Partition hook lengths

T. Amdeberhan, G. E. Andrews, K. Ono, & A. Singh

7	5	4	3	1
5	3	2	1	
1				

The Partition function $p(n)$

Definition

A **partition** of an integer n is any nonincreasing sequence

$$\Lambda := \{\lambda_1, \lambda_2, \ldots, \lambda_t\}$$

of positive integers which sum to n.

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Notation

The partition function

$$p(n) := \#\text{ partitions of } n.$$

$$4 = 3 + 1 = 2 + 2 = 2 + 1 + 1 = 1 + 1 + 1 + 1 \quad \Longrightarrow \quad p(4) = 5.$$

Partitions in Number Theory

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Theorem (Hardy and Ramanujan)

We have that

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Theorem (Ramanujan)

For every n, we have that

$$\begin{aligned} p(5n+4) &\equiv 0 \pmod{5}, \\ p(7n+5) &\equiv 0 \pmod{7}, \\ p(11n+6) &\equiv 0 \pmod{11}. \end{aligned}$$

Partitions in representation theory

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where $d(\Lambda) := \#\{\text{standard tableaux}\}$ (i.e. $\uparrow$ rows & columns).

3. *In terms of* **hook numbers**, *we have*

$$d(\Lambda) := \frac{n!}{\prod_{h \in \mathcal{H}(\Lambda)} h(i,j)}.$$

Hook numbers?

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Definition

Each partition $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_m$ has a *Young diagram*

$$\begin{array}{cccccl} \bullet & \bullet & \bullet & \cdots\bullet & & \leftarrow \lambda_1 \text{ nodes} \\ \bullet & \bullet & \ldots & \bullet & & \leftarrow \lambda_2 \text{ nodes} \\ \vdots & \vdots & \vdots & & & \\ \bullet & \ldots & \bullet & & & \leftarrow \lambda_m \text{ nodes,} \end{array}$$

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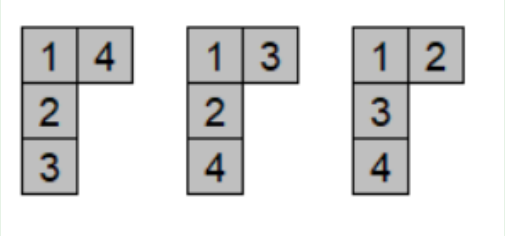
where λ'_j is the number of nodes in column j.

Example

- The partition $\Lambda = 2 + 1 + 1$ has the 3 standard tableaux



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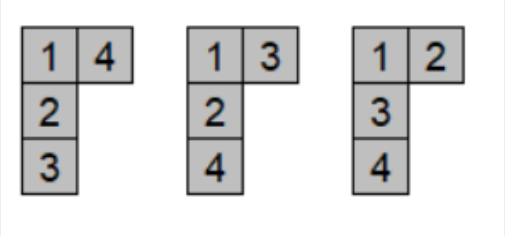

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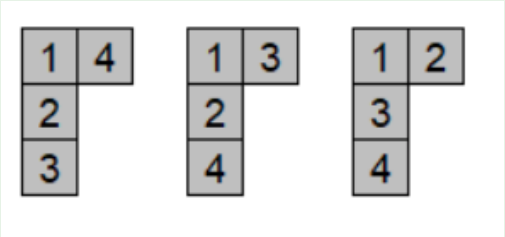

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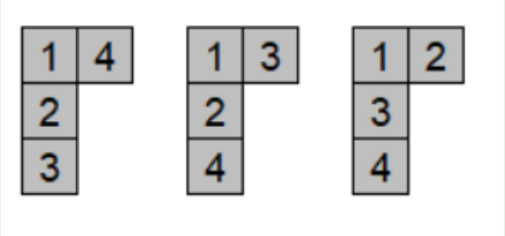

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We have the "hook multiset" $\mathcal{H}(\Lambda) = \{1, 1, 2, 4\}$, and so

$$d(\Lambda) = \frac{4!}{4 \cdot 2 \cdot 1 \cdot 1} = 3.$$

Hook numbers in representation theory

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Remark

For primes p, the p-hooks "control" the reduction modulo p

$$\overline{\rho_\Lambda}: \ S_n \to \mathrm{GL}_{d(\Lambda)}(\mathbb{Z}) \pmod{p} = \mathrm{GL}_{d(\Lambda)}(\mathbb{Z}/p\mathbb{Z}).$$

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Theorem (Granville-O, '98)

Every finite simple group has an irreducible representation that remains irreducible "mod p" for $p \geq 5$.

Hooks and Infinite Products

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Theorem (Nekrasov-Okounkov, 2003)

For any complex number z, we have

$$\prod_{n=1}^{\infty}(1-q^n)^{z-1} = \sum_{\Lambda} q^{|\Lambda|} \prod_{h\in\mathcal{H}(\Lambda)} \left(1-\frac{z}{h^2}\right).$$

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Remark

"Partition formula" for powers of Dedekind's eta function!

An Application of Nekrasov-Okounkov

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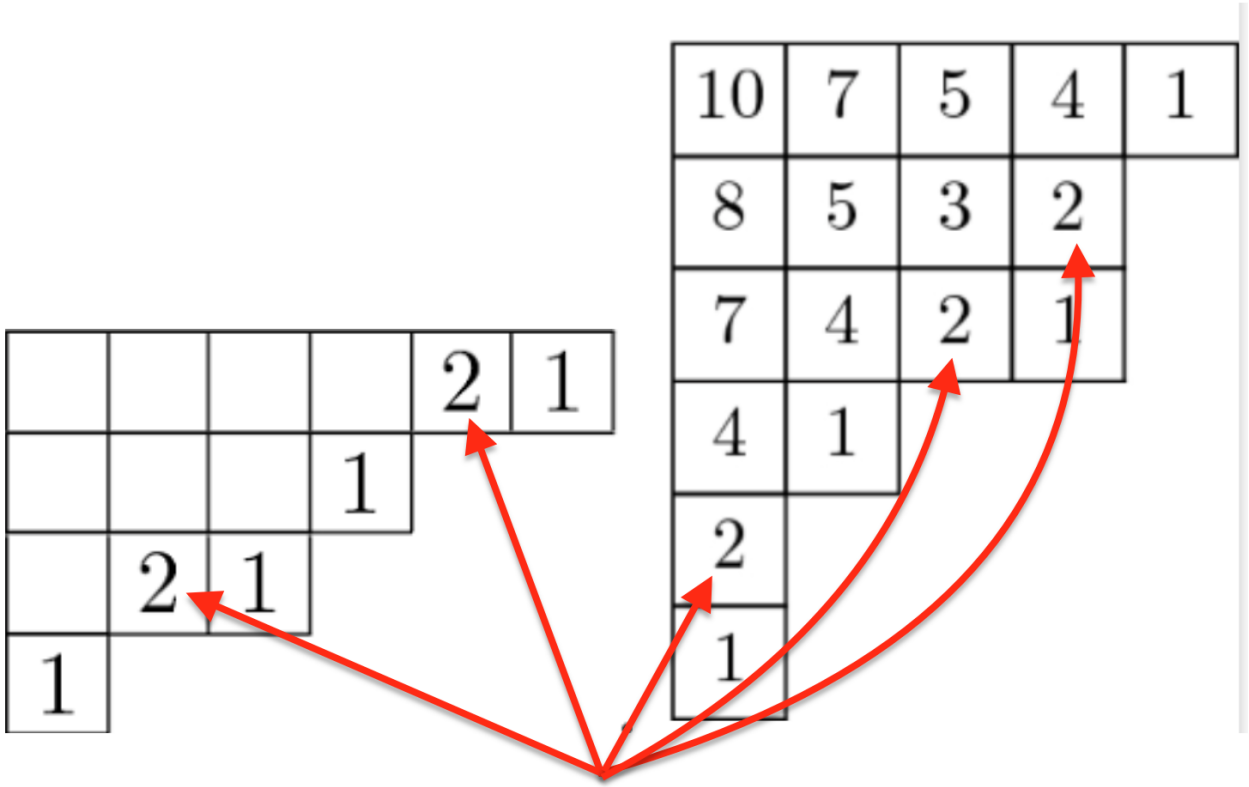

2-hooks follow from gaps >1 or repeated parts

An Application of N-O cont.

- The only partitions without a 2-hook are triangular:

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- New proof of Jacobi's identity

$$\prod_{n=1}^{\infty} (1-q^n)^{4-1} = \prod_{n=1}^{\infty} (1-q^n)^3 = \sum_{k \geq 0} (-1)^k (2k+1) q^{\frac{k^2+k}{2}}.$$

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Problems

1. *How are t-hooks distributed among size n partitions?*

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2. *Is there a limiting distribution as $n \to +\infty$?*

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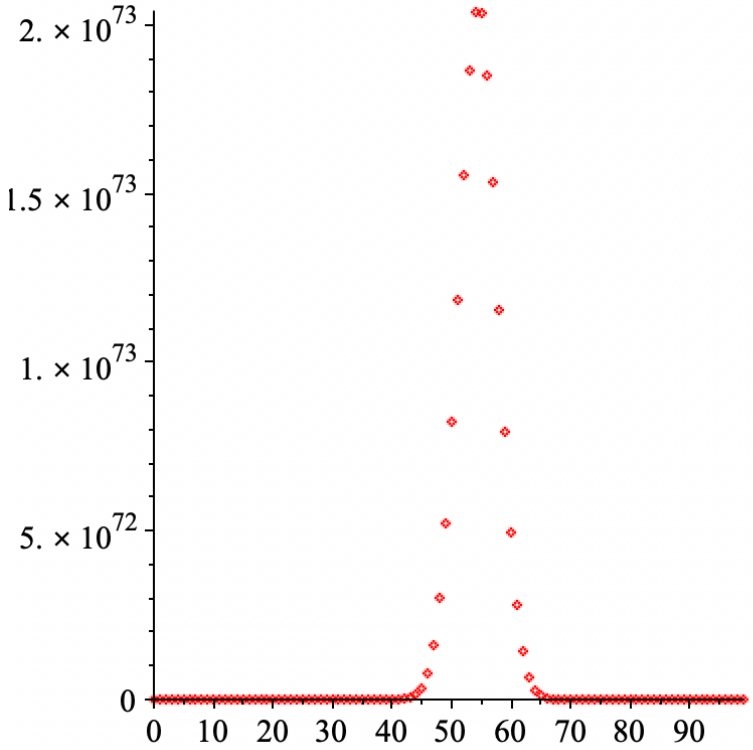

Distribution of t-hooks

Theorem (Griffin, O, Tsai (2022))

Among size n partitions, t-hooks are asymptotically **normal** *with mean* $\mu_t(n) \sim \frac{\sqrt{6n}}{\pi} - \frac{t}{2}$ *and variance* $\sigma_t^2(n) \sim \frac{(\pi^2-6)\sqrt{6n}}{2\pi^3}$.

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A key tool in the proof:

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A key tool in the proof:

Theorem (Han, 2008)

If $n_t(\Lambda) := \#\{t \in \mathcal{H}(\Lambda)\}$, then

$$\sum_{\Lambda} x^{n_t(\Lambda)} q^{|\Lambda|} = \prod_{n=1}^{\infty} \frac{(1+(x-1)q^{tn})^t}{1-q^n}.$$

Question

How about self-conjugate partitions?

Self-conjugate partitions

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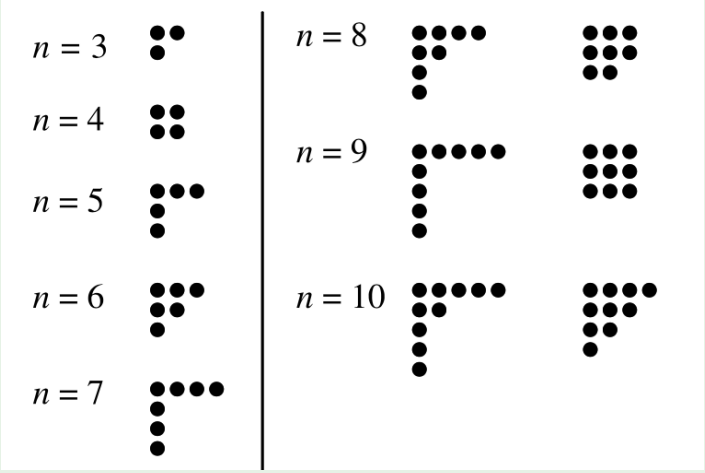
Examples (Self-conjugates of size $3 \leq n \leq 10$)



Elementary Observation

Theorem (EZ)

If $sc(n) = \#\{\text{self-conjugate partitions of size } n\}$, *then*

$$\sum_{n=0}^{\infty} sc(n)q^n = \prod_{n=0}^{\infty}(1+q^{2n+1}).$$

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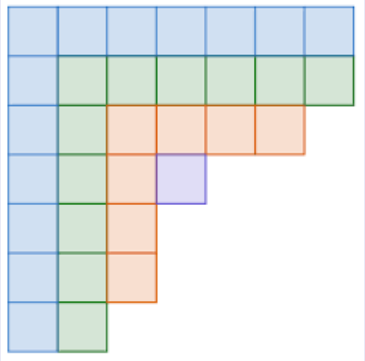

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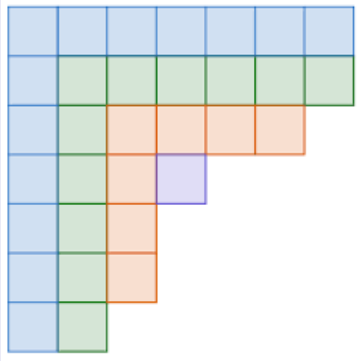

- Each "right angle" has **odd length** and cannot be repeated. □

Natural Questions

Problems ($\mathcal{SC} = \{\text{self-conjugate partitions}\}$)

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3. *How are t-hooks among partitions $\mathcal{SC}(n)$ distributed as $n \to +\infty$?*

BBCFW Conjecture (2023)

Conjecture (Ballantine, Burson, Craig, Folsom, Wen)

For $t \geq 2$ and positive integers n, then let

$$a_t^\star(n) := \sum_{\Lambda \in \mathcal{SC}(n)} n_t(\Lambda) = \#\{t\text{-hooks in all size } n \text{ self-conj } \Lambda\}.$$

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If $m \geq 1$, then for all n we have

$$a_{2m}^\star(n) \equiv 0 \pmod{2m}.$$

Some notation

Notation (Pochhammer Symbol)

$$(a;q)_n := \begin{cases} 1 & \text{if } n = 0, \\ (1-a)(1-aq)\cdots(1-aq^{n-1}) & \text{if } n \in \mathbb{Z}_+, \\ \prod_{j=0}^{\infty}(1-aq^j) & \text{if } n = +\infty. \end{cases}$$

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Remark (EZ Observation revisited)

$$\sum_{n=0}^{\infty} sc(n)q^n = (-q;q^2)_\infty.$$

Theorem 1 (AAOS)

If t is **even**, *then we have that*

$$\sum_{\Lambda \in \mathcal{SC}} x^{n_t(\Lambda)} q^{|\Lambda|} = (-q;q^2)_\infty \cdot ((1-x^2)q^{2t};q^{2t})_\infty^{\frac{t}{2}}.$$

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Theorem 2 (AAOS)

If t is **odd**, *then we have that*

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where

$$H^\star(x;q) := \frac{(-q;q^2)_\infty}{(-q^t;q^{2t})_\infty} \cdot \left[\left(1-\frac{1}{x}\right) \sum_{n\geq 0} \frac{(x^2-1)^n q^{2n^2+n}}{(q^2;q^2)_n(-q;q^2)_{n+1}} + \frac{1}{x} \sum_{n\geq 0} \frac{(x^2-1)^n q^{2n^2-n}}{(q^2;q^2)_n(-q;q^2)_n} \right].$$

Number of t-hooks: Even case

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Furthermore, we have that

$$a_t^{\star}(n) = t \sum_{j\geq 1} sc(n-2tj).$$

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Corollary (AAOS)

The BBCFW Conjecture is true.

Example: $\mathcal{SC}(16)$

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15	8	6	5	4	3	2	1
8	1						
6							
5							
4							
3							
2							
1							

13	8	6	4	3	2	1
8	3	1				
6	1					
4						
3						
2						
1						

11	8	5	4	2	1
8	5	2	1		
5	2				
4	1				
2					
1					

9	8	4	3	2
8	7	3	2	1
4	3			
3	2			
2	1			

7	6	5	4
6	5	4	3
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Figure 1. Hook lengths of the self-conjugate partitions of 16

t	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	≥ 16
$a_t^*(16)$	14	14	12	12	8	6	2	8	1	0	1	0	1	0	1	0

Table 1. Values of $a_t^*(16)$

Example continued

- The first few terms of the generating function for $sc(n)$:

$$\sum_{n=0}^{\infty} sc(n)q^n = (-q;q^2)_\infty$$
$$= 1 + q + q^3 + q^4 + q^5 + q^6 + q^7 + 2q^8 + 2q^9 + 2q^{10} + 2q^{11} + 3q^{12} + \cdots,$$

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$$a_t^\star(n) = t\left(sc(n-2t) + sc(n-4t) + sc(n-6t) + \ldots\right).$$

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- And so, we have

$$a_2^\star(16) = 2\left(sc(16-4) + sc(16-8) + sc(16-12) + sc(0)\right) = 14,$$
$$a_4^\star(16) = 4\left(sc(16-8) + sc(16-16)\right) = 12,$$
$$a_6^\star(16) = 6sc(16-12) = 6,$$
$$a_8^\star(16) = 8sc(16-16) = 8.$$

Number of t-hooks: Odd case

Theorem 4 (AAOS)

If t is **odd**, *then we have that*

$$\sum_{n\geq 1} a_t^{\star}(n)\, q^n = \frac{q^t(1+(t-1)q^t+tq^{2t})}{(1-q^{2t})(1+q^t)} \cdot (-q;q^2)_{\infty}.$$

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Furthermore, we have that

$$\begin{aligned} a_t^{\star}(n) &= \sum_{j\geq 1} \left((-1)^{j-1} j \cdot sc(n-tj) + t \cdot sc(n-2tj)\right) \\ &= \sum_{j\geq 1} \left((-1)^{j-1} j \cdot q^*(n-tj) + t \cdot q^*(n-2tj)\right). \end{aligned}$$

t-HOOK DISTRIBUTIONS IN $\mathcal{SC}$

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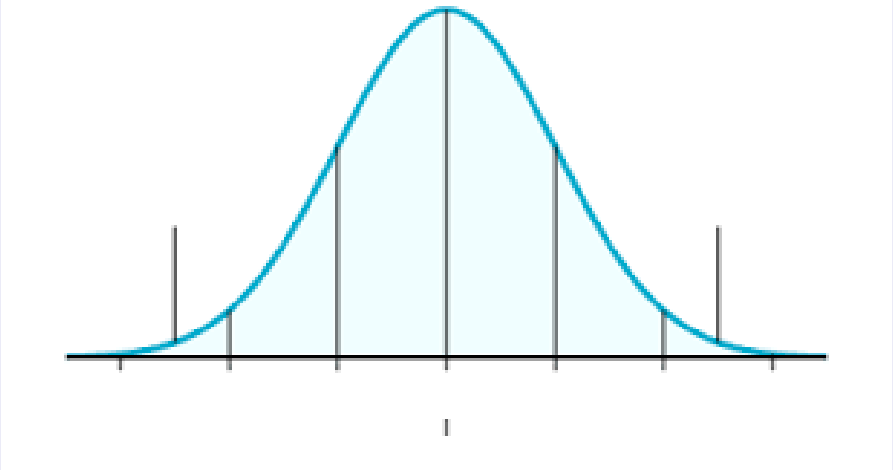
THEOREM 5 (CRAIG + OS)

As $n \to +\infty$, the t-hooks are asymptotically **normal** *in $\mathcal{SC}(n)$.*



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Theorem (Classic)

If $c_t(n)$ denotes the number of t-core partitions of size n, then

$$\sum_{n=0}^{\infty} c_t(n) q^n = \frac{(q^t; q^t)_\infty^t}{(q; q)_\infty}.$$

Self-Conjugate t-cores

Theorem (Garvan, Kim, Stanton)

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(1) If t *is* **even**, *then we have that*

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Littlewood's Bijections

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"Theorem" (Littlewood)

*(1) The "Littlewood algorithm" associates to each $\Lambda \in \mathcal{P}$ a t-core $\omega \in \mathcal{P}_t$ and a t-***quotient*** $(\nu^{(0)}, \nu^{(1)}, \ldots, \nu^{(t-1)}) \in \mathcal{P}^t$.*

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(2) This map defines a **bijection** *$\phi_t : \mathcal{P} \to \mathcal{P}_t \times \mathcal{P}^t$ given by*

$$\phi_t(\Lambda) := (w; \nu^{(0)}, \ldots, \nu^{(t-1)}),$$

where $|\Lambda| = |\omega| + t \sum_{i=0}^{t-1} |\nu^{(i)}|$.

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(4) We have that

$$|\Lambda| = \begin{cases} |\omega| + 2t \sum_{i=0}^{t/2-1} |\nu^{(i)}| & \text{if } t \text{ is even} \\ |\omega| + 2t \sum_{i=0}^{(t-1)/2-1} |\nu^{(i)}| + t|\mu| & \text{if } t \text{ is odd.} \end{cases}$$

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Remark

By (5), $n_t(\Lambda)$ is determined from 1-hooks of the quotient $\underline{\nu}$.

Counting 1-hooks

Theorem (AAOS)

(1) We have that

$$\sum_{\Lambda\in\mathcal{SC}} x^{n_1(\Lambda)}\, q^{|\Lambda|} = \frac{(-q;q^2)_\infty}{2x}\left[\left(1-\sqrt{\frac{1-x}{1+x}}\right)(-\sqrt{1-x^2};-q)_\infty + \left(1+\sqrt{\frac{1-x}{1+x}}\right)(\sqrt{1-x^2};-q)_\infty\right].$$

(2) We have that

$$\sum_{\Lambda\in\mathcal{SC}} x^{n_1(\Lambda)}\, q^{|\Lambda|} = (-q;q^2)_\infty \cdot \left[\left(1-\frac{1}{x}\right)\sum_{n\geq 0}\frac{(x^2-1)^n q^{2n^2+n}}{(q^2;q^2)_n(-q;q^2)_{n+1}} + \frac{1}{x}\sum_{n\geq 0}\frac{(x^2-1)^n q^{2n^2-n}}{(q^2;q^2)_n(-q;q^2)_n}\right].$$

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Remark (Two types of self-conjugates)

Figure 4. Self-conjugate partitions of 17 of Type 1 and Type 2

Proof of 1-hooks in $\mathcal{SC}$

- We have that

$$\sum_{\Lambda \in \mathcal{SC}} x^{n_1(\Lambda)} q^{|\Lambda|} = 1 + D_1(x;q) + D_2(x;q),$$

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- One sees that these series are

$$D_1(x;q) = \sum_{n \geq 1} xq^{n^2} \prod_{j=1}^{n-1} (1 + x^2 q^{2j} + x^2 q^{4j} + \cdots) \qquad \text{(lonely 1-hook + paired 1-hooks)}$$

$$D_2(x;q) = \sum_{n \geq 1} q^{n^2} (x^2 q^{2n} + x^2 q^{4n} + \cdots) \prod_{j=1}^{n-1} (1 + x^2 q^{2j} + x^2 q^{4j} + \cdots). \qquad \text{(paired 1-hooks)}$$

Proof of 1-hooks in $\mathcal{SC}$ cont.

- By combining these, we obtain

$$\sum_{\Lambda \in \mathcal{SC}} x^{n_1(\Lambda)} q^{|\Lambda|} = 1 + x \cdot \sum_{n \geq 1} \frac{q^{n^2} \cdot (1-(1-x)q^{2n}) \cdot ((1-x^2)q^2; q^2)_{n-1}}{(q^2; q^2)_n}.$$

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- If we let

$$F(A;q) := \sum_{n \geq 0} \frac{q^{n^2}(A;q^2)_n}{(q^2;q^2)_n},$$

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then we obtain

$$\sum_{\Lambda\in\mathcal{SC}} x^{n_1(\Lambda)}q^{|\Lambda|} = 1 + \frac{1}{x}\cdot\left[(F(1-x^2;q)-1)+(x-1)\cdot(G(1-x^2;q)-1)\right].$$

Proof of 1-hooks in $\mathcal{SC}$ cont.

- Heine's ${}_2\phi_1$ basic hypergeometric series transformation gives:

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Lemma 3.2. *The following are true.*
(1) If we let $F(A;q) := \sum_{n\geq 0} \frac{q^{n^2}(A;q^2)_n}{(q^2;q^2)_n}$, *then we have that*

$$F(A;q) = \frac{1}{2} \cdot (-q;q^2)_\infty \left[(-\sqrt{A};-q)_\infty + (\sqrt{A};-q)_\infty\right].$$

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$$G(A;q) = \frac{1}{2\sqrt{A}} \cdot (-q;q^2)_\infty \left[(-\sqrt{A};-q)_\infty - (\sqrt{A};-q)_\infty\right].$$

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- This completes the proof of the 1-hook $\mathcal{SC}$ gen fun formulas. $\square$

Generating Functions

Theorem 1 (AAOS)

If t is **even**, *then we have that*

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Proving the generating functions

- For $\Lambda \in \mathcal{SC}$, Pétréolle's "Littlewood bijection" gives

$$\phi_t(\Lambda) := (w; \nu^{(0)}, \ldots, \nu^{(t-1)}) \in \mathcal{P}_t \times \mathcal{P}^t.$$

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- Assemble these pieces! $\square$

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If t is **even**, *then we have that*

$$\sum_{\Lambda \in \mathcal{SC}} n_t(\Lambda) q^{|\Lambda|} = \sum_{n \geq 1} a_t^{\star}(n)\, q^n = t \cdot \frac{q^{2t} \cdot (-q; q^2)_\infty}{1 - q^{2t}}.$$

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Furthermore, we have that

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In particular, the BBCFW Conjecture is true.

Proof of Theorem 3

- As t is even, Theorem 1 gives

$$\sum_{\Lambda \in \mathcal{SC}} x^{n_t(\Lambda)} q^{|\Lambda|} = (-q; q^2)_\infty \cdot ((1-x^2)q^{2t}; q^{2t})_\infty^{\frac{t}{2}}.$$

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- Differentiating with respect to x and letting $x=1$ gives

$$\sum_{n \geq 1} a_t^\star(n) q^n = t \cdot \frac{q^{2t} \cdot (-q;q^2)_\infty}{1-q^{2t}} = t \cdot (q^{2t} + q^{4t} + q^{6t} + \cdots) \cdot (-q;q^2)_\infty.$$

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- Differentiating with respect to x and letting $x=1$ gives

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- The recursive formula follows from

$$\sum_{n\geq 0} sc(n)q^n = \prod_{n=0}^{\infty}(1+q^{2n+1}) = (-q;q^2)_\infty.$$

Motivation: t-hooks for **all partitions**

Theorem (Griffin, O, Tsai (2022))

Among size n partitions, t-hooks are asymptotically **normal** *with mean $\mu_t(n) \sim \frac{\sqrt{6n}}{\pi} - \frac{t}{2}$ and variance $\sigma_t^2(n) \sim \frac{(\pi^2-6)\sqrt{6n}}{2\pi^3}$.*

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Theorem (Han, 2008)

$$\sum_{\Lambda} q^{|\Lambda|} x^{\#\{t \in \mathcal{H}(\Lambda)\}} = \prod_{n=1}^{\infty} \frac{(1+(x-1)q^{tn})^t}{1-q^n}.$$

Generating Functions for $\mathcal{SC}$

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Number of t-hooks: Even case

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If t is **even**, *then we have that*

$$\sum_{\Lambda \in \mathcal{SC}} n_t(\Lambda) q^{|\Lambda|} = \sum_{n \geq 1} a_t^{\star}(n)\, q^n = t \cdot \frac{q^{2t} \cdot (-q; q^2)_\infty}{1 - q^{2t}}.$$

Number of t-hooks: Even case

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Furthermore, we have that

$$a_t^{\star}(n) = t \sum_{j \geq 1} sc(n - 2tj).$$

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In particular, the BBCFW Conjecture is true.

Number of t-hooks: Odd case

Theorem (AAOS)

If t is **odd**, *then we have that*

$$\sum_{n\geq 1} a_t^{\star}(n)\, q^n = \frac{q^t(1+(t-1)q^t+tq^{2t})}{(1-q^{2t})(1+q^t)} \cdot (-q;q^2)_{\infty}.$$

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Furthermore, we have that

$$\begin{aligned} a_t^{\star}(n) &= \sum_{j\geq 1} \left((-1)^{j-1} j \cdot sc(n-tj) + t \cdot sc(n-2tj)\right) \\ &= \sum_{j\geq 1} \left((-1)^{j-1} j \cdot q^*(n-tj) + t \cdot q^*(n-2tj)\right). \end{aligned}$$

t-HOOK DISTRIBUTIONS IN $\mathcal{SC}$

THEOREM (CRAIG+OS)

As $n \to +\infty$, we have that t-hooks are asymptotically **normally distributed** *in $\mathcal{SC}(n)$.*