Trigonometric sum and difference formulas

This worksheet will give you practice in using the formulas for the trigonometric functions of the sum of two angles.

$$\cos(\alpha+\beta) = \cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta) \tag{i}$$

$$\sin(\alpha+\beta) = \sin(\alpha)\cos(\beta) - \cos(\alpha)\sin(\beta) \tag{ii}$$

$$\tan(\alpha+\beta) = \frac{\tan(\alpha)+\tan(\beta)}{1-\tan(\alpha)\tan(\beta)} \tag{iii}$$

1. Suppose Fred is afraid he'll freak out on his precalculus test and he won't be able to remember whether there's a "+" or "−" sign in the formula for $\cos(\alpha+\beta)$. Show how he could plug in values of $\alpha = \pi/4$ and $\beta = \pi/4$ to check that he got the sign right. (Is the "−" sign correct in equation (i) above? If not, fix it now.)

2. Check the sign in the second equation by plugging in values of $\alpha = \pi/4$ and $\beta = \pi/4$. Is the "−" sign correct in this equation? If not, fix it now.

3. Fill in the blank:
   Because $\cos(x)$ is an EVEN function, we know that $\cos(-\theta) =$ ____________.

4. Fill in the blank:
   Because $\sin(x)$ is an ODD function, we know that $\sin(-\theta) =$ ____________.

5. Substitute $\beta = -\theta$ in equation (i) to get a formula for the cosine of the difference of two angles, $\cos(\alpha-\theta)$. (Simplify, using your answers to parts 3 and 4.)

6. Substitute $\beta = -\theta$ in equation (ii) to get a formula for the sine of the difference of two angles, $\sin(\alpha-\theta)$. (Simplify, using your answers to parts 3 and 4.)

7. Use the fact that $\dfrac{3}{12} + \dfrac{4}{12} = \dfrac{7}{12}$ to find the exact value of $\cos(\dfrac{7\pi}{12})$.

(Check your answer using your knowledge of the unit circle: In which quadrant of the unit circle is $\frac{7\pi}{12}$? What should the sign of your answer be? Should $\cos(\frac{7\pi}{12})$ be more or less than $\cos(\frac{8\pi}{12})$? Is it?)

8. Find the exact value of $\sin(\dfrac{\pi}{12})$ by using formula (ii).

(How is this related to $\cos(\frac{7\pi}{12})$? Draw the two triangles in the unit circle to see why.)

9. Use the formulas on the first page to find a formula for

$$\sin(\frac{\pi}{2} - \theta)$$

in terms of trig. functions of θ alone. (Yes, you've seen this formula before!)

10. Use the formulas on the first page to find a formula for

$$\tan(\frac{\pi}{2} - \theta)$$

in terms of trig. functions of θ alone.

Derive the "double-angle formulas."

11. Use equation (ii) with $\alpha = \theta$ and $\beta = \theta$ to find a formula for $\sin(2\theta)$ in terms of θ.

$$\sin(2\theta) = \underline{\hspace{5cm}}.$$

12. Use equation (i) with $\alpha = \theta$ and $\beta = \theta$ to find a formula for $\cos(2\theta)$ in terms of θ.

$$\cos(2\theta) = \underline{\hspace{5cm}}.$$

13. Because we know that $\sin^2(\theta) + \cos^2(\theta) = 1$, there are two other forms of the formula for $\cos(2\theta)$. One comes from your formula above by substituting $\sin^2(\theta) = 1 - \cos^2(\theta)$ and the other comes from substituting $\cos^2(\theta) = 1 - \sin^2(\theta)$. Write these two forms here:

$$\cos(2\theta) = \underline{\hspace{5cm}}.$$

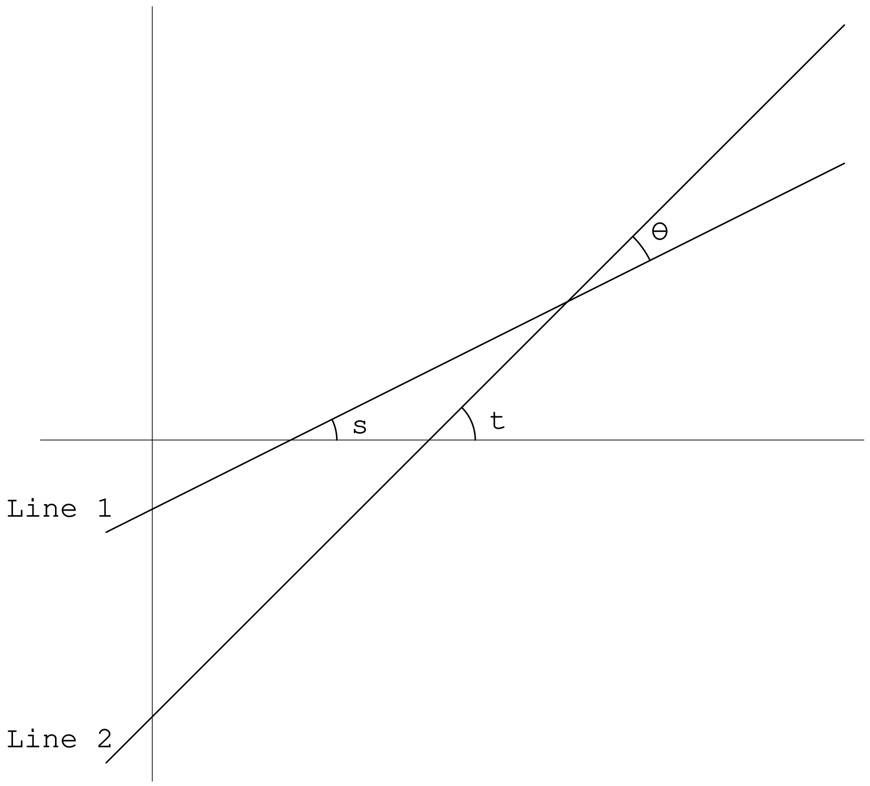
$$\cos(2\theta) = \underline{\hspace{5cm}}.$$

14. Use equation (iii) with $\alpha = \theta$ and $\beta = \theta$ to find a formula for $\tan(2\theta)$ in terms of θ.

$$\tan(2\theta) = \underline{\hspace{5cm}}.$$

(Challenge: For what values of θ will the denominator be zero? Why?)

Suppose we have two non-vertical intersecting lines (as shown in the diagram), and let θ denote the acute angle between them. If the first line has slope m_1 and the second has a slope of m_2, find a formula for the $\tan(\theta)$ in terms of m_1 and m_2.



If you need some hints:

(a) Look at the angles s, t, and θ. How are they related? (If you're not seeing it: Draw a horizontal line through the point where Line 1 intersects Line 2, and measure s and t there.)

(b) Consider Line 1. It's slope is m_1. Can you draw a triangle and relate m_1 to any trigonometric function(s) of s? Do the same with m_2.

(c) If you've done things right, you should be ready to use one of the equations from the first page.

For Instructors:

Suggestions on how to introduce worksheet #13

YES there is an error in equation (ii)! The students should identify this while working through the second problem. Make sure they all find and correct the error.

Students don't need to know much to do this worksheet. The important equations are on the front page and the students merely manipulate and interpret these.

You do NOT need to introduce double-angle formulas before using this worksheet, because the students will derive these formulas in the worksheet.

Be sure students are familiar with

- The unit circle definition of sine and cosine
- The definition of the tangent function in terms of sine and cosine
- The relationship between angles and lengths (via trig. functions) in a right triangle
- The meaning of "even" and "odd" symmetries of functions