The Ariki–Koike Algebras and q-Appell Functions

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Outline

1. Background
2. Kleshchev bipartitions in $\Lambda^{a,2}$ with residue statistics
3. Two variable generating function formula for $\Lambda_0^{1,2}$
4. Generating function for $\Lambda_0^{1,2}$ and combinatorial proof

Integer partitions

- $\lambda = (\lambda_1, \lambda_2, \lambda_3, \ldots)$: partition of n if

$$\lambda_1 \geqslant \lambda_2 \geqslant \lambda_3 \geqslant \cdots > 0 \quad \text{and} \quad n = \lambda_1 + \lambda_2 + \lambda_3 + \cdots .$$

  Write $|\lambda| := \lambda_1 + \lambda_2 + \lambda_3 + \cdots$, and $\ell(\lambda) :=$ number of parts.

  Example. $n = 4$: $(4), (3, 1), (2, 2), (2, 1, 1), (1, 1, 1, 1)$.

$$\begin{aligned} 4 &= 4 \\ &= 3 + 1 \\ &= 2 + 2 \\ &= 2 + 1 + 1 \\ &= 1 + 1 + 1 + 1 \end{aligned}$$

- $p(n) :=$ total number of partitions of n.

$$\sum_{n \geqslant 0} p(n) q^n = \frac{1}{(1-q)(1-q^2)(1-q^3)\cdots}.$$

The Rogers–Ramanujan identities

- The Rogers–Ramanujan identities:

$$\sum_{n\geqslant 0} \frac{q^{n^2}}{(1-q)\cdots(1-q^n)} = \prod_{n\geqslant 0} \frac{1}{(1-q^{5n+1})(1-q^{5n+4})},$$

$$\sum_{n\geqslant 0} \frac{q^{n^2+n}}{(1-q)\cdots(1-q^n)} = \prod_{n\geqslant 0} \frac{1}{(1-q^{5n+2})(1-q^{5n+3})}.$$

  First given by Rogers (1894), and rediscovered later by Ramanujan (1919).

- The Andrews–Gordon generalization:

$$\sum_{N_1\geqslant\cdots\geqslant N_{k-1}\geqslant 0} \frac{q^{N_1^2+\cdots+N_{k-1}^2}}{\prod_{j=1}^{k-1}(1-q)\cdots(1-q^{N_j-N_{j+1}})} = \prod_{\substack{n\geqslant 1 \\ n\not\equiv 0,\pm k \pmod{2k+1}}} \frac{1}{1-q^n}.$$

- Appear in other areas, e.g., representation theory of Lie algebras.

The Ariki–Koike Algebras

- Ariki–Koike Algebras $\mathcal{H}_{\mathbb{C},v;Q_1,\ldots,Q_m}\big(G(m,1,n)\big)$: Iwahori-Hecke algebras associated to the complex reflection groups $G(m,1,n) \cong (\mathbb{Z}/m\mathbb{Z})^n \rtimes S_n$, where $v, Q_1, \ldots, Q_m$ are parameters.

  Introduced by Ariki–Koike (1994) and Broué–Malle (1993) independently.

- Ariki–Mathas (2000): Simple modules of the algebras are labeled by partitions of a certain type, namely Kleshchev multipartitions.

  These partitions are defined recursively, and in general no simple description is known except for $v = -1$.

To state their result, we need to define Kleshchev multipartitions. I'm going to define them for a special case.

Kleshchev multipartitions (Special case)

For $1 \leqslant a \leqslant m$, assume

$$v = -1, Q_1 = \cdots = Q_a = -1, Q_{a+1} = \cdots = Q_m = 1,$$

and set

$$t_1 = \cdots = t_a = 0, t_{a+1} = \cdots = t_m = 1.$$

- A Kleshchev multipartition $\lambda = (\lambda^{(1)}, \ldots, \lambda^{(m)})$ is a multipartition satisfying the following conditions:
  1. each $\lambda^{(i)}$ is a strict partition, i.e., partition into distinct parts;
  2. $\lambda_1^{(i)} \leqslant \ell\big(\lambda^{(i+1)}\big) + (t_{i+1} - t_i)$ for $1 \leqslant i \leqslant m-1$.

- $\Lambda^{a,m} := \{$ Kleshchev multipartitions $\}$.

Exmaple: $a = 2, m = 3$

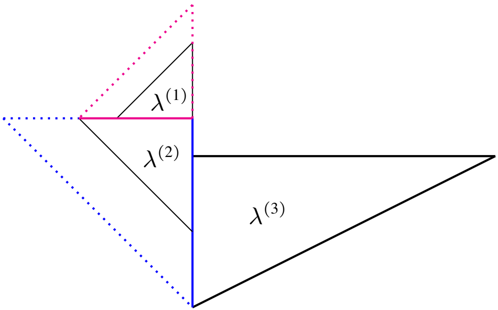


Figure: $\lambda = (\lambda^{(1)}, \lambda^{(2)}, \lambda^{(3)}) \in \Lambda^{2,3}$

Notation

$$(x;q)_n := (1-x)(1-xq)\cdots(1-xq^{n-1}),$$
$$(x;q)_\infty := \lim_{n\to\infty}(x;q)_n,$$

$$(x_1,\ldots,x_k;q)_n := (x_1;q)_n\cdots(x_k;q)_n,$$
$$(x_1,\ldots,x_k;q)_\infty := (x_1;q)_\infty\cdots(x_k;q)_\infty,$$

$$\begin{bmatrix} n \\ k \end{bmatrix} := \begin{bmatrix} n \\ k \end{bmatrix}_q := \frac{(q;q)_n}{(q;q)_k(q;q)_{n-k}} \text{ for } 0 \leqslant k \leqslant n.$$

Theorem (Ariki–Mathas (2000))

$$\sum_{\lambda \in \Lambda^{a,m}} q^{|\lambda|} = \frac{(q^{a+1}, q^{m-a+1}, q^{m+2}; q^{m+2})_\infty}{(q;q^2)_\infty (q;q)_\infty}.$$

Their proof is based on the categorification theorem and the Weyl–Kac character formula.

Recently, a q-series proof is given by Chern–Li–Stanton–Xue–Y.

Theorem (Chern–Li–Stanton–Xue–Y. (2024))

$$\sum_{N_1,\dots,N_m \geqslant 0} \frac{q^{\sum_{i=1}^m \binom{N_i+1}{2}}}{(q;q)_{N_m}} \begin{bmatrix} N_2 + \delta_{a+1,2} \\ N_1 \end{bmatrix} \begin{bmatrix} N_3 + \delta_{a+1,3} \\ N_2 \end{bmatrix} \cdots \begin{bmatrix} N_m + \delta_{a+1,m} \\ N_{m-1} \end{bmatrix}.$$

$$= \frac{(q^{a+1}, q^{m-a+1}, q^{m+2}; q^{m+2})_\infty}{(q;q^2)_\infty (q;q)_\infty}.$$

Blocks of Ariki–Koike algebras

- Lyle–Mathas (2007) classified the blocks of the Ariki–Koike algebras.

  This classification is given in combinatorial terms, but they didn't even compute the generating function for simple modules in a fixed block.

- Question: Find a generating function formula for the number of simple modules in a fixed block.

- The $m = 2$ case is done by Chern–Li–Stanton–Xue–Y. But, this question is still open for $m > 2$.

2-Residues

For a partition λ,

- 2-residue of a node $x = (i,j) \in \lambda$:

$$Res(x) := (j - i) \mod 2.$$

0	1	0	1	0
1	0	1	0	
0	1	0	1	
1	0			

Figure: 2-residue diagram of $(5,4,4,2)$

- Statistic $\omega(\lambda)$:

$$\omega(\lambda) := (\#\text{ nodes with residue } 0) - (\#\text{ nodes with residue } 1).$$

- Remark: This ω statistic is the same as the BG-rank of Berkovich and Garvan introduced in 2008.

2-Residues for Kleshchev multipartitions

For $\lambda = (\lambda^{(1)}, \ldots, \lambda^{(m)}) \in \Lambda^{a,m}$,

- 2-residue of a node $x = (i, j) \in \lambda^{(s)}$:

$$Res(x) := (j - i + t_s) \mod 2.$$

- Statistic $\omega(\lambda)$:

$$\omega(\lambda) := \big(\omega(\lambda^{(1)}) + \cdots + \omega(\lambda^{(a)})\big) - \big(\omega(\lambda^{(a+1)}) + \cdots + \omega(\lambda^{(m)})\big).$$

0	1	0	1
1	0	1	

1	0	1	0	1
0	1	0		
1	0			

Figure: $\big((4,3),(5,3,2)\big) \in \Lambda^{1,2}, \omega = -1$

- $\Lambda^{a,m}_{\omega} := \{\lambda \in \Lambda^{a,m} \mid \omega(\lambda) = \omega\}$.

Theorem (Lyle–Mathas (2007))

Simple modules in a fixed block of the Ariki–Koike algebras are labeled by $\Lambda_{\omega}^{a,m}$.

Theorem (Chern–Li–Stanton–Xue–Y. (2024))

$$\sum_{\lambda\in\Lambda^{1,2}} x^{\omega(\lambda)} q^{|\lambda|} = (-q^2, -xq, -q/x; q^2)_\infty,$$

$$\sum_{\lambda\in\Lambda^{2,2}} x^{\omega(\lambda)} q^{|\lambda|} = \frac{1}{2}\Big((-q, -x, -q^2/x; q^2)_\infty + (q, x, q^2/x; q^2)_\infty\Big).$$

Corollary

$$\sum_{\lambda\in\Lambda^{1,2}_{\omega}} q^{|\lambda|} = q^{\omega^2}\frac{(-q^2;q^2)_\infty}{(q^2;q^2)_\infty}, \qquad \sum_{\lambda\in\Lambda^{2,2}_{\omega}\cup\Lambda^{2,2}_{1-\omega}} q^{|\lambda|} = q^{\omega(\omega-1)}\frac{(-q;q^2)_\infty}{(q^2;q^2)_\infty}.$$

NOTE. These results can be derived by combining results of Ariki–Mathas–Lyle and the Weyl–Kac character formula computations for affine Lie algebras.

Combinatorial Questions

Recall

$$\sum_{\lambda \in \Lambda_{\omega}^{1,2}} q^{|\lambda|} = q^{\omega^2} \frac{(-q^2; q^2)_\infty}{(q^2; q^2)_\infty}.$$

Any combinatorial explanations on the following identities?

$$|\Lambda_{\omega}^{1,2}(n)| = |\Lambda_{-\omega}^{1,2}(n)|,$$
$$|\Lambda_{\omega}^{1,2}(n)| = |\Lambda_{0}^{1,2}(n-\omega^2)|,$$
$$|\Lambda_{\omega}^{1,2}(n)| = \overline{p}\big((n-\omega^2)/2\big).$$

Identities arising from Kleshchev bipartitions

Theorem (Chern–Li–Stanton–Xue–Y., (2024))

$$\sum_{r,s\geqslant 0} \frac{q^{r^2+s^2+r+s}(q^2;q^2)_{r+s+1}}{(q^2;q^2)_r(q^2;q^2)_r(q^2;q^2)_s(q^2;q^2)_{s+1}} = \frac{(-q^2;q^2)_\infty}{(q^2;q^2)_\infty},$$

$$\sum_{r,s\geqslant 0} \frac{q^{r^2+s^2+2s}(q^2;q^2)_{r+s}}{(q^2;q^2)_r(q^2;q^2)_s(q^2;q^2)_r(q^2;q^2)_s} + \sum_{r,s\geqslant 0} \frac{q^{r^2+s^2+2r+2s+2}(q^2;q^2)_{r+s+1}}{(q^2;q^2)_r(q^2;q^2)_{r+1}(q^2;q^2)_s(q^2;q^2)_{s+1}} = \frac{(-q;q^2)_\infty}{(q^2;q^2)_\infty}.$$

Generalizations

Theorem (Li–Seo–Stanton–Y. (preprint))

$$\sum_{r,s\geqslant 0} \frac{z^r q^{r^2+s^2+r+s}(zq^2;q^2)_{r+s+1}}{(zq^2;q^2)_r(q^2;q^2)_r(q^2;q^2)_s(zq^2;q^2)_{s+1}} = \frac{(-q^2;q^2)_\infty}{(zq^2;q^2)_\infty},$$

$$\sum_{r,s\geqslant 0} \frac{z^s q^{r^2+s^2+2s}(zq^2;q^2)_{r+s}}{(zq^2;q^2)_r(q^2;q^2)_r(zq^2;q^2)_s(q^2;q^2)_s}$$

$$+\sum_{r,s\geqslant 1} \frac{z^s q^{r^2+s^2}(zq^2;q^2)_{r+s-1}}{(zq^2;q^2)_r(q^2;q^2)_{r-1}(zq^2;q^2)_s(q^2;q^2)_{s-1}} = \frac{(-q;q^2)_\infty}{(zq^2;q^2)_\infty}.$$

Non-negativity

Define

$$g_{r,s}(z) = \frac{(zq;q)_{r+s+1}}{(zq;q)_r(q;q)_r(q;q)_s(zq;q)_{s+1}}.$$

Note

$$g_{r,s}(1) = \frac{1}{(q;q)_r(q;q)_s}\begin{bmatrix} r+s+1 \\ s+1 \end{bmatrix}.$$

Proposition

As a formal power series in q and z, $g_{r,s}(z)$ has non-negative coefficients.

NOTE. We know what statistic z keeps track of in the r, s double sum.

Jackson's Transformation Formula

Let

$$\Psi_1(a;b;c,c';x,y;\lambda) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(a;q)_{m+n}(b;q)_m\, x^m y^n q^{\lambda n(n-1)}}{(q,c;q)_m(q,c';q)_n},$$

$$\Phi(A,B;C;X) = \sum_{m=0}^{\infty} \frac{(A;q)_m(B;q)_m}{(q;q)_m(C;q)_m} X^m,$$

$${}_1\Phi_1(A;C;Y;\lambda) = \sum_{n=0}^{\infty} \frac{(A;q)_n}{(q;q)_n(C;q)_n} Y^n q^{\lambda n(n-1)}.$$

Theorem (Jackson (1944))

$$\Psi_1(a;b;c,c';x,y;\lambda)$$
$$= \sum_{r=0}^{\infty} \frac{(a,b;q)_r\, x^r y^r a^r q^{(1+\lambda)r(r-1)}}{(q,c,c';q)_r} \Phi(aq^r;bq^r;cq^r;x)\,{}_1\Phi_1(aq^r;c'q^r;yq^{2\lambda r};\lambda).$$

Sketch of Proof

- The double sum, once q^2 is replaced by q, is

$$\lim_{b\to\infty} \Psi_1(a;b;c,c';x/b,y;\lambda)$$

with $a = c' = zq^2, c = zq, x = -c = -zq, y = q, \lambda = 1/2$.

-

$$
\begin{aligned}
{}_1\Phi_1(aq^r;aq^r;q^{r+1};1/2) &= (-q^{r+1};q)_\infty, \\
\lim_{b\to\infty} \Phi(aq^r,bq^r;cq^r;-c/b) &= \frac{1}{1-cq^r}(-cq^{r+1};q)_\infty.
\end{aligned}
$$

- The right hand side of Jackson's transformation identity:

$$
\begin{aligned}
&\sum_{r=0}^{\infty} \frac{(cq;q)_r q^{4\binom{r}{2}} c^{2r} q^{2r}}{(q;q)_r (c;q)_r (cq;q)_r} (-q^{r+1};q)_\infty \frac{(-cq^{r+1};q)_\infty}{1-cq^r} \\
&= \frac{(-q;q)_\infty}{(c;q)_\infty} = \frac{(-q;q)_\infty}{(zq;q)_\infty}.
\end{aligned}
$$

Recall

$$\sum_{\lambda\in\Lambda_0^{1,2}} q^{|\lambda|} = \frac{(-q^2;q^2)_\infty}{(q^2;q^2)_\infty}$$

and

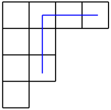
$$\sum_{r,s\geqslant 0} \frac{q^{r^2+s^2+r+s}(q^2;q^2)_{r+s+1}}{(q^2;q^2)_r(q^2;q^2)_r(q^2;q^2)_s(q^2;q^2)_{s+1}} = \frac{(-q^2;q^2)_\infty}{(q^2;q^2)_\infty}.$$

Theorem

$$\sum_{\lambda\in\Lambda_0^{1,2}} q^{|\lambda|} = \sum_{r,s\geqslant 0} \frac{q^{r^2+s^2+r+s}(q^2;q^2)_{r+s+1}}{(q^2;q^2)_r(q^2;q^2)_r(q^2;q^2)_s(q^2;q^2)_{s+1}}.$$

t-Core Partitions

- Hook and hook length:



- t-Core if no hook lengths are divisible by t.

  EXAMPLE. The partition $(4, 2, 2, 1)$ is 6-core.

7	5	2	1
4	2		
3	1		
1			

- There is a well-known algorithm for getting a t-core partition $\lambda_{t\text{-core}}$ from an arbitrary partition λ. This algorithm can be described using an abacus diagram.

$\lambda = (4, 2, 2, 1)$:

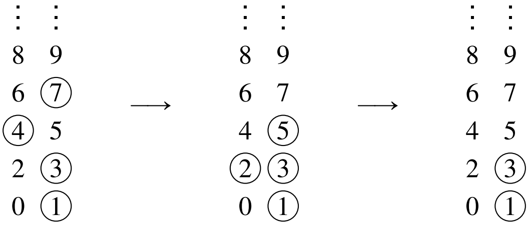


Figure: 2-abacus of $(4, 2, 2, 1)$

The 2-core partition of λ can be easily constructed from the abacus on the right side, namely $\lambda_{2\text{-core}} = (2, 1)$.

2-Core Partitions

Let

$$\Delta_j := (j, j-1, \dots, 1).$$

Proposition

λ is a 2-core partition if and only if $\lambda = \Delta_j$ for some $j \geqslant 1$.

Let

$$\mathcal{P} := \{ \text{ partitions } \}.$$

Theorem (Littlewood decomposition)

$$\sum_{\substack{\lambda \in \mathcal{P} \\ \lambda_{2\text{-core}} = \Delta_j}} q^{|\lambda|} = \frac{q^{\binom{j+1}{2}}}{(q^2; q^2)_\infty^2}.$$

Results on strict partitions

Let

$$\mathcal{D} := \{ \text{ strict partitions } \}.$$

Theorem (Li–Seo–Stanton–Y. (preprint))

$$\sum_{\substack{\lambda \in \mathcal{D} \\ \lambda_{\text{2-core}} = \Delta_j}} q^{|\lambda|} = \frac{q^{\binom{j+1}{2}}}{(q^2; q^2)_\infty}.$$

NOTE. Huang, Senger, Wear and Wu proved the above theorem combinatorially, and their proof is essentially the same as ours.

Berkovich–Uncu's Result

Theorem (Berkovich–Uncu)

$$\sum_{\substack{\lambda\in\mathcal{D},\ \lambda_1\leqslant N\\ \lambda_{\text{2-core}}=\Delta_j}} q^{|\lambda|} = q^{\binom{j+1}{2}} \begin{bmatrix} N \\ \lfloor (N-j)/2 \rfloor \end{bmatrix}_{q^2}.$$

NOTE. We can also prove this finite form using our proof.

NOTE. Recently, Dhar and Mukhopadhyay gave another combinatorial proof.

Theorem

$$\sum_{\lambda \in \mathcal{D}} x^{\omega(\lambda)} q^{|\lambda|} = (-xq, -q^3/x; q^4)_\infty (-q^2; q^2)_\infty .$$

Key Ingredients of the proof:

$$\omega(\lambda) = \omega(\lambda_{\text{2-core}}),$$

$$\omega(\Delta_j) = (-1)^{j+1} \left\lceil \frac{j}{2} \right\rceil .$$

Generating function for $\Lambda_0^{1,2}$

Recall

Theorem

$$\sum_{\lambda \in \Lambda_0^{1,2}} q^{|\lambda|} = \sum_{r,s \geqslant 0} \frac{q^{r^2+s^2+r+s}(q^2;q^2)_{r+s+1}}{(q^2;q^2)_r (q^2;q^2)_r (q^2;q^2)_s (q^2;q^2)_{s+1}} = \frac{(-q^2;q^2)_\infty}{(q^2;q^2)_\infty}.$$

Sketch of Proof

Let $\lambda = (\lambda^{(1)}, \lambda^{(2)}) \in \Lambda_0^{1,2}$.

- Since

$$\omega(\lambda) = \omega(\lambda^{(1)}) - \omega(\lambda^{(2)}) = \omega(\lambda^{(1)}_{\text{2-core}}) - \omega(\lambda^{(2)}_{\text{2-core}}) = 0,$$

$$\lambda^{(1)}_{\text{2-core}} = \lambda^{(2)}_{\text{2-core}}.$$

- Let

$$s_i := \#\text{ beads in the } i\text{-runner in the } 2\text{-abacus for } \lambda^{(2)}.$$

  Then

$$s_1 + s_2 = \ell(\lambda^{(2)}) \quad \text{and} \quad \lambda^{(1)}_1 \leqslant \ell(\lambda^{(2)}) + 1 = s_1 + s_2 + 1.$$

- $\lambda^{(2)}$ has its 2-core equal to Δ_j when $(s_1, s_2) = (s + j, s)$ or $(s, s + j + 1)$ for some $s \geqslant 0$.

$$\sum_{s\geqslant 0}\ \sum_{\substack{\lambda^{(2)}_{\text{2-core}}=\Delta_j\\ (s_1,s_2)=(s,s+j)}} q^{|\lambda^{(2)}|} \sum_{\lambda^{(1)}_{\text{2-core}}=\Delta_j} q^{|\lambda^{(1)}|} = \sum_{s\geqslant 0} \frac{q^{2s(s+j+1)+2\binom{j+1}{2}}}{(q^2;q^2)_s(q^2;q^2)_{s+j}} \begin{bmatrix} 2s+j+1 \\ s \end{bmatrix}_{q^2},$$

and

$$\sum_{s\geqslant 0}\ \sum_{\substack{\lambda^{(2)}_{\text{2-core}}=\Delta_j\\ (s_1,s_2)=(s+j+1,s)}} q^{|\lambda^{(2)}|} \sum_{\lambda^{(1)}_{\text{2-core}}=\Delta_j} q^{|\lambda^{(1)}|} = \sum_{s\geqslant 0} \frac{q^{2(s+1)(s+j+1)+\binom{j+1}{2}}}{(q^2;q^2)_s(q^2;q^2)_{s+j+1}} \begin{bmatrix} 2s+j+2 \\ s+1 \end{bmatrix}_{q^2}.$$

These are the $r \geqslant s$ and $r < s$ cases, respectively, in the theorem.

Combinatorial Questions

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$$\sum_{\lambda\in\Lambda_{\omega}^{1,2}} q^{|\lambda|} = q^{\omega^2}\frac{(-q^2;q^2)_\infty}{(q^2;q^2)_\infty}.$$

Any combinatorial explanations on the following identities?

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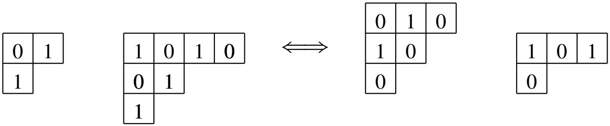
$$|\Lambda_{\omega}^{1,2}(n)| = \overline{p}\big((n-\omega^2)/2\big).$$

Combinatorial Proof of $|\Lambda^{1,2}_{\omega}(n)| = |\Lambda^{1,2}_{-\omega}(n)|$

1. $\lambda^{(1)}_1 < \ell(\lambda^{(2)})$:
   Move the first column of $\lambda^{(2)}$ to the top of the first row of $\lambda^{(1)}$.
2. $\lambda^{(1)}_1 \geqslant \ell(\lambda^{(2)})$:
   Move the first row of $\lambda^{(1)}$ to the left of the first column of $\lambda^{(2)}$.



Thank you!