Some things you should know about

# Vectors, Dot Products, and Cross Products

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## VECTORS

A vector is a quantity that has both **magnitude** (length) and **direction**. It is often depicted as a arrow (the direction of the arrow is the direction of the vector, and the length of the arrow is the magnitude of the vector).

A vector $\vec{v}$ can be written as

$$\begin{array}{lll} \vec{v} = v_1\,\vec{\imath} + v_2\,\vec{\jmath} & \text{or} \quad \vec{v} = (v_1, v_2) & \text{in 2-D} \\ \vec{v} = v_1\,\vec{\imath} + v_2\,\vec{\jmath} + v_3\,\vec{k} & \text{or} \quad \vec{v} = (v_1, v_2, v_3) & \text{in 3-D.} \end{array}$$

The numbers $v_1$, $v_2$, and $v_3$ are called the *components* of the vector, and give the coordinates of the head of the vector when the tail is put at the origin.

A **displacement vector** indicates the direction and distance you would go to move from one point to another. For example, the displacement vector from the point $(x_1, y_1, z_1)$ to the point $(x_2, y_2, z_2)$ is

$$(x_2 - x_1)\,\vec{\imath} + (y_2 - y_1)\,\vec{\jmath} + (z_2 - z_1)\,\vec{k}.$$

The **magnitude** or **norm** of a vector is its length, and can be calculated in terms of components by

$$\begin{array}{rll} & ||\vec{v}|| = \sqrt{v_1^2 + v_2^2} & \text{in 2-D} \\ \text{and} & ||\vec{v}|| = \sqrt{v_1^2 + v_2^2 + v_3^2} & \text{in 3-D .} \end{array}$$

Two vectors are **parallel** if they are (scalar) multiples of one another. Algebraically, this means that $\vec{v}$ and $\vec{w}$ are parallel if there is a constant $\alpha$ so that

$$\vec{v} = \alpha\vec{w},$$

which may be written (in 3-D) as

$$(v_1, v_2, v_3) = (\alpha w_1, \alpha w_2, \alpha w_3).$$

## The DOT PRODUCT

The following are two equivalent definitions of the **dot product** between the vectors $\vec{v}$ and $\vec{w}$:

$$\begin{aligned} \vec{v}\cdot\vec{w} &= v_1w_1 + v_2w_2 + v_3w_3 \\ \vec{v}\cdot\vec{w} &= ||\vec{v}||\,||\vec{w}||\cos(\theta), \end{aligned}$$

where $\theta$ is the angle between $\vec{v}$ and $\vec{w}$. The first is more useful when doing computations with components and the second is more useful for interpreting dot products geometrically.

Note that the dot product yields a SCALAR, and that it can be zero even when neither of the vectors is zero (a dot product of zero just means that the vectors are perpendicular).

Dot products are mostly used for three purposes:

- finding **lengths** (norms) of vectors
$$||\vec{v}|| = \sqrt{\vec{v}\cdot\vec{v}}$$
- detecting **perpendicularity** (orthogonality)
$$\vec{v}\cdot\vec{w} = 0 \text{ means " } \vec{v} \text{ is perpendicular to } \vec{w} \text{ "}$$
- finding **projections** (resolving a vector into components in different directions)

  The component of $\vec{v}$ in the direction of $\vec{w}$ is
$$\begin{aligned}\vec{v}_{\text{parallel}} &= (\vec{v}\cdot\frac{\vec{w}}{||\vec{w}||})\frac{\vec{w}}{||\vec{w}||} \\ &= (\frac{\vec{v}\cdot\vec{w}}{|\vec{w}||^2})\vec{w}\end{aligned}$$

  and the component of $\vec{v}$ perpendicular to $\vec{w}$ is
$$\vec{v}_{\text{perp}} = \vec{v} - \vec{v}_{\text{parallel}}$$

### PLANES in 3-D

The equation for a plane through the point $\vec{x}_0 = (x_0, y_0, z_0)$ with perpendicular or "normal vector" $\vec{n} = (a, b, c)$ can be written as

$$\begin{aligned} 0 &= \vec{n}\cdot(\vec{x}-\vec{x}_0) \\ \text{i.e. } - \quad 0 &= a(x-x_0)+b(y-y_0)+c(z-z_0). \end{aligned}$$

### The CROSS PRODUCT

The cross product $\vec{u}\times\vec{v}$ yields a VECTOR whose direction is perpendicular to both $\vec{u}$ and $\vec{v}$ (oriented by the right-hand-rule) and whose length is

$$\|\vec{u}\times\vec{v}\| = ||\vec{u}||\,||\vec{v}||\sin(\theta),$$

where $\theta$ again represents the angle between $\vec{u}$ and $\vec{v}$. It can be calculated in term of components using

$$\begin{aligned}\vec{u}\times\vec{v} &= \det\begin{pmatrix}\vec{\imath} & \vec{\jmath} & \vec{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3\end{pmatrix} \\ &= (u_2v_3-u_3v_2)\vec{\imath}+(u_3v_1-u_1v_3)\vec{\jmath}+(u_1v_2-u_2v_1)\vec{k}.\end{aligned}$$

Cross products are defined *only* in three dimensions.
Notice that the cross product is **not commutative**:

$$\vec{u}\times\vec{v} = -\vec{v}\times\vec{u}.$$

Cross products are useful when you need a vector that is perpendicular to another pair ($\vec{u}\times\vec{v}$ is perpendicular to both $\vec{u}$ and $\vec{v}$) and for computing the area of a parallelogram ($\|\vec{u}\times\vec{v}\|$ is the area of a parallelogram with edges $\vec{u}$ and $\vec{v}$).