Plane partitions and spin adapted quantum states

PSS

joint work with Ada Stelzer and Svala Sverrisdóttir - arXiv:2601.06295

FS

motivated by [Faulstich, Sverrisdóttir '26] - arXiv:2601.16646

$B(a,b,c) := \{ a \times b \times c$ plane partitions $\}$

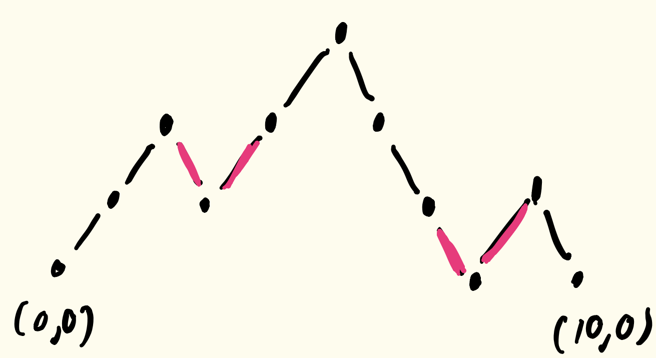
an $a \times b$ array of non-negative integers with max entry at most c

$$a\underset{b}{\begin{bmatrix} & \geq & \\ & & \end{bmatrix}}\,\vee|$$

ex: $\begin{bmatrix} 5 & 5 & 4 & 3 \\ 5 & 3 & 2 & 2 \\ 2 & 1 & 1 & 0 \end{bmatrix}$ is $3 \times 4 \times 5$ plane partition

$D(n,r) := \{$ Dyck paths of length $2n$ with $r-1$ valleys $\}$

ex: $n=5,\ r=3$



$\in D(5,3)$

$\# D(n,r) = \frac{1}{n}\binom{n}{r}\binom{n}{r-1}$ "Narayana number"

$$S_{m,k} = \mathbb{C}\begin{bmatrix} z_{11} & \cdots & z_{1,m-k} \\ \vdots & & \vdots \\ z_{k1} & \cdots & z_{k,m-k} \end{bmatrix} \Big/ \langle f^{a,b,c}_{p,q,r} \rangle$$

$$a \le b \le c$$
$$p \le q \le r$$

$$f^{a,b,c}_{p,q,r} := \sum_{\sigma \in S\{a,b,c\}} z_{p,\sigma(a)} z_{q,\sigma(b)} z_{r,\sigma(c)}$$

ex: $f^{1,1,2}_{1,1,1} = 6 z_{11}^2 z_{12} = \text{permanent}\left(\begin{bmatrix} z_{11} & z_{11} & z_{12} \\ z_{11} & z_{11} & z_{12} \\ z_{11} & z_{11} & z_{12} \end{bmatrix}\right)$

Thm: (PSS)

$$\begin{array}{ccc} \dim_{\mathbb{C}} S_{m,k} & = & \#D(m+1,k+1) \\ ⓐ \parallel & \nearrow\!\!\swarrow ⓑ & \\ \#B(k, m-k, 2) & & \end{array} \qquad (\ast)$$

Thm: (FS) $S_{m,k} \underset{\text{ring}}{\cong} R^{SU(2)}_{m,2k}$, "spin adapted electronic quantum states"

pf:
- Show $S_{m,k} \hookrightarrow R^{SU(2)}_{m,2k}$
- $\dim_{\mathbb{C}} R^{SU(2)}_{m,2k} = \#D(m+1,k+1)$
- Use $(\ast)$ to $\twoheadrightarrow$

pf of ⓐ:

Prop 1: The set of all monomials not divisible by any element of

$$\{ z_{p,a} z_{q,b} z_{r,c} : p \le q \le r, a \le b \le c \}$$

forms a vector space basis for $S_{m,k}$.

ex: any degree 2 monomial is in the basis,

$z_{12} z_{21}^2$ is in the basis

pf: We show that the $f^{a,b,c}_{p,q,r}$ form a Gröbner basis for $I_{m,k}$ under diagonal term order.

Now, monomials in $\mathbb{C}[z_{11}, \ldots, z_{k,m-k}] \longleftrightarrow \mathrm{Mat}_{k,m-k}(\mathbb{Z}_{\ge 0})$ by taking exponents.

ex: $k=2, m=4$ $\quad z_{12} z_{21}^2 \longleftrightarrow \begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix}$

Def: The width of $M \in \mathrm{Mat}_{k,m-k}(\mathbb{Z}_{\ge 0})$ is the max weight of a SE lattice path from $(1,1) \longrightarrow (k, m-k)$ in M.

ex: $M = \begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix}$, width$(M) = 3$

weight $= 1+2 = 3$

weight $= 1+1 = 2$

Prop 1 implies that $\dim_{\mathbb{C}} S_{m,k} = \# \{ M \in \mathrm{Mat}_{k,m-k}(\mathbb{Z}_{\ge 0}) : \mathrm{width}(M) < 3 \}$.

Def: $SSYT(\lambda, a) := \left\{ \begin{array}{c} \text{semistandard Young tableaux of shape} \\ \lambda, \text{ content } a \end{array} \right\}$

ex: $\lambda = (4,2,1)$, $a = 4$

$$\begin{ytableau} 1 & 1 & 2 & 3 \\ 2 & 3 \\ 4 \end{ytableau} \in SSYT(\lambda, a)$$

$\leq$ (along rows), $<$ (down columns)

Thm: (Robinson–Schensted–Knuth)

$$\{ M \in Mat_{a,b}(\mathbb{Z}_{\geq 0}) : \text{width}(M) \leq c \} \xleftrightarrow{RSK} \bigsqcup_{\substack{\lambda \\ \text{at most } c \\ \text{columns}}} SSYT(\lambda, a) \times SSYT(\lambda, b)$$

ex:

$$\begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix} \longleftrightarrow \begin{pmatrix} 1 & 1 & 2 & 2 \\ 1 & 2 & 1 & 1 \end{pmatrix} \longleftrightarrow \left(\begin{ytableau} 1 & 1 & 1 \\ 2 \end{ytableau}, \begin{ytableau} 1 & 1 & 2 \\ 2 \end{ytableau} \right)$$

width = 3

Thm: (folklore, Sam Hopkins "RSK via local transformations")

$$B(a,b,c) \longleftrightarrow \bigsqcup_{\substack{\lambda \\ \leq c \text{ columns}}} SSYT(\lambda, a) \times SSYT(\lambda, b)$$

ex:

$$\left(\begin{ytableau} 1 & 1 & 1 \\ 2 \end{ytableau}, \begin{ytableau} 1 & 1 & 2 \\ 2 \end{ytableau} \right)$$

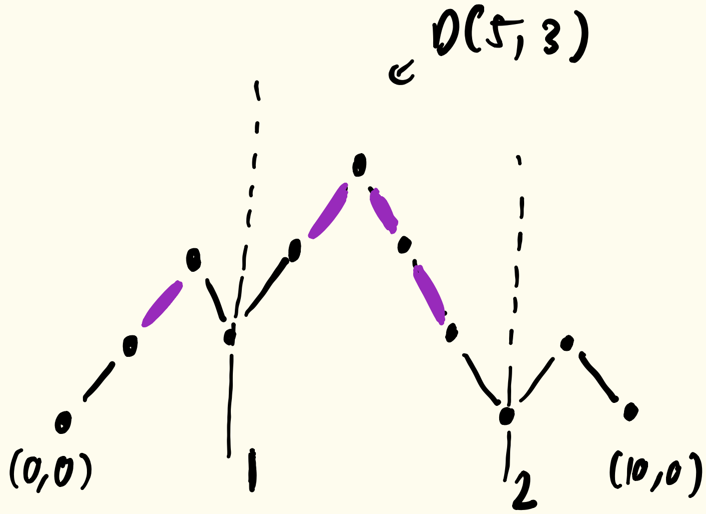
$$\updownarrow$$

$$\underset{(3)}{\ydiagram{3}} \subseteq \underset{(3,1)}{\ydiagram{3,1}} \supseteq \underset{(2)}{\ydiagram{2}} \longleftrightarrow \begin{array}{cc} & \begin{array}{cc} & 2 \end{array} \\ \begin{bmatrix} 3 & 2 \\ 3 & 1 \end{bmatrix} & \begin{array}{c} 2 \\ \end{array} \end{array} \in B(2,2,3)$$

We've shown that $\dim_{\mathbb{C}} S_{m,k} = \# B(k, m-k, 2)$.

<u>pf of (b)</u>: Want a bijection $B(k, m-k, 2) \longleftrightarrow D(m+1, k+1)$

<u>ex</u>: $k = 2$, $m = 4$



$$\longleftrightarrow \quad \begin{array}{cc} 2 & \left[\begin{array}{cc} 2 & 2 \\ 1 & 0 \end{array}\right] \\ & 2 \end{array}$$

• ignore everything after $\vdots_2$, ignore the first , and

  – remainder in

$\#2$s in row $k-i+1 = \#$ before $\vdots_i$

$\#1$s in row $k-i+1 = \#$ before $\vdots_i$ $-$ before $\vdots_i$