Law of Sines, Law of Cosines

The theme for this worksheet is:

> Given partial information about the side-lengths and angles of a triangle, how can you find the remaining lengths and angles?

The definitions of $\sin(\theta)$ and $\cos(\theta)$ can be used when you have a right triangle, but for other triangles you need to know

- The "sum of angles" theorem: $\alpha + \beta + \gamma = \pi$
- The "Law of Sines": $\dfrac{\sin(\alpha)}{a} = \dfrac{\sin(\beta)}{b} = \dfrac{\sin(\gamma)}{c}$
- The "Law of Cosines": $c^2 = a^2 + b^2 - 2ab\cos(\gamma)$

for a triangle with angles of measure α, β, γ opposite sides of length a, b, and c, respectively.

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1. Sketch a triangle below in which $\gamma = \pi/2$ (i.e., a right triangle). Label the sides "a", "b", and "c" and the angles "α", "β", and "γ" according to the convention in the Law of Sines. What does the second equality in the Law of Sines say about the value of $\sin(\beta)$?

   This should agree with what you already know about right triangles!

2. If we have a triangle in which $\gamma = \pi/2$, what does the Law of Cosines say?

   This should agree with what you already know about right triangles!

3. The Law of Cosines tells you the length of a side when you know the opposite angle and the lengths of the other two sides. As stated above, the Law gives you a formula for c in terms of γ, a, and b. Write here the corresponding formulas for a in terms of α, b, and c, and for b in terms of β, a, and c.

   $a^2 =$

   $b^2 =$

4. Notice that the Law of Cosines looks like the Pythagorean theorem with an extra term, $-2ab\cos(\gamma)$, tacked onto the "$a^2 + b^2$."

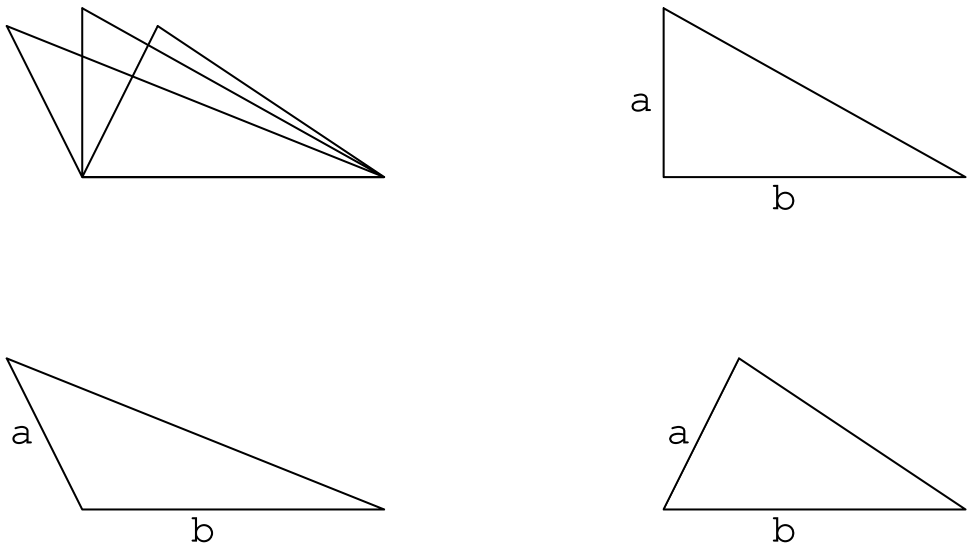
   (a) If $\gamma = \frac{\pi}{2}$, what is the value of $-2ab\cos(\gamma)$?

   (b) Suppose γ is an obtuse angle ($\frac{\pi}{2} < \gamma < \pi$). Is $-2ab\cos(\gamma)$ positive or negative?

   (c) Suppose γ is an acute angle ($0 < \gamma < \frac{\pi}{2}$). Is $-2ab\cos(\gamma)$ positive or negative?

   (d) The pictures below show three triangles, each of which has one side of length a and another of length b. For each picture, decide whether the length "c" of the third side satisfies

   $c^2 = a^2 + b^2$ or $c^2 < a^2 + b^2$ or $c^2 > a^2 + b^2$.



   (e) How are your answers to questions 4a–4c related to question 4d?

In the remaining problems, partial information about a triangle is given. For each, make a quick sketch of the triangle or triangles (if any) which agree with the given information and use your calculator determine the remaining sides and angles. (In each case, α is the angle opposite the side of length a, β is the angle opposite the side of length b, and γ is the angle opposite the side of length c.)

5. $b = 4$, $c = 6$, $\beta = \frac{\pi}{9}$

6. $\alpha = \frac{\pi}{3}$, $\beta = \frac{\pi}{2}$, and $\gamma = \frac{\pi}{4}$

7. $a = 4$, $c = 5$, $\alpha = \frac{2\pi}{3}$

8. $a = 3$, $b = 2$, $\alpha = \frac{3\pi}{10}$

9. $a = 4$, $b = 3$, $c = 6$

10. $a = 4$, $b = 5$, $\alpha = \frac{\pi}{3}$

11. $\alpha = \frac{\pi}{8}$, $\beta = \frac{\pi}{4}$, $a = 2$

12. $\alpha = \frac{\pi}{8}$, $\beta = \frac{\pi}{4}$, $c = 2$

13. $\alpha = \frac{\pi}{3}$, $\beta = \frac{\pi}{2}$, and $\gamma = \frac{\pi}{6}$

For Instructors:

Suggestions on how to use worksheet #15

- Make sure students realize that there are **two answers** for problem number 5
- Note that in these problems angles are expressed in **radians**, whereas the problems in the book are all in degrees.
- Remind students that they can **check their answers**: the angles should add up to π (or 180^o) and all three sides should agree with the law of sines. (Of course, this check won't tell them if there is another solution which they didn't find, and it isn't useful if they think there is "no solution".)
- You might consider asking each team to be responsible for solving one of the triangle questions (and checking their answer). Then toward the end of class, have each team write their solution on the board for the others to compare.