NAME: _____

FINAL EXAM
MA3160, Fall '05

Please **show work** or give reasoning for **every** answer. (No credit will be given for correct answers without an indication of how you arrived at your conclusion.)

If you obtain an answer or part of an answer with your **calculator**, please indicate what you punched into your calculator and what the output was.

If you use a **formula**, please write down the formula that you are using.

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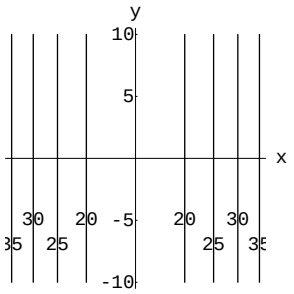
1. Suppose the level curves for a function $z = f(x, y)$ look like those shown at right. The contours are parallel to the y-axis and get closer and closer together as you move away from the y-axis.

   $z = f(x, y)$ contours



   (a) Sketch a possible cross-section corresponding to $x = 5$.
   Label axes with "x", "y", and/or "z".

   (b) Sketch a possible cross-section corresponding to $y = 5$.
   Label axes with "x", "y", and/or "z".

   (c) Give a possible formula for $f(x, y)$.

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2. Suppose $f(x, y)$ is a linear function. What does its graph look like? What do the level curves look like?

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3. The table contains function values for a linear function of two variables, $l(x, y)$.

			x	
		4	6	8
	0	2	6	
y	1	7		
	2			

   (a) Finish filling in the table.

   (b) Find $\nabla l(4, 0)$, the gradient of the function at the point $(4, 0)$.

   (c) Write an equation for $l(x, y)$.

   (d) Find $l_{\vec{v}}$, the directional derivative of the function l in the direction $\vec{v} = 2\vec{\imath} + \vec{\jmath}$.

4. Suppose Barney's calculus exam score S depends on how much he studies (x) and his caffeine intake (y), so the number of points he receives is

$$S = g(x, y)$$

where x is measured in hours and y is measured in grams.

(a) Explain in words the meaning of the statement " $g(5, 0) = 82$. "
(Include units for all three numbers.)

(b) If $g_x(5, 0) = 6$, what are the units of the "6"?

(c) Given that $g(5, 0) = 82$, $g_x(5, 0) = 6$, and $g_y(5, 0) = 2$ estimate the value of $g(5.5, 0)$.

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5. Consider the function

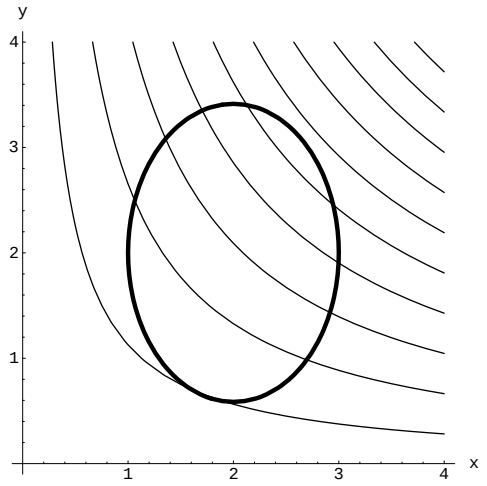
$$f(x, y) = 4 - x^2 - 2y^2.$$

(a) Give an equation for the level curve of f which goes through the point $(0, 1)$.

(b) Calculate a vector perpendicular to the level curve of f at the point $(0, 1)$.

(c) Calculate a vector perpendicular to the surface $z = f(x, y)$ at the point $(0, 1, 2)$.
(Hint: This surface is the level surface of some function of three variables.)

6. Suppose we want to find the maximum and minimum values of $f(x, y) = xy$ subject to the constraint that $10 + 2x^2 + y^2 \leq 8x + 4y$.



Shown to the right is a graph of some level curves of $f(x, y)$.

The dark oval is the graph of $10 + 2x^2 + y^2 = 8x + 4y$.

The interior of the oval is where $10 + 2x^2 + y^2 \leq 8x + 4y$.

(a) Identify the approximate location of the minimum of f subject to the constraint, put a big dot there, and label it "MIN." Do likewise for the maximum ("MAX").

(b) Is the maximum at a critical point of f?

(c) Is it appropriate to solve for the maximum using using Lagrange multipliers?

(d) Write down any/all equations you would need to solve in order to find (exactly) the location of the maximum.
(You don't need to solve the equations; just write them out explicitly in terms of x and y, and possibly λ.)

(e) Based on the graphs, do you expect to find one, two, or more solutions to these equations? (Why?)

(f) Is the second derivative test (i.e., the discriminant "D") applicable here, to test for a maximum/minimum/sa
at each solution to your equations?

7. Kooky Curt was doing his section 16.3 homework and had written down the following integral:

$$\int_0^{\sqrt{2}} \int_1^{z^2} \int_{5+x^2}^{5-x^2} \sin(x) dx\, dy\, dz.$$

How do you know that Curt set up the integral WRONG, without even knowing what the question was?

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8. Sketch the region of integration (in the x-y plane) for each of the following:
(Label the boundaries and/or describe the region in words if your picture is not clear.)

(a) $\int_0^1 \int_y^5 y \sin(x) dx dy$

(b) $\int_0^{\pi} \int_1^2 r^2 dr d\theta$

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9. Suppose you want to compute the total population of a square city whose population density is given in the table shown, where x is the distance east of the city center and y is the distance north of the city center ($-10 \leq x \leq 10$, $-10 \leq y \leq 10$).

The population density is people per square mile, and x and y are measured in miles. Assume that the density decreases as you move away from the city center.

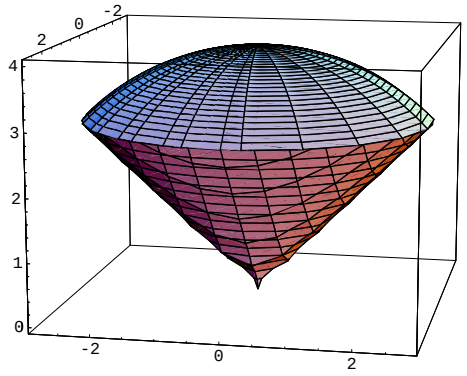
Population Density

		x				
		-10	-5	0	5	10
	-10	10	10	20	20	20
	-5	80	150	200	150	50
y	0	90	200	400	200	80
	5	80	150	200	150	70
	10	0	5	15	15	10

Use a Riemann sum to get an overestimate of the total population in the city ($-10 \leq x \leq 10$, $-10 \leq y \leq 10$).

The questions on this page deal with setting up iterated integrals for $\int_W f dV$, where $f(x, y, z) = \sqrt{x^2 + y^2 + z^2}$ and W is the region above the cone described by $z = \sqrt{x^2 + y^2}$ and under the sphere of radius 4 centered at the origin. (Note: the cone make an angle of $\pi/4$ with the z-axis.)

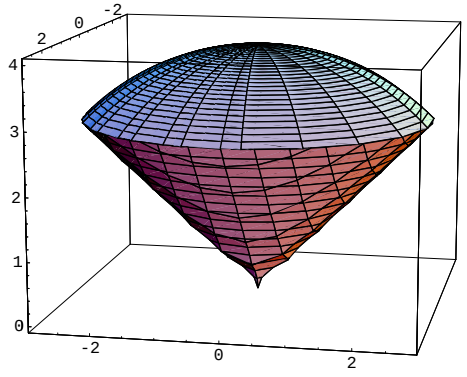
10. **Cartesian Coordinates**



(a) Label the top, bottom, and "rim" with equations (in terms of x, y, z).

(b) Set up the iterated integral for $\int_W f dV$ in Cartesian coordinates.

11. **Cylindrical or Spherical Coordinates**
(Choose one and circle your choice.)

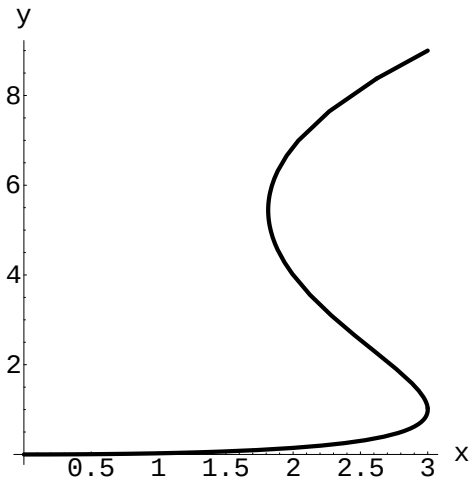


(a) Label the top and bottom with equations (in terms of r, θ, z or ρ, ϕ, θ).

(b) Set up the iterated integral for $\int_W f dV$ in cylindrical or spherical coordinates.

12. Suppose the position of a car is given by

$$\begin{aligned} x(t) &= t^3 - 5t^2 + 7t \\ y(t) &= t^2 \qquad\quad 0 \le t \le 3 \end{aligned}$$



where x and y are measured in meters and t is measured in seconds. (The car's path is shown at right.)

(a) Where is the car at time $t = 2$?

(b) What is the velocity of the car at time $t = 2$?

(c) What is the speed of the car at time $t = 2$?

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13. Is $\vec{G}(x, y) = 2xy\vec{\imath} + (x^2 + 1)\vec{\jmath}$ a gradient field? How can you tell?

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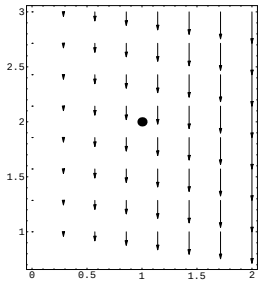
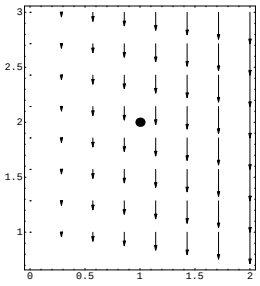
14. Consider the vector field $\vec{v}$ graphed below. Is the divergence of $\vec{v}$ positive, negative, or zero at the point shown? Justify your answer without referring to formulas (make it clear that you understand the geometric definition of the divergence).



15. Consider the vector field $\vec{v}(x, y, z)$ graphed below. In what direction does the **curl** of $\vec{v}$ point? Justify your answer without referring to formulas (make it clear that you understand the geometric definition of the curl).



Note: $\vec{v}(x, y, z)$ is independent of z and has no z-component.

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16. Here we investigate the line integral of the function

$$\vec{F}(x, y, z) = -z\vec{j} + y\vec{k}$$

around C, a circle of radius 5 in the y-z plane, parameterized by

$$\begin{array}{lrcll} C: & x & = & 0 & \\ & y & = & 5\cos(t) & 0 \le t \le 2\pi \\ & z & = & 5\sin(t). & \end{array}$$

(a) Set up a line integral for $\displaystyle\int_C \vec{F} \cdot d\vec{r}$.

(b) Can we use the Fundamental Theorem of Line Integrals to compute this line integral? (If "yes," show how. If "no," explain why not.)

17. Find the flux of $\vec{v} = z\vec{\imath} + x\vec{\jmath} + y\vec{k}$ out of a sphere of radius 3 centered at the origin.
(You can set up the flux integral or use the divergence theorem. Just make it clear how you get to your answer.)

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18. In problem 16, you set up a line integral to compute the line integral $\int_C \vec{F} \cdot d\vec{r}$ for the vector field

$$\vec{F}(x, y, z) = -z\vec{\jmath} + y\vec{k}$$

around the curve C given by

$$\begin{array}{rrcll} C: & x & = & 0 & \\ & y & = & 5\cos(t) & 0 \le t \le 2\pi \\ & z & = & 5\sin(t). & \end{array}$$

(a) Stokes' theorem says that the line integral $\int_C \vec{F} \cdot d\vec{r}$ is equal to the surface integral of **what function** over **what surface**?
Give an explicit formula for the integrand and sketch/describe the surface.

(b) Identify the normal to the surface (with the correct orientation for Stokes' theorem), set up the double integral, and evaluate it.