds?

5. Through what angle does the wheel turn in $\frac{1}{2}$ second?

6. Consider a single point on the tire.
   How far does this point move during $\frac{1}{2}$ second?
   (answer in meters)

7. Sketch a picture below and determine how the answers to the previous questions are related. (Make sure you understand how changing any single parameter will change your answers.) Which answers depend on the radius of the wheel?

Now we'll analyze a bicycle moving on the road at a speed of $9m/s$.

8. Easy: How far (in meters) does the bicycle travel over a time of one second?

9. How far does the bicycle travel during one revolution of the wheel?

10. How many revolutions per second does the wheel make? (Hint: Combine your previous two answers. Writing units for each quantity will help.)

11. How much time does it take for one revolution of the bicycle wheel?
(Hint: Time per revolution is measured in seconds per revolution. How is this related to the previous answer?)

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12. What is the difference between linear speed and angular speed? Refer to the bicycle example. Do they both depend on the radius of the wheel?

13. Assuming you measure angular speed ω in radians per second. How do you find the linear speed v? (Invent a formula. Check the units.)

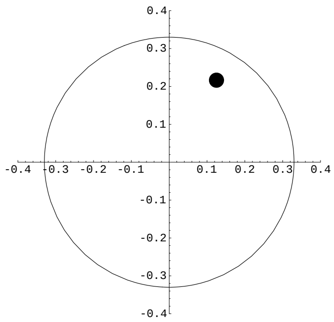
14. Suppose you measure angular speed in revolutions per second. Then how would you compute the linear speed? (Invent a formula. Check the units.)

15. Suppose you measure angular speed in degrees per second. Then how would you compute the linear speed? (Invent a formula. Check the units.)

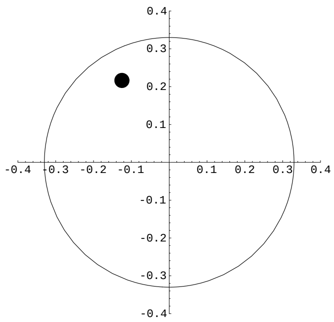
16. We hope you can see at least one reason why using radians is easier than using degrees or revolutions to measure angles. Now GO BACK to the previous two formulas and CROSS THEM OUT so you know not to use them (because we will be using RADIANS).

Now we'll take a look at a reflector attached to the tire of the stationary bicycle wheel. The reflector is attached to a certain spoke at a distance of $.25m$ from the center of the wheel. The following pictures show the reflector at equally-spaced times. (We've added a coordinate system with the origin at the center of the wheel. The units are meters, of course.)

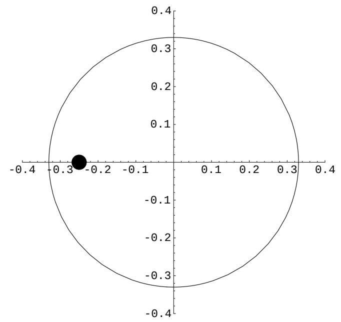
For each picture,
a. Draw the right triangle with corners at the reflector, the center of the wheel, and the point on the x-axis immediately below/above the reflector.
b. Find the angle θ (in radians) that the spoke makes with the horizontal line through the center of the wheel. Label it in the picture.
c. Find x and y coordinates of the reflector. Label the legs of the triangle with these lengths.
d. Find $\sin(\theta)$ and $\cos(\theta)$.
(You should be able to find all these numbers based ONLY on the pictures and information given; you don't need to have memorized any special angles and you really shouldn't need your calculator!)



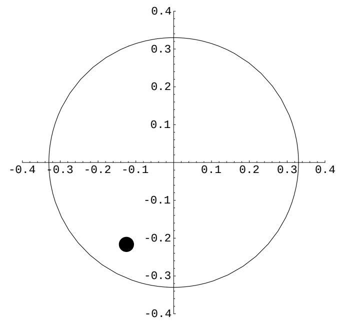
$$\begin{array}{ll|ll|ll} t = & 0.03 & x = & \frac{1}{8} & \sin(\theta) = & \_\_\_\_ \\ \theta = & \frac{\pi}{3} & y = & \frac{\sqrt{3}}{8} & \cos(\theta) = & \_\_\_\_ \end{array}$$



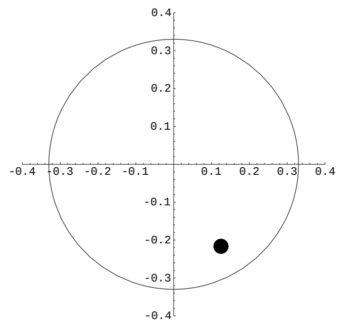
$$\begin{array}{ll|ll|ll} t = & 0.06 & x = & \_\_\_\_ & \sin(\theta) = & \_\_\_\_ \\ \theta = & \frac{2\pi}{3} & y = & \_\_\_\_ & \cos(\theta) = & \_\_\_\_ \end{array}$$



$$\begin{array}{ll|ll|ll} t = & 0.09 & x = & \_\_\_\_ & \sin(\theta) = & \_\_\_\_ \\ \theta = & \_\_\_\_ & y = & \_\_\_\_ & \cos(\theta) = & \_\_\_\_ \end{array}$$

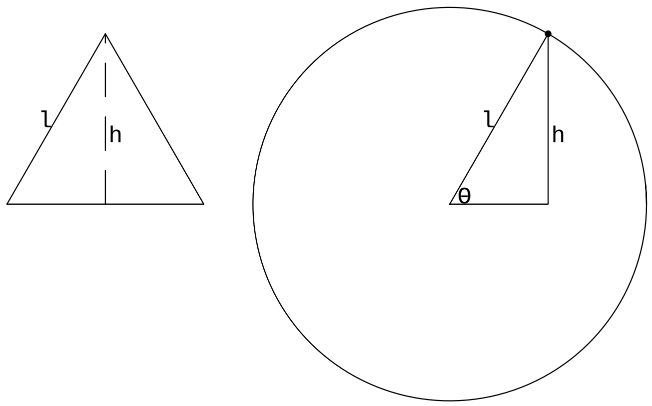


$$\begin{array}{ll|ll|ll} t = & 0.12 & x = & \_\_\_\_ & \sin(\theta) = & \_\_\_\_ \\ \theta = & \_\_\_\_ & y = & \_\_\_\_ & \cos(\theta) = & \_\_\_\_ \end{array}$$



$$\begin{array}{ll|ll|ll} t = & 0.15 & x = & \_\_\_\_ & \sin(\theta) = & \_\_\_\_ \\ \theta = & \_\_\_\_ & y = & \_\_\_\_ & \cos(\theta) = & \_\_\_\_ \end{array}$$

The picture below shows an equilateral triangle (i.e., a triangle whose three sides are all the same length, "l") and a copy of the left half of that triangle inscribed in a circle. The purpose of the following questions is to help you understand how to find the exact values of $\sin(\frac{\pi}{3})$ and $\cos(\frac{\pi}{3})$.



17. Notice that all three angles in the equilateral triangle are the same. Using the fact that the sum of all angles in a triangle is π, what is the angle θ?

18. What is the length of the base of the smaller triangle (in terms of "l")? Label it in the picture of the smaller triangle.

19. Use the Pythagorean theorem to find a formula for the height "h" (the length of the dotted line) in terms of "l". Label it in the picture of the smaller triangle.

20. Notice that the smaller triangle is a right triangle (convince yourself of this if you are not sure!). Find the sine of the angle $\theta = \frac{\pi}{3}$:

$$\sin(\theta) = \frac{y\text{-coordinate of point}}{\text{radius of circle}}$$

$$= \underline{\qquad\qquad}$$

21. What is the cosine of the angle $\theta = \frac{\pi}{3}$?

If you have more time:

22. Find the angular speed of the bicycle on page 3.

23. What would the linear speed of the bicycle be if it were riding along a road?

CYCLOIDS:

24. Find an equation $y(t)$ for the height of the reflector as a function of time.

25. Find an equation $x(t)$ for the x-coordinate of the reflector as a function of time, if the bike is stationary.

26. Find an equation $x(t)$ for the x-coordinate of the reflector as a function of time, if the bike is moving along a road.

For Instructors:

Suggestions on how to introduce worksheet #10

Be sure students are familiar with

- Circumference of a circle (as function of radius or diameter)
- Definitions of radians, revolutions, degrees
- Angular speed (angle per time) vs. linear speed (distance per time) (Two kids on a merry-go-round is a good example; they both have the same angular speed but not necessarily the same linear speed.)
- Definitions of sine and cosine as the y- and x-coordinates of a point on the unit circle.
- How to scale the unit circle to deal with circles of arbitrary radius.
- Finding the sine and cosine of angles in a right triangle by computing ratios of lengths.