Commutative algebra proof of certain partition identities

Seminar in partition theory, q-series, and related topics
Michigan Technological University

Alapan Ghosh

Department of Mathematics
Indian Institute of Technology Guwahati
Guwahati-781039, Assam

Feb 19, 2026

Outline

Outline

Partition

Definition 1 (Partition of a positive integer).

A partition of a positive integer n is a finite nonincreasing sequence of positive integers $(\lambda_1, \lambda_2, \ldots, \lambda_\ell)$ such that

$$\lambda_1 + \lambda_2 + \cdots + \lambda_\ell = n.$$

Partition

Definition 1 (Partition of a positive integer).

A partition of a positive integer n is a finite nonincreasing sequence of positive integers $(\lambda_1, \lambda_2, \ldots, \lambda_\ell)$ such that

$$\lambda_1 + \lambda_2 + \cdots + \lambda_\ell = n.$$

The integers λ_i are called the parts of the partition.

Partition

Definition 1 (Partition of a positive integer).

A partition of a positive integer n is a finite nonincreasing sequence of positive integers $(\lambda_1, \lambda_2, \ldots, \lambda_\ell)$ such that

$$\lambda_1 + \lambda_2 + \cdots + \lambda_\ell = n.$$

The integers λ_i are called the parts of the partition. The number of partitions of n is denoted by $p(n)$. By convention $p(0) := 1$.

Partition

Definition 1 (Partition of a positive integer).

A partition of a positive integer n is a finite nonincreasing sequence of positive integers $(\lambda_1, \lambda_2, \ldots, \lambda_\ell)$ such that

$$\lambda_1 + \lambda_2 + \cdots + \lambda_\ell = n.$$

The integers λ_i are called the parts of the partition. The number of partitions of n is denoted by $p(n)$. By convention $p(0) := 1$.

Example 2.

For example, $p(4) = 5$ with the corresponding partitions $(4),\ (3,1),\ (2,2),\ (2,1,1), (1,1,1,1)$.

Generating function

Definition 3 (Generating function).

For $a_n \in \mathbb{C},\ n \geq 0$, the generating function $f(q)$ for the sequence $(a_n)_{n\geq 0}$ is the power series

$$f(q) = \sum_{n\geq 0} a_n q^n.$$

Generating function

Definition 3 (Generating function).

For $a_n \in \mathbb{C},\ n \geq 0$, the generating function $f(q)$ for the sequence $(a_n)_{n\geq 0}$ is the power series

$$f(q) = \sum_{n\geq 0} a_n q^n.$$

The generating function for the partition function $p(n)$, for $|q| < 1$, is given by

$$\sum_{n=0}^{\infty} p(n)q^n = \prod_{n=1}^{\infty} \frac{1}{1-q^n} = 1 + q + 2q^2 + 3q^3 + 5q^4 + 7q^5 + \cdots .$$

Partition identities

Definition 4 (Partition identities).

A partition identity is an equality for every nonnegative integer n between the number of the partitions of n satisfying a certain condition A and the number of those satisfying another condition B.

Partition identities

Definition 4 (Partition identities).

A partition identity is an equality for every nonnegative integer n between the number of the partitions of n satisfying a certain condition A and the number of those satisfying another condition B.

Theorem 5 (Euler, Marcum-Michaelem Bousquet (1748)).

The number of partitions of n into odd parts equals to the number of partitions of n into distinct parts.

Partition identities

Definition 4 (Partition identities).

A partition identity is an equality for every nonnegative integer n between the number of the partitions of n satisfying a certain condition A and the number of those satisfying another condition B.

Theorem 5 (Euler, Marcum-Michaelem Bousquet (1748)).

The number of partitions of n into odd parts equals to the number of partitions of n into distinct parts.

Example 6.

- (5), (3,1,1) and (1,1,1,1,1)
- (5), (4,1), and (3,2)

Rogers and Ramanujan

Figure: Srinivasa Ramanujan

Figure: Leonard James Rogers

The Rogers-Ramanujan identities

- Rogers [Proc. Lond. Math. Soc. (1894)]

The Rogers-Ramanujan identities

- Rogers [Proc. Lond. Math. Soc. (1894)]
- Ramanujan [J. Indian Math. Soc. 6 (1914), Cambridge University Press (1927), The American Mathematical Society (2000)]

The Rogers-Ramanujan identities

- Rogers [Proc. Lond. Math. Soc. (1894)]
- Ramanujan [J. Indian Math. Soc. 6 (1914), Cambridge University Press (1927), The American Mathematical Society (2000)]
- Schur [Sitzungsberichte der Preussischen Akadamie der Wissenschaften (1917)]
- P. A. MacMahon [Cambridge (1918)]

The Rogers-Ramanujan identities

- Rogers [Proc. Lond. Math. Soc. (1894)]
- Ramanujan [J. Indian Math. Soc. 6 (1914), Cambridge University Press (1927), The American Mathematical Society (2000)]
- Schur [Sitzungsberichte der Preussischen Akadamie der Wissenschaften (1917)]
- P. A. MacMahon [Cambridge (1918)]

Theorem 7 (The first Rogers-Ramanujan identity).

The partitions of an integer n in which the difference between any two parts is at least 2 are equinumerous with the partitions of n into parts congruent to 1 or 4 modulo 5.

$A_{r,i}(n)$ and $B_{r,i}(n)$

Theorem 8 (The second Rogers-Ramanujan identity).

The partitions of an integer n in which each part exceeds 1 and the difference between any two parts is at least 2 are equinumerous with the partitions of n into parts congruent to 2 or 3 modulo 5.

$A_{r,i}(n)$ and $B_{r,i}(n)$

Theorem 8 (The second Rogers-Ramanujan identity).

The partitions of an integer n in which each part exceeds 1 and the difference between any two parts is at least 2 are equinumerous with the partitions of n into parts congruent to 2 or 3 modulo 5.

- Gordon [Amer. J. Math. (1961)]

$A_{r,i}(n)$ and $B_{r,i}(n)$

Theorem 8 (The second Rogers-Ramanujan identity).

The partitions of an integer n in which each part exceeds 1 and the difference between any two parts is at least 2 are equinumerous with the partitions of n into parts congruent to 2 or 3 modulo 5.

- Gordon [Amer. J. Math. (1961)]
- Andrews [Proc. Natl. Acad. Sci. (1974)]

$A_{r,i}(n)$ and $B_{r,i}(n)$

Theorem 8 (The second Rogers-Ramanujan identity).

The partitions of an integer n in which each part exceeds 1 and the difference between any two parts is at least 2 are equinumerous with the partitions of n into parts congruent to 2 or 3 modulo 5.

- Gordon [Amer. J. Math. (1961)]
- Andrews [Proc. Natl. Acad. Sci. (1974)]
- Afsharijoo [Ann. Comb. (2021)]

$A_{r,i}(n)$ and $B_{r,i}(n)$

Theorem 8 (The second Rogers-Ramanujan identity).

The partitions of an integer n in which each part exceeds 1 and the difference between any two parts is at least 2 are equinumerous with the partitions of n into parts congruent to 2 or 3 modulo 5.

- Gordon [Amer. J. Math. (1961)]
- Andrews [Proc. Natl. Acad. Sci. (1974)]
- Afsharijoo [Ann. Comb. (2021)]

Let r and i be positive integers with $1 \leq i \leq r$.

Definition 9.

Let $A_{r,i}(n)$ denote the number of partitions of n into parts which are not congruent to 0 or $\pm i$ modulo $2r + 1$.

Rogers-Ramanujan-Gordon identities

Definition 10.

Let $B_{r,i}(n)$ denote the number of partitions $(\lambda_1, \lambda_2, \ldots, \lambda_s)$ of n satisfying the following conditions:

1. $\lambda_m - \lambda_{m+r-1} \geq 2$ and
2. at most $i - 1$ parts are equal to 1.

Rogers-Ramanujan-Gordon identities

Definition 10.

Let $B_{r,i}(n)$ denote the number of partitions $(\lambda_1, \lambda_2, \ldots, \lambda_s)$ of n satisfying the following conditions:

1. $\lambda_m - \lambda_{m+r-1} \geq 2$ and
2. at most $i - 1$ parts are equal to 1.

Theorem 11 (Rogers-Ramanujan-Gordon identities).

Let r and i be positive integers with $1 \leq i \leq r$. Then $A_{r,i}(n) = B_{r,i}(n)$, for all $n \geq 0$.

- The case $r = 2$ in Theorem 11 gives the Rogers-Ramanujan identities.

Rogers-Ramanujan-Gordon identities

Definition 10.

Let $B_{r,i}(n)$ denote the number of partitions $(\lambda_1, \lambda_2, \ldots, \lambda_s)$ of n satisfying the following conditions:

1. $\lambda_m - \lambda_{m+r-1} \geq 2$ and
2. at most $i - 1$ parts are equal to 1.

Theorem 11 (Rogers-Ramanujan-Gordon identities).

Let r and i be positive integers with $1 \leq i \leq r$. Then $A_{r,i}(n) = B_{r,i}(n)$, for all $n \geq 0$.

- The case $r = 2$ in Theorem 11 gives the Rogers-Ramanujan identities.
- The case $r = 1$ leads to trivial identity $1 = 1$. Hence we take $r \geq 2$.

Basil Gordon

Figure: Basil Gordon

The Göllnitz-Gordon identities

- Gordon [Duke Math. J. 31 (1965)]

The Göllnitz-Gordon identities

- Gordon [Duke Math. J. 31 (1965)]
- Göllnitz [J. Reine Angew. Math. (1967)]

The Göllnitz-Gordon identities

- Gordon [Duke Math. J. 31 (1965)]
- Göllnitz [J. Reine Angew. Math. (1967)]
- Ramanujan's Lost Notebook [Springer (2008)].

The Göllnitz-Gordon identities

- Gordon [Duke Math. J. 31 (1965)]
- Göllnitz [J. Reine Angew. Math. (1967)]
- Ramanujan's Lost Notebook [Springer (2008)].

Theorem 12 (The first Göllnitz-Gordon identity).

The number of partitions of any positive integer n into parts congruent to 1, 4, 7 modulo 8 is equal to the number of partitions $(\lambda_1, \lambda_2, \ldots, \lambda_s)$ of n such that $\lambda_m - \lambda_{m+1} \geq 2$, and $\lambda_m - \lambda_{m+1} \geq 3$ if λ_m is even.

The Göllnitz-Gordon identities

- Gordon [Duke Math. J. 31 (1965)]
- Göllnitz [J. Reine Angew. Math. (1967)]
- Ramanujan's Lost Notebook [Springer (2008)].

Theorem 12 (The first Göllnitz-Gordon identity).

The number of partitions of any positive integer n into parts congruent to 1, 4, 7 modulo 8 is equal to the number of partitions $(\lambda_1, \lambda_2, \ldots, \lambda_s)$ of n such that $\lambda_m - \lambda_{m+1} \geq 2$, and $\lambda_m - \lambda_{m+1} \geq 3$ if λ_m is even.

Theorem 13 (The second Göllnitz-Gordon identity).

The number of partitions of any positive integer n into parts congruent to 3, 4, 5 modulo 8 is equal to the number of partitions $(\lambda_1, \lambda_2, \ldots, \lambda_s)$ of n such that $\lambda_s \geq 3$, $\lambda_m - \lambda_{m+1} \geq 2$, and $\lambda_m - \lambda_{m+1} \geq 3$ if λ_m is even.

$C_{r,i}(n)$ and $D_{r,i}(n)$

- Andrews [Proc. Amer. Math. Soc. (1967)].

$C_{r,i}(n)$ and $D_{r,i}(n)$

- Andrews [Proc. Amer. Math. Soc. (1967)].

Let r and i be positive integers with $1 \leq i \leq r$.

Definition 14.

Let $C_{r,i}(n)$ denote the number of partitions of n into parts which are not congruent to 2 modulo 4 and also not congruent to 0 or $\pm(2i-1)$ modulo $4r$.

$C_{r,i}(n)$ and $D_{r,i}(n)$

- Andrews [Proc. Amer. Math. Soc. (1967)].

Let r and i be positive integers with $1 \leq i \leq r$.

Definition 14.

Let $C_{r,i}(n)$ denote the number of partitions of n into parts which are not congruent to 2 modulo 4 and also not congruent to 0 or $\pm(2i-1)$ modulo $4r$.

Definition 15.

Let $D_{r,i}(n)$ denote the number of partitions $(\lambda_1, \lambda_2, \ldots, \lambda_s)$ of n satisfying the following conditions:

1. No odd part is repeated,
2. $\lambda_m - \lambda_{m+r-1} \geq 2$ if λ_m is odd,
3. $\lambda_m - \lambda_{m+r-1} \geq 3$ if λ_m is even, and
4. at most $i-1$ parts are equal to 1 or 2.

Göllnitz-Gordon-Andrews identities

Theorem 16 (Göllnitz-Gordon-Andrews identities).

Let r and i be positive integers with $1 \leq i \leq r$. Then $C_{r,i}(n) = D_{r,i}(n)$, for all $n \geq 0$.

Göllnitz-Gordon-Andrews identities

Theorem 16 (Göllnitz-Gordon-Andrews identities).

Let r and i be positive integers with $1 \leq i \leq r$. Then $C_{r,i}(n) = D_{r,i}(n)$, for all $n \geq 0$.

- The case $r = 2$ in theorem 16 gives the Göllnitz-Gordon identities.

Göllnitz-Gordon-Andrews identities

Theorem 16 (Göllnitz-Gordon-Andrews identities).

Let r and i be positive integers with $1 \leq i \leq r$. Then $C_{r,i}(n) = D_{r,i}(n)$, for all $n \geq 0$.

- The case $r = 2$ in theorem 16 gives the Göllnitz-Gordon identities.
- The case $r = 1$ leads to trivial identity $1 = 1$. We take $r \geq 2$.

Purpose of this seminar?

Outline

Graded ring

Definition 17 (Graded ring).

A graded ring is a ring A together with a family $(A_j)_{j\geq 0}$ of subgroups of the additive group of A, such that $A = \bigoplus_{j=0}^{\infty} A_j$ and $A_{j_1}A_{j_2} \subseteq A_{j_1+j_2}$ for all $j_1, j_2 \geq 0$.

Graded ring

Definition 17 (Graded ring).

A graded ring is a ring A together with a family $(A_j)_{j\geq 0}$ of subgroups of the additive group of A, such that $A = \bigoplus_{j=0}^{\infty} A_j$ and $A_{j_1}A_{j_2} \subseteq A_{j_1+j_2}$ for all $j_1, j_2 \geq 0$.

Remark 18.

1. *A_0 is a subring of A.*

Graded ring

Definition 17 (Graded ring).

A graded ring is a ring A together with a family $(A_j)_{j\geq 0}$ of subgroups of the additive group of A, such that $A = \bigoplus_{j=0}^{\infty} A_j$ and $A_{j_1}A_{j_2} \subseteq A_{j_1+j_2}$ for all $j_1, j_2 \geq 0$.

Remark 18.

1. *A_0 is a subring of A.*
2. *Each A_j is an A_0-module.*

Graded ring

Definition 17 (Graded ring).

A graded ring is a ring A together with a family $(A_j)_{j\geq 0}$ of subgroups of the additive group of A, such that $A = \bigoplus_{j=0}^{\infty} A_j$ and $A_{j_1}A_{j_2} \subseteq A_{j_1+j_2}$ for all $j_1, j_2 \geq 0$.

Remark 18.

1. *A_0 is a subring of A.*
2. *Each A_j is an A_0-module.*
3. *A non-zero element of A_j is said to be homogeneous element of degree j.*

Homogeneous ideal

Definition 19 (Homogeneous ideal).

An ideal I of a graded ring A is called homogeneous if it is generated by homogeneous elements.

Homogeneous ideal

Definition 19 (Homogeneous ideal).

An ideal I of a graded ring A is called homogeneous if it is generated by homogeneous elements.

Remark 20.

1. *A homogeneous ideal I is the direct sum of its homogeneous parts $I_j := I \cap A_j$, i.e., $I = \bigoplus_{j=0}^{\infty} I_j$.*

Homogeneous ideal

Definition 19 (Homogeneous ideal).

An ideal I of a graded ring A is called homogeneous if it is generated by homogeneous elements.

Remark 20.

1. *A homogeneous ideal I is the direct sum of its homogeneous parts $I_j := I \cap A_j$, i.e., $I = \bigoplus_{j=0}^{\infty} I_j$.*
2. *If I is a homogeneous ideal of a graded ring A, then the quotient ring $\frac{A}{I}$ is also a graded ring, decomposed as*

$$\frac{A}{I} = \bigoplus_{j=0}^{\infty} \frac{A_j}{I_j}.$$

Graded algebra

Definition 21 (Graded algebra).

Let $\mathbb{F}$ be a field. A graded ring $A = \bigoplus_{j=0}^{\infty} A_j$ is called a graded $\mathbb{F}$-algebra if it is also an $\mathbb{F}$-algebra, and A_j is a vector space for all $j \geq 0$ with $A_0 = \mathbb{F}$.

Graded algebra

Definition 21 (Graded algebra).

Let $\mathbb{F}$ be a field. A graded ring $A = \bigoplus_{j=0}^{\infty} A_j$ is called a graded $\mathbb{F}$-algebra if it is also an $\mathbb{F}$-algebra, and A_j is a vector space for all $j \geq 0$ with $A_0 = \mathbb{F}$.

Definition 22 (Weight of a polynomial).

The weight of the monomial $x_{i_1}^{\alpha_1} \cdots x_{i_m}^{\alpha_m} \in \mathbb{F}[x_1, x_2, \ldots]$ is defined as $\sum_{k=1}^{m} i_k \alpha_k$. A polynomial $f(x) \in \mathbb{F}[x_1, x_2, \ldots]$ is said to be a homogeneous polynomial of weight a if every monomial of $f(x)$ has the same weight a.

Hilbert-Poincaré series

Example 23 (Gradation by weight).

Let $\mathbb{F}$ be a field of characteristic zero. Then $A := \mathbb{F}[x_1, x_2, \ldots]$ is a graded algebra. A is graded by weight, i.e., $A = \bigoplus_{j=0}^{\infty} A_j$, where A_j is the set of polynomials of weight j along with zero polynomial.

Hilbert-Poincaré series

Example 23 (Gradation by weight).

Let $\mathbb{F}$ be a field of characteristic zero. Then $A := \mathbb{F}[x_1, x_2, \ldots]$ is a graded algebra. A is graded by weight, i.e., $A = \bigoplus_{j=0}^{\infty} A_j$, where A_j is the set of polynomials of weight j along with zero polynomial.

Definition 24 (Hilbert-Poincaré series).

Let $\mathbb{F}$ be a field of characteristic zero and $A = \bigoplus_{j=0}^{\infty} A_j$ be a graded $\mathbb{F}$-algebra such that $\dim_{\mathbb{F}}(A_j) < \infty$. Then the Hilbert-Poincaré series of A is

$$\mathrm{HP}_A(q) := \sum_{j \geq 0} \dim_{\mathbb{F}}(A_j) q^j.$$

My Collaborators

Figure: Prof. Rupam Barman

Figure: Dr. Gurinder Singh

Outline

Göllnitz-Gordon-Andrews identities and Generating function of $C_{r,i}(n)$

Theorem 25 (Göllnitz-Gordon-Andrews identities).

Let r and i be positive integers with $1 \leq i \leq r$. Then $C_{r,i}(n) = D_{r,i}(n)$, for all $n \geq 0$.

- The case $r = 2$ in Theorem 25 gives the Göllnitz-Gordon identities.
- The case $r = 1$ leads to trivial identity $1 = 1$. We take $r \geq 2$.

Göllnitz-Gordon-Andrews identities and Generating function of $C_{r,i}(n)$

Theorem 25 (Göllnitz-Gordon-Andrews identities).

Let r and i be positive integers with $1 \leq i \leq r$. Then $C_{r,i}(n) = D_{r,i}(n)$, for all $n \geq 0$.

- The case $r = 2$ in Theorem 25 gives the Göllnitz-Gordon identities.
- The case $r = 1$ leads to trivial identity $1 = 1$. We take $r \geq 2$.

For $1 \leq i \leq r$, define

$$C_i(q) := \prod_{\substack{m \geq 1,\ m \not\equiv 2 \pmod 4 \\ m \not\equiv 0,\ 2r \pm (2i-1) \pmod{4r}}} \frac{1}{(1 - q^m)}.$$

Göllnitz-Gordon-Andrews identities and Generating function of $C_{r,i}(n)$

Theorem 25 (Göllnitz-Gordon-Andrews identities).

Let r and i be positive integers with $1 \le i \le r$. Then $C_{r,i}(n) = D_{r,i}(n)$, for all $n \ge 0$.

- The case $r = 2$ in Theorem 25 gives the Göllnitz-Gordon identities.
- The case $r = 1$ leads to trivial identity $1 = 1$. We take $r \ge 2$.

For $1 \le i \le r$, define

$$C_i(q) := \prod_{\substack{m \ge 1,\ m \not\equiv 2 \pmod 4 \\ m \not\equiv 0,\ 2r \pm (2i-1) \pmod{4r}}} \frac{1}{(1-q^m)}.$$

- $C_{r-i+1}(q)$ is the generating function of $C_{r,i}(n)$.

Generalization of $C_i(q)$

For $g \geq 1$, the series $C_{(r-1)g+i}(q)$ is defined recursively by Coulson et al. [Ramanujan J. (2017)] as follows: For $i = 1$

$$C_{(r-1)g+1}(q) = C_{(r-1)(g-1)+r}(q)$$

Generalization of $C_i(q)$

For $g \geq 1$, the series $C_{(r-1)g+i}(q)$ is defined recursively by Coulson et al. [Ramanujan J. (2017)] as follows: For $i = 1$

$$C_{(r-1)g+1}(q) = C_{(r-1)(g-1)+r}(q)$$

and for $i = 2, \ldots, r$

$$C_{(r-1)g+i}(q) = \frac{C_{(r-1)(g-1)+r-i+1}(q) - C_{(r-1)(g-1)+r-i+2}(q)}{q^{2g(i-1)}} - \frac{C_{(r-1)g+i-1}(q)}{q}.$$

$E_{r,i,J}(n)$ as a generalization of $D_{r,i}(n)$

Let r, i, and $J \geq 0$ be integers such that $1 \leq i \leq r$. Let $E_{r,i,J}(n)$ denote the number of partitions $(\lambda_1, \lambda_2, \ldots, \lambda_s)$ of n satisfying the following conditions:

1. No odd part is repeated,

$E_{r,i,J}(n)$ as a generalization of $D_{r,i}(n)$

Let r, i, and $J \geq 0$ be integers such that $1 \leq i \leq r$. Let $E_{r,i,J}(n)$ denote the number of partitions $(\lambda_1, \lambda_2, \ldots, \lambda_s)$ of n satisfying the following conditions:

1. No odd part is repeated,
2. $\lambda_m - \lambda_{m+r-1} \geq 2$ if λ_m is odd,

$E_{r,i,J}(n)$ as a generalization of $D_{r,i}(n)$

Let r, i, and $J \geq 0$ be integers such that $1 \leq i \leq r$. Let $E_{r,i,J}(n)$ denote the number of partitions $(\lambda_1, \lambda_2, \ldots, \lambda_s)$ of n satisfying the following conditions:

1. No odd part is repeated,
2. $\lambda_m - \lambda_{m+r-1} \geq 2$ if λ_m is odd,
3. $\lambda_m - \lambda_{m+r-1} \geq 3$ if λ_m is even,

$E_{r,i,J}(n)$ as a generalization of $D_{r,i}(n)$

Let r, i, and $J \geq 0$ be integers such that $1 \leq i \leq r$. Let $E_{r,i,J}(n)$ denote the number of partitions $(\lambda_1, \lambda_2, \ldots, \lambda_s)$ of n satisfying the following conditions:

1. No odd part is repeated,
2. $\lambda_m - \lambda_{m+r-1} \geq 2$ if λ_m is odd,
3. $\lambda_m - \lambda_{m+r-1} \geq 3$ if λ_m is even,
4. all parts are greater than $2J$,

$E_{r,i,J}(n)$ as a generalization of $D_{r,i}(n)$

Let r, i, and $J \geq 0$ be integers such that $1 \leq i \leq r$. Let $E_{r,i,J}(n)$ denote the number of partitions $(\lambda_1, \lambda_2, \ldots, \lambda_s)$ of n satisfying the following conditions:

1. No odd part is repeated,
2. $\lambda_m - \lambda_{m+r-1} \geq 2$ if λ_m is odd,
3. $\lambda_m - \lambda_{m+r-1} \geq 3$ if λ_m is even,
4. all parts are greater than $2J$,and
5. at most $i-1$ parts are equal to $2J+1$ or $2J+2$.

$E_{r,i,J}(n)$ as a generalization of $D_{r,i}(n)$

Let r, i, and $J \geq 0$ be integers such that $1 \leq i \leq r$. Let $E_{r,i,J}(n)$ denote the number of partitions $(\lambda_1, \lambda_2, \ldots, \lambda_s)$ of n satisfying the following conditions:

1. No odd part is repeated,
2. $\lambda_m - \lambda_{m+r-1} \geq 2$ if λ_m is odd,
3. $\lambda_m - \lambda_{m+r-1} \geq 3$ if λ_m is even,
4. all parts are greater than $2J$,and
5. at most $i - 1$ parts are equal to $2J + 1$ or $2J + 2$.

Let $\mathcal{E}_{r,i,J}(q)$ denote the generating function of $E_{r,i,J}(n)$.

J-generalization

Theorem 26 (Coulson et al., Ramanujan J. (2017)).

For any nonnegative integer J *and* $1 \le i \le r$, *we have*

$$C_{(r-1)J+\ell}(q) = \mathcal{E}_{r,i,J}(q),$$

where $\ell = r - i + 1$.

J-generalization

Theorem 26 (Coulson et al., Ramanujan J. (2017)).

For any nonnegative integer J and $1 \leq i \leq r$, we have

$$C_{(r-1)J+\ell}(q) = \mathcal{E}_{r,i,J}(q),$$

where $\ell = r - i + 1$.

- We note that $E_{r,i,0}(n) = D_{r,i}(n)$.

J-generalization

Theorem 26 (Coulson et al., Ramanujan J. (2017)).

For any nonnegative integer J and $1 \le i \le r$, we have

$$C_{(r-1)J+\ell}(q) = \mathcal{E}_{r,i,J}(q),$$

where $\ell = r - i + 1$.

- We note that $E_{r,i,0}(n) = D_{r,i}(n)$.
- The case $J = 0$ in Theorem 26 gives the Göllnitz-Gordon-Andrews identities.

J-generalization

Theorem 26 (Coulson et al., Ramanujan J. (2017)).

For any nonnegative integer J and $1 \leq i \leq r$, we have

$$C_{(r-1)J+\ell}(q) = \mathcal{E}_{r,i,J}(q),$$

where $\ell = r - i + 1$.

- We note that $E_{r,i,0}(n) = D_{r,i}(n)$.
- The case $J = 0$ in Theorem 26 gives the Göllnitz-Gordon-Andrews identities.
- It is not known if $C_{(r-1)J+\ell}(q)$ for $J \geq 1$ is generating function of some partition function.

Notations

- $r \geq 2$.

Notations

- $r \geq 2$.
- Let $\mathbb{F}$ be a field of characteristic zero.

Notations

- $r \geq 2$.
- Let $\mathbb{F}$ be a field of characteristic zero.
- $S := \mathbb{F}[x_1, x_2, x_3, \ldots]$.

Notations

- $r \geq 2$.
- Let $\mathbb{F}$ be a field of characteristic zero.
- $S := \mathbb{F}[x_1, x_2, x_3, \ldots]$.
- $S_k := \mathbb{F}[x_k, x_{k+1}, x_{k+2}, \ldots]$.

Notations

- $r \geq 2$.
- Let $\mathbb{F}$ be a field of characteristic zero.
- $S := \mathbb{F}[x_1, x_2, x_3, \ldots]$.
- $S_k := \mathbb{F}[x_k, x_{k+1}, x_{k+2}, \ldots]$.
- S_k is a graded algebra, graded by weight.

Notations

- $r \geq 2$.
- Let $\mathbb{F}$ be a field of characteristic zero.
- $S := \mathbb{F}[x_1, x_2, x_3, \ldots]$.
- $S_k := \mathbb{F}[x_k, x_{k+1}, x_{k+2}, \ldots]$.
- S_k is a graded algebra, graded by weight.
- $S_1 = S$.

Notations

- $r \geq 2$.
- Let $\mathbb{F}$ be a field of characteristic zero.
- $S := \mathbb{F}[x_1, x_2, x_3, \ldots]$.
- $S_k := \mathbb{F}[x_k, x_{k+1}, x_{k+2}, \ldots]$.
- S_k is a graded algebra, graded by weight.
- $S_1 = S$.
- By $(S_k)_j$ we mean the homogeneous part of degree j of the graded algebra S_k.

The ideal $L_{r,i,J}$

Let $a,\ b,\ c,\ n_1$, and n_2 be integers. For $r \geq 2$, $1 \leq i \leq r$, and a fixed nonnegative integer J, we take the ideal

$$L_{r,i,J} := \Big(x_{2J+1}^2,\ x_{2J+1}x_{2J+2}^{i-1},\ x_{2J+2}^{i},\ x_{2a-1}^2,\ x_{2b-1}x_{2b}^{r-1},\ x_{2c}^{r-n_1}x_{2c+2}^{n_1},$$
$$x_{2c}^{r-n_2-1}x_{2c+1}x_{2c+2}^{n_2} : 2a-1, 2b-1, 2c \geq 2J+2; 0 \leq n_1 \leq r-1; 0 \leq n_2 \leq r-2\Big).$$

The ideal $L_{r,i,J}$

Let a, b, c, n_1, and n_2 be integers. For $r \geq 2$, $1 \leq i \leq r$, and a fixed nonnegative integer J, we take the ideal

$$L_{r,i,J} := \Big(x_{2J+1}^2,\ x_{2J+1}x_{2J+2}^{i-1},\ x_{2J+2}^{i},\ x_{2a-1}^2,\ x_{2b-1}x_{2b}^{r-1},\ x_{2c}^{r-n_1}x_{2c+2}^{n_1},$$
$$x_{2c}^{r-n_2-1}x_{2c+1}x_{2c+2}^{n_2} : 2a-1, 2b-1, 2c \geq 2J+2; 0 \leq n_1 \leq r-1; 0 \leq n_2 \leq r-2\Big).$$

- $L_{r,i,J}$ is a homogeneous ideal of S_{2J+1}.

The ideal $L_{r,i,J}$

Let $a,\ b,\ c,\ n_1$, and n_2 be integers. For $r \geq 2$, $1 \leq i \leq r$, and a fixed nonnegative integer J, we take the ideal

$$L_{r,i,J} := \Big(x_{2J+1}^2,\ x_{2J+1}x_{2J+2}^{i-1},\ x_{2J+2}^{i},\ x_{2a-1}^{2},\ x_{2b-1}x_{2b}^{r-1},\ x_{2c}^{r-n_1}x_{2c+2}^{n_1},$$
$$x_{2c}^{r-n_2-1}x_{2c+1}x_{2c+2}^{n_2} : 2a-1, 2b-1, 2c \geq 2J+2; 0 \leq n_1 \leq r-1; 0 \leq n_2 \leq r-2\Big).$$

- $L_{r,i,J}$ is a homogeneous ideal of S_{2J+1}.
- $\frac{S_{2J+1}}{L_{r,i,J}}$ is a graded algebra.

The ideal $L_{r,i,J}$

Let $a,\ b,\ c,\ n_1$, and n_2 be integers. For $r \geq 2$, $1 \leq i \leq r$, and a fixed nonnegative integer J, we take the ideal

$$L_{r,i,J} := \Big(x_{2J+1}^2,\ x_{2J+1}x_{2J+2}^{i-1},\ x_{2J+2}^{i},\ x_{2a-1}^2,\ x_{2b-1}x_{2b}^{r-1},\ x_{2c}^{r-n_1}x_{2c+2}^{n_1},\ x_{2c}^{r-n_2-1}x_{2c+1}x_{2c+2}^{n_2} : 2a-1, 2b-1, 2c \geq 2J+2; 0 \leq n_1 \leq r-1; 0 \leq n_2 \leq r-2 \Big).$$

- $L_{r,i,J}$ is a homogeneous ideal of S_{2J+1}.
- $\frac{S_{2J+1}}{L_{r,i,J}}$ is a graded algebra.

$$\dim_{\mathbb{F}} \left(\frac{S_{2J+1}}{L_{r,i,J}} \right)_j = \dim_{\mathbb{F}} \left(\frac{(S_{2J+1})_j}{(L_{r,i,J})_j} \right) \leq \dim_{\mathbb{F}}((S_{2J+1})_j) \leq \dim_{\mathbb{F}}((S)_j) = p(j) < \infty.$$

Correspondence between partitions and monomials

Let

$$T_j = \{x_{l_1}x_{l_2}\cdots x_{l_m} \in S \mid l_1 \geq l_2 \geq \cdots \geq l_m,\ m \in \mathbb{N}, \text{and} \sum_{p=1}^{m} l_p = j\}$$

Correspondence between partitions and monomials

Let

$$T_j = \{x_{l_1} x_{l_2} \cdots x_{l_m} \in S \mid l_1 \geq l_2 \geq \cdots \geq l_m,\ m \in \mathbb{N}, \text{and} \sum_{p=1}^{m} l_p = j\}$$

and

$$T'_j = \{(l_1, l_2, \ldots, l_m) \mid l_1 \geq l_2 \geq \cdots \geq l_m,\ m \in \mathbb{N}, \text{and} \sum_{p=1}^{m} l_p = j\}.$$

Correspondence between partitions and monomials

Let

$$T_j = \{x_{l_1}x_{l_2}\cdots x_{l_m} \in S \mid l_1 \geq l_2 \geq \cdots \geq l_m,\ m \in \mathbb{N}, \text{and} \sum_{p=1}^{m} l_p = j\}$$

and

$$T'_j = \{(l_1, l_2, \ldots, l_m) \mid l_1 \geq l_2 \geq \cdots \geq l_m,\ m \in \mathbb{N}, \text{and} \sum_{p=1}^{m} l_p = j\}.$$

Then,

$$(l_1, l_2, \ldots, l_m) \longleftrightarrow x_{l_1}x_{l_2}\cdots x_{l_m}$$

is a bijection between T'_j and T_j.

Correspondence between partitions and monomials

Let

$$T_j = \{x_{l_1}x_{l_2}\cdots x_{l_m} \in S \mid l_1 \geq l_2 \geq \cdots \geq l_m,\ m \in \mathbb{N}, \text{and} \sum_{p=1}^{m} l_p = j\}$$

and

$$T'_j = \{(l_1, l_2, \ldots, l_m) \mid l_1 \geq l_2 \geq \cdots \geq l_m,\ m \in \mathbb{N}, \text{and} \sum_{p=1}^{m} l_p = j\}.$$

Then,

$$(l_1, l_2, \ldots, l_m) \longleftrightarrow x_{l_1}x_{l_2}\cdots x_{l_m}$$

is a bijection between T'_j and T_j.Also T_j is a basis of $(S)_j$. Hence,

$$\dim_{\mathbb{F}}((S)_j) = p(j).$$

Generating function of $E_{r,i,J}(j)$

This indicates the existence of Hilbert-Poincaré series of $\frac{S_{2J+1}}{L_{r,i,J}}$, which by definition is as follows:

$$\mathrm{HP}_{\frac{S_{2J+1}}{L_{r,i,J}}}(q) = \sum_{j\geq 0} \dim_{\mathbb{F}} \left(\frac{S_{2J+1}}{L_{r,i,J}}\right)_j q^j.$$

Generating function of $E_{r,i,J}(j)$

This indicates the existence of Hilbert-Poincaré series of $\frac{S_{2J+1}}{L_{r,i,J}}$, which by definition is as follows:

$$\mathrm{HP}_{\frac{S_{2J+1}}{L_{r,i,J}}}(q) = \sum_{j\geq 0} \dim_{\mathbb{F}} \left(\frac{S_{2J+1}}{L_{r,i,J}}\right)_j q^j.$$

- $L_{r,i,J}$ is generated by $x_{2J+1}^2, x_{2J+1}x_{2J+2}^{i-1}, x_{2J+2}^i$ and the monomials of the form
$$x_{2a-1}^2,\ x_{2b-1}x_{2b}^{r-1},\ x_{2c}^{r-n_1}x_{2c+2}^{n_1},\ x_{2c}^{r-n_2-1}x_{2c+1}x_{2c+2}^{n_2}$$
such that $2a-1, 2b-1, 2c \geq 2J+2$, $0 \leq n_1 \leq r-1$, and $0 \leq n_2 \leq r-2$.

Generating function of $E_{r,i,J}(j)$

This indicates the existence of Hilbert-Poincaré series of $\frac{S_{2J+1}}{L_{r,i,J}}$, which by definition is as follows:

$$\mathrm{HP}_{\frac{S_{2J+1}}{L_{r,i,J}}}(q) = \sum_{j\geq 0} \dim_{\mathbb{F}} \left(\frac{S_{2J+1}}{L_{r,i,J}}\right)_j q^j.$$

- $L_{r,i,J}$ is generated by $x_{2J+1}^2, x_{2J+1}x_{2J+2}^{i-1}, x_{2J+2}^i$ and the monomials of the form
$$x_{2a-1}^2,\ x_{2b-1}x_{2b}^{r-1},\ x_{2c}^{r-n_1}x_{2c+2}^{n_1},\ x_{2c}^{r-n_2-1}x_{2c+1}x_{2c+2}^{n_2}$$
such that $2a-1, 2b-1, 2c \geq 2J+2$, $0 \leq n_1 \leq r-1$, and $0 \leq n_2 \leq r-2$.
- $\dim_{\mathbb{F}}\left(\frac{S_{2J+1}}{L_{r,i,J}}\right)_j = E_{r,i,J}(j)$.

Generating function of $E_{r,i,J}(j)$

This indicates the existence of Hilbert-Poincaré series of $\frac{S_{2J+1}}{L_{r,i,J}}$, which by definition is as follows:

$$\mathrm{HP}_{\frac{S_{2J+1}}{L_{r,i,J}}}(q) = \sum_{j\geq 0} \dim_{\mathbb{F}}\left(\frac{S_{2J+1}}{L_{r,i,J}}\right)_j q^j.$$

- $L_{r,i,J}$ is generated by $x_{2J+1}^2,\ x_{2J+1}x_{2J+2}^{i-1},\ x_{2J+2}^i$ and the monomials of the form

  $$x_{2a-1}^2,\ x_{2b-1}x_{2b}^{r-1},\ x_{2c}^{r-n_1}x_{2c+2}^{n_1},\ x_{2c}^{r-n_2-1}x_{2c+1}x_{2c+2}^{n_2}$$

  such that $2a-1, 2b-1, 2c \geq 2J+2$, $0 \leq n_1 \leq r-1$, and $0 \leq n_2 \leq r-2$.
- $\dim_{\mathbb{F}}\left(\frac{S_{2J+1}}{L_{r,i,J}}\right)_j = E_{r,i,J}(j)$.

Therefore,

$$\mathrm{HP}_{\frac{S_{2J+1}}{L_{r,i,J}}}(q) = \sum_{j\geq 0} E_{r,i,J}(j)q^j = \mathcal{E}_{r,i,J}(q). \tag{1}$$

The ideals L_k and L_k^ℓ

Let $a,\ b,\ c,\ n_1,\ n_2$, and ℓ be integers such that $1 \leq \ell \leq r$. We define the following two ideals of S_k for $k \geq 2J+1$:

$$L_k := \Big(x_{2a-1}^2,\ x_{2b-1}x_{2b}^{r-1},\ x_{2c}^{r-n_1}x_{2c+2}^{n_1},\ x_{2c}^{r-n_2-1}x_{2c+1}x_{2c+2}^{n_2} :$$
$$2a-1, 2b-1, 2c \geq k,\ 0 \leq n_1 \leq r-1,\ \text{and } 0 \leq n_2 \leq r-2\Big)$$

The ideals L_k and L_k^ℓ

Let $a,\ b,\ c,\ n_1,\ n_2$, and ℓ be integers such that $1 \le \ell \le r$. We define the following two ideals of S_k for $k \ge 2J+1$:

$$L_k := \Big(x_{2a-1}^2,\ x_{2b-1}x_{2b}^{r-1},\ x_{2c}^{r-n_1}x_{2c+2}^{n_1},\ x_{2c}^{r-n_2-1}x_{2c+1}x_{2c+2}^{n_2} : \\ 2a-1, 2b-1, 2c \ge k,\ 0 \le n_1 \le r-1,\ \text{and } 0 \le n_2 \le r-2\Big)$$

and

$$L_k^\ell := \begin{cases} \Big(x_k^\ell,\ x_k^{\ell-1}x_{k+2}^{r-\ell+1},\ x_k^{\ell-2}x_{k+2}^{r-\ell+2}, \ldots, x_k x_{k+2}^{r-1},\ x_k^{\ell-1}x_{k+1}x_{k+2}^{r-\ell}, \\ \quad x_k^{\ell-2}x_{k+1}x_{k+2}^{r-\ell+1}, \ldots,\ x_k x_{k+1} x_{k+2}^{r-2},\ L_{k+1}\Big) & \text{if } k \text{ is even;} \\ \Big(x_k^2,\ x_k x_{k+1}^{\ell-1},\ L_{k+1}^\ell\Big) & \text{if } k \text{ is odd.} \end{cases}$$

The ideals L_k and L_k^ℓ

Let $a,\ b,\ c,\ n_1,\ n_2$, and ℓ be integers such that $1 \le \ell \le r$. We define the following two ideals of S_k for $k \ge 2J+1$:

$$L_k := \big(x_{2a-1}^2,\ x_{2b-1}x_{2b}^{r-1},\ x_{2c}^{r-n_1}x_{2c+2}^{n_1},\ x_{2c}^{r-n_2-1}x_{2c+1}x_{2c+2}^{n_2} : \\ 2a-1, 2b-1, 2c \ge k,\ 0 \le n_1 \le r-1,\ \text{and } 0 \le n_2 \le r-2\big)$$

and

$$L_k^\ell := \begin{cases} \big(x_k^\ell,\ x_k^{\ell-1}x_{k+2}^{r-\ell+1},\ x_k^{\ell-2}x_{k+2}^{r-\ell+2}, \ldots, x_k x_{k+2}^{r-1},\ x_k^{\ell-1}x_{k+1}x_{k+2}^{r-\ell}, \\ \quad x_k^{\ell-2}x_{k+1}x_{k+2}^{r-\ell+1}, \ldots,\ x_k x_{k+1} x_{k+2}^{r-2},\ L_{k+1}\big) & \text{if } k \text{ is even;} \\ \big(x_k^2,\ x_k x_{k+1}^{\ell-1},\ L_{k+1}^\ell\big) & \text{if } k \text{ is odd.} \end{cases}$$

- L_k and L_k^ℓ are homogeneous ideals of S_k.

The ideals L_k and L_k^ℓ

Let $a,\ b,\ c,\ n_1,\ n_2$, and ℓ be integers such that $1 \le \ell \le r$. We define the following two ideals of S_k for $k \ge 2J+1$:

$$L_k := \Big(x_{2a-1}^2,\ x_{2b-1}x_{2b}^{r-1},\ x_{2c}^{r-n_1}x_{2c+2}^{n_1},\ x_{2c}^{r-n_2-1}x_{2c+1}x_{2c+2}^{n_2} : \\ 2a-1, 2b-1, 2c \ge k,\ 0 \le n_1 \le r-1,\ \text{and}\ 0 \le n_2 \le r-2\Big)$$

and

$$L_k^\ell := \begin{cases} \Big(x_k^\ell,\ x_k^{\ell-1}x_{k+2}^{r-\ell+1},\ x_k^{\ell-2}x_{k+2}^{r-\ell+2}, \ldots, x_k x_{k+2}^{r-1},\ x_k^{\ell-1}x_{k+1}x_{k+2}^{r-\ell}, \\ \quad x_k^{\ell-2}x_{k+1}x_{k+2}^{r-\ell+1}, \ldots,\ x_k x_{k+1}x_{k+2}^{r-2},\ L_{k+1}\Big) & \text{if } k \text{ is even;} \\ \Big(x_k^2,\ x_k x_{k+1}^{\ell-1},\ L_{k+1}^\ell\Big) & \text{if } k \text{ is odd.} \end{cases}$$

- L_k and L_k^ℓ are homogeneous ideals of S_k.
- $\frac{S_k}{L_k}$ and $\frac{S_k}{L_k^\ell}$ are graded algebras.

Notations and Links

- $\mathrm{HP}_{\frac{S_k}{L_k^\ell}}(q) := \mathrm{HP}_\ell^k$

Notations and Links

- $\mathrm{HP}_{\frac{S_k}{L_k^{\ell}}}(q) := \mathrm{HP}_{\ell}^{k}$
- $\mathrm{HP}_{\frac{A}{I}}(q) := \mathrm{HP}\left(\frac{A}{I}\right)$.

Notations and Links

- $\mathrm{HP}_{\frac{S_k}{L_k^\ell}}(q) := \mathrm{HP}_\ell^k$
- $\mathrm{HP}_{\frac{A}{I}}(q) := \mathrm{HP}\left(\frac{A}{I}\right)$.
- $\mathrm{HP}\left(\frac{S_{2J+1}}{L_{r,i,J}}\right) = \mathrm{HP}\left(\frac{S_{2J+1}}{L_{2J+1}^i}\right) = \mathrm{HP}_i^{2J+1}$.

Notations and Links

- $\mathrm{HP}_{\frac{S_k}{L_k^{\ell}}}(q) := \mathrm{HP}_{\ell}^{k}$
- $\mathrm{HP}_{\frac{A}{I}}(q) := \mathrm{HP}\left(\frac{A}{I}\right)$.
- $\mathrm{HP}\left(\frac{S_{2J+1}}{L_{r,i,J}}\right) = \mathrm{HP}\left(\frac{S_{2J+1}}{L_{2J+1}^{i}}\right) = \mathrm{HP}_{i}^{2J+1}$.
- $\mathrm{HP}_{\frac{S_{2J+1}}{L_{r,i,J}}}(q) = \mathcal{E}_{r,i,J}(q)$.

Notations and Links

- $\mathrm{HP}_{\frac{S_k}{L_k^{\ell}}}(q) := \mathrm{HP}_{\ell}^{k}$

- $\mathrm{HP}_{\frac{A}{I}}(q) := \mathrm{HP}\left(\frac{A}{I}\right)$.

- $\mathrm{HP}\left(\frac{S_{2J+1}}{L_{r,i,J}}\right) = \mathrm{HP}\left(\frac{S_{2J+1}}{L_{2J+1}^{i}}\right) = \mathrm{HP}_{i}^{2J+1}$.

- $\mathrm{HP}_{\frac{S_{2J+1}}{L_{r,i,J}}}(q) = \mathcal{E}_{r,i,J}(q)$.

Hence, we have

$$\mathrm{HP}_{i}^{2J+1} = \mathcal{E}_{r,i,J}(q).$$

Notations and Links

- $\mathrm{HP}_{\frac{S_k}{L_k^\ell}}(q) := \mathrm{HP}_\ell^k$
- $\mathrm{HP}_{\frac{A}{I}}(q) := \mathrm{HP}\left(\frac{A}{I}\right)$.
- $\mathrm{HP}\left(\frac{S_{2J+1}}{L_{r,i,J}}\right) = \mathrm{HP}\left(\frac{S_{2J+1}}{L_{2J+1}^i}\right) = \mathrm{HP}_i^{2J+1}$.
- $\mathrm{HP}_{\frac{S_{2J+1}}{L_{r,i,J}}}(q) = \mathcal{E}_{r,i,J}(q)$.

Hence, we have

$$\mathrm{HP}_i^{2J+1} = \mathcal{E}_{r,i,J}(q).$$

To prove Theorem 26, i.e., $C_{(r-1)J+\ell}(q) = \mathcal{E}_{r,i,J}(q)$, it is enough to prove $C_{(r-1)J+\ell} = \mathrm{HP}_i^{2J+1}$.

General recursion formula for HP_ℓ^k

Lemma 27 (Barman-Ghosh-Singh (Submitted)).

Let J be a nonnegative integer. Let k, r, and ℓ be positive integers with $r \geq 2$ and $1 \leq \ell \leq r$. Then for any odd $k \geq 2J+1$, we have

$$\mathrm{HP}_\ell^k = \sum_{j=1}^{\ell-1} q^{(k+1)j-1} \mathrm{HP}_{r-j+1}^{k+2} + \sum_{j=1}^{\ell} q^{(k+1)(j-1)} \mathrm{HP}_{r-j+1}^{k+2}.$$

General recursion formula for HP_i^{2J+1}

Lemma 28 (Barman-Ghosh-Singh (Submitted)).

Let J be a nonnegative integer, and r, i be integers with $r \geq 2,\ 1 \leq i \leq r$. Then we have the following recursion formula

$$\mathrm{HP}_i^{2J+1} = \sum_{j=1}^{r} N^J_{i,j,(r-1)d+j} \mathrm{HP}^{2d+1}_{r-j+1},$$

where $d \geq J+1$.

General recursion formula for HP_i^{2J+1}

Lemma 28 (Barman-Ghosh-Singh (Submitted)).

Let J be a nonnegative integer, and r, i be integers with $r \geq 2,\ 1 \leq i \leq r$. Then we have the following recursion formula

$$\mathrm{HP}_i^{2J+1} = \sum_{j=1}^{r} N^J_{i,j,(r-1)d+j} \mathrm{HP}_{r-j+1}^{2d+1},$$

where $d \geq J+1$.Here, the coefficients $N^J_{i,j,(r-1)d+j} \in \mathbb{F}[[q]]$ satisfy the following recursion formula for $1 \leq j \leq r$

$$N^J_{i,j,(r-1)(d+1)+j} = q^{2(d+1)(j-1)} \sum_{m=1}^{r-j+1} N^J_{i,m,(r-1)d+m} + q^{2(d+1)j-1} \sum_{m=1}^{r-j} N^J_{i,m,(r-1)d+m}$$

General recursion formula for HP_i^{2J+1}

Lemma 28 (Barman-Ghosh-Singh (Submitted)).

Let J be a nonnegative integer, and r, i be integers with $r \geq 2,\ 1 \leq i \leq r$. Then we have the following recursion formula

$$\mathrm{HP}_i^{2J+1} = \sum_{j=1}^{r} N^J_{i,j,(r-1)d+j} \mathrm{HP}^{2d+1}_{r-j+1},$$

where $d \geq J+1$.Here, the coefficients $N^J_{i,j,(r-1)d+j} \in \mathbb{F}[[q]]$ satisfy the following recursion formula for $1 \leq j \leq r$

$$N^J_{i,j,(r-1)(d+1)+j} = q^{2(d+1)(j-1)} \sum_{m=1}^{r-j+1} N^J_{i,m,(r-1)d+m} + q^{2(d+1)j-1} \sum_{m=1}^{r-j} N^J_{i,m,(r-1)d+m}$$

with the following initial conditions (for $d = J+1$)

$$N^J_{i,j,(r-1)(J+1)+j} = \begin{cases} q^{2(J+1)j-1} + q^{2(J+1)(j-1)} & \textit{if } 1 \leq j \leq i-1; \\ q^{2(J+1)(j-1)} & \textit{if } j = i; \\ 0 & \textit{if } i+1 \leq j \leq r. \end{cases}$$

General recursion formula for $C_{(r-1)J+\ell}$

Lemma 29 (Coulson et al., Ramanujan J. (2017)).

Let J *be a nonnegative integer and* $r,\ \ell$ *be integers with* $r \geq 2,\ 1 \leq \ell \leq r$. *Then for* $d \geq J+1$ *we have the following recursion formula*

$$C_{(r-1)J+\ell} = \sum_{j=1}^{r} M^{J}_{\ell,j,(r-1)d+j} C_{(r-1)d+j}.$$

General recursion formula for $C_{(r-1)J+\ell}$

Lemma 29 (Coulson et al., Ramanujan J. (2017)).

Let J *be a nonnegative integer and* $r,\ \ell$ *be integers with* $r \geq 2,\ 1 \leq \ell \leq r$. *Then for* $d \geq J+1$ *we have the following recursion formula*

$$C_{(r-1)J+\ell} = \sum_{j=1}^{r} M^{J}_{\ell,j,(r-1)d+j} C_{(r-1)d+j}.$$

Here, the coefficients $M^{J}_{\ell,j,(r-1)d+j} \in \mathbb{F}[[q]]$ *satisfy the following recursion formula for* $1 \leq j \leq r$

$$M^{J}_{\ell,j,(r-1)(d+1)+j} = q^{2(d+1)(j-1)} \sum_{m=1}^{r-j+1} M^{J}_{\ell,m,(r-1)d+m} + q^{2(d+1)j-1} \sum_{m=1}^{r-j} M^{J}_{\ell,m,(r-1)d+m}$$

General recursion formula for $C_{(r-1)J+\ell}$

Lemma 29 (Coulson et al., Ramanujan J. (2017)).

Let J *be a nonnegative integer and* $r,\ \ell$ *be integers with* $r \geq 2,\ 1 \leq \ell \leq r$. *Then for* $d \geq J+1$ *we have the following recursion formula*

$$C_{(r-1)J+\ell} = \sum_{j=1}^{r} M^J_{\ell,j,(r-1)d+j} C_{(r-1)d+j}.$$

Here, the coefficients $M^J_{\ell,j,(r-1)d+j} \in \mathbb{F}[[q]]$ *satisfy the following recursion formula for* $1 \leq j \leq r$

$$M^J_{\ell,j,(r-1)(d+1)+j} = q^{2(d+1)(j-1)} \sum_{m=1}^{r-j+1} M^J_{\ell,m,(r-1)d+m} + q^{2(d+1)j-1} \sum_{m=1}^{r-j} M^J_{\ell,m,(r-1)d+m}$$

with the following initial conditions (for $d = J+1$*)*

$$M^J_{\ell,j,(r-1)(J+1)+j} = \begin{cases} q^{2(J+1)j-1} + q^{2(J+1)(j-1)} & \text{if } 1 \leq j \leq r-\ell; \\ q^{2(J+1)(j-1)} & \text{if } j = r-\ell+1; \\ 0 & \text{if } r-\ell+2 \leq j \leq r. \end{cases}$$

Equality of the coefficients $M^J_{\ell,v,(r-1)d+v}$ and $N^J_{i,v,(r-1)d+v}$

Lemma 30 (Barman-Ghosh-Singh (Submitted)).

For all $d \geq J + 1$, $r \geq 2$, *and* $1 \leq v \leq r$, *we have*

$$M^J_{\ell,v,(r-1)d+v} = N^J_{i,v,(r-1)d+v},$$

where $\ell = r - i + 1$.

Equality of $C_{(r-1)J+\ell}$ and HP_i^{2J+1}

Theorem 31 (Barman-Ghosh-Singh (Submitted)).

For $r \geq 2$, $1 \leq i \leq r$, *and* $J \geq 0$, *we have*

$$C_{(r-1)J+\ell} = \mathrm{HP}_i^{2J+1},$$

where $\ell = r - i + 1$.

Outline

Rogers-Ramanujan-Gordon identities and Generating function of $A_{r,i}(n)$

Theorem 32 (Rogers-Ramanujan-Gordon identities).

Let r and i be positive integers with $1 \leq i \leq r$. Then $A_{r,i}(n) = B_{r,i}(n)$, for all $n \geq 0$.

- The case $r = 2$ in Theorem 32 gives the Rogers-Ramanujan identities.
- The case $r = 1$ leads to trivial identity $1 = 1$. Hence we take $r \geq 2$.

Rogers-Ramanujan-Gordon identities and Generating function of $A_{r,i}(n)$

Theorem 32 (Rogers-Ramanujan-Gordon identities).

Let r and i be positive integers with $1 \leq i \leq r$. Then $A_{r,i}(n) = B_{r,i}(n)$, for all $n \geq 0$.

- The case $r = 2$ in Theorem 32 gives the Rogers-Ramanujan identities.
- The case $r = 1$ leads to trivial identity $1 = 1$. Hence we take $r \geq 2$.

For $1 \leq i \leq r$, define

$$\mathcal{A}_i(q) := \prod_{\substack{m \geq 1, \\ m \not\equiv 0, \ \pm(r-i+1) \pmod{2r+1}}} \frac{1}{(1-q^m)}.$$

Rogers-Ramanujan-Gordon identities and Generating function of $A_{r,i}(n)$

Theorem 32 (Rogers-Ramanujan-Gordon identities).

Let r and i be positive integers with $1 \le i \le r$. Then $A_{r,i}(n) = B_{r,i}(n)$, for all $n \ge 0$.

- The case $r = 2$ in Theorem 32 gives the Rogers-Ramanujan identities.
- The case $r = 1$ leads to trivial identity $1 = 1$. Hence we take $r \ge 2$.

For $1 \le i \le r$, define

$$\mathcal{A}_i(q) := \prod_{\substack{m \ge 1, \\ m \not\equiv 0,\ \pm(r-i+1) \pmod{2r+1}}} \frac{1}{(1-q^m)}.$$

- We note that $\mathcal{A}_{r-i+1}(q)$ is the generating function of $A_{r,i}(n)$.

Generalization of $\mathcal{A}_i(q)$

For $g \geq 1$, the series $\mathcal{A}_{(r-1)g+i}(q)$ is defined recursively [J. Lepowsky and M. Zhu, The Ramanujan Journal (2012)] as follows: For $i = 1$

$$\mathcal{A}_{(r-1)g+1}(q) = \mathcal{A}_{(r-1)(g-1)+r}(q)$$

Generalization of $\mathcal{A}_i(q)$

For $g \geq 1$, the series $\mathcal{A}_{(r-1)g+i}(q)$ is defined recursively [J. Lepowsky and M. Zhu, The Ramanujan Journal (2012)] as follows: For $i = 1$

$$\mathcal{A}_{(r-1)g+1}(q) = \mathcal{A}_{(r-1)(g-1)+r}(q)$$

and for $i = 2, \ldots, r$

$$\mathcal{A}_{(r-1)g+i}(q) = \frac{\mathcal{A}_{(r-1)(g-1)+r-i+1}(q) - \mathcal{A}_{(r-1)(g-1)+r-i+2}(q)}{q^{g(i-1)}}.$$

J-generalization

Let r, i, and $J \geq 0$ be integers such that $1 \leq i \leq r$. Let $B_{r,i,J}(n)$ denote the number of partitions $(\lambda_1, \lambda_2, \ldots, \lambda_s)$ of n satisfying the following conditions:

1. $\lambda_m - \lambda_{m+r-1} \geq 2$,
2. all parts are greater than J, and
3. at most $i - 1$ parts are equal to $J + 1$.

J-generalization

Let r, i, and $J \geq 0$ be integers such that $1 \leq i \leq r$. Let $B_{r,i,J}(n)$ denote the number of partitions $(\lambda_1, \lambda_2, \ldots, \lambda_s)$ of n satisfying the following conditions:

1. $\lambda_m - \lambda_{m+r-1} \geq 2$,
2. all parts are greater than J, and
3. at most $i-1$ parts are equal to $J+1$.

Let $\mathcal{B}_{r,i,J}(q)$ denote the generating function of $B_{r,i,J}(n)$.

J-generalization

Let r, i, and $J \geq 0$ be integers such that $1 \leq i \leq r$. Let $B_{r,i,J}(n)$ denote the number of partitions $(\lambda_1, \lambda_2, \ldots, \lambda_s)$ of n satisfying the following conditions:

1. $\lambda_m - \lambda_{m+r-1} \geq 2$,
2. all parts are greater than J, and
3. at most $i-1$ parts are equal to $J+1$.

Let $\mathcal{B}_{r,i,J}(q)$ denote the generating function of $B_{r,i,J}(n)$.

Theorem 33 (Coulson et al., Ramanujan J. (2017)).

For any nonnegative integer J and $1 \leq i \leq r$, we have

$$\mathcal{A}_{(r-1)J+\ell}(q) = \mathcal{B}_{r,i,J}(q), \tag{2}$$

where $\ell = r - i + 1$.

Some observations

- The case $J = 0$ in Theorem 33 gives the Rogers-Ramanujan-Gordon identities.

Some observations

- The case $J = 0$ in Theorem 33 gives the Rogers-Ramanujan-Gordon identities.
- We note that $B_{r,i,0}(n) = B_{r,i}(n)$.

Some observations

- The case $J = 0$ in Theorem 33 gives the Rogers-Ramanujan-Gordon identities.
- We note that $B_{r,i,0}(n) = B_{r,i}(n)$.
- It is not in the literature whether $\mathcal{A}_{(r-1)J+\ell}(q)$ for $J \geq 1$ is generating function for some partition function.

The ideal $P_{r,i,J}$

Let a and t be integers. For $r \geq 2$, $1 \leq i \leq r$, and a fixed nonnegative integer J, we take the ideal

$$P_{r,i,J} := \left(x_{J+1}^{i},\ x_{J+1}^{i} x_{J+2}^{r-i+1}, x_{J+1}^{i-1} x_{J+2}^{r-i+2}, \ldots, x_{J+1} x_{J+2}^{r-1},\ x_a^{r-t} x_{a+1}^{t} \right.$$
$$\left. : a \geq J+2;\ 0 \leq t \leq r-1 \right)$$

The ideal $P_{r,i,J}$

Let a and t be integers. For $r \geq 2$, $1 \leq i \leq r$, and a fixed nonnegative integer J, we take the ideal

$$P_{r,i,J} := \big(x_{J+1}^{i},\ x_{J+1}^{i} x_{J+2}^{r-i+1}, x_{J+1}^{i-1} x_{J+2}^{r-i+2}, \ldots, x_{J+1} x_{J+2}^{r-1},\ x_a^{r-t} x_{a+1}^{t} \\ : a \geq J+2;\ 0 \leq t \leq r-1\big)$$

- $P_{r,i,J}$ is an homogeneous ideal of S_{J+1}.
- $\frac{S_{J+1}}{P_{r,i,J}}$ is a graded algebra.

The ideal $P_{r,i,J}$

Let a and t be integers. For $r \geq 2$, $1 \leq i \leq r$, and a fixed nonnegative integer J, we take the ideal

$$P_{r,i,J} := \Big(x_{J+1}^{i},\ x_{J+1}^{i}x_{J+2}^{r-i+1}, x_{J+1}^{i-1}x_{J+2}^{r-i+2}, \ldots, x_{J+1}x_{J+2}^{r-1},\ x_a^{r-t}x_{a+1}^{t}$$
$$: a \geq J+2;\ 0 \leq t \leq r-1\Big)$$

- $P_{r,i,J}$ is an homogeneous ideal of S_{J+1}.
- $\frac{S_{J+1}}{P_{r,i,J}}$ is a graded algebra.

$$\dim_{\mathbb{F}} \left(\frac{S_{J+1}}{P_{r,i,J}}\right)_j \leq p(j) < \infty.$$

Generating function of $B_{r,i,J}$

This indicates the existence of Hilbert-Poincaré series of $\frac{S_{J+1}}{P_{r,i,J}}$, which by definition is as follows:

$$\mathrm{HP}_{\frac{S_{J+1}}{P_{r,i,J}}}(q) = \sum_{j \geq 0} \dim_{\mathbb{F}} \left(\frac{S_{J+1}}{P_{r,i,J}} \right)_j q^j.$$

Generating function of $B_{r,i,J}$

This indicates the existence of Hilbert-Poincaré series of $\frac{S_{J+1}}{P_{r,i,J}}$, which by definition is as follows:

$$\mathrm{HP}_{\frac{S_{J+1}}{P_{r,i,J}}}(q) = \sum_{j\geq 0} \dim_{\mathbb{F}} \left(\frac{S_{J+1}}{P_{r,i,J}}\right)_j q^j.$$

- We know that $P_{r,i,J}$ is generated by $x_{J+1}^i, x_{J+1}^i x_{J+2}^{r-i+1}, \ldots, x_{J+1} x_{J+2}^{r-1}$ and the monomials of the form $x_a^{r-t} x_{a+1}^t$, such that $a \geq J+2$ and $0 \leq t \leq r-1$.
- $\dim_{\mathbb{F}} \left(\frac{S_{J+1}}{P_{r,i,J}}\right)_j = B_{r,i,J}(j)$.

Generating function of $B_{r,i,J}$

This indicates the existence of Hilbert-Poincaré series of $\frac{S_{J+1}}{P_{r,i,J}}$, which by definition is as follows:

$$\mathrm{HP}_{\frac{S_{J+1}}{P_{r,i,J}}}(q) = \sum_{j\geq 0} \dim_{\mathbb{F}} \left(\frac{S_{J+1}}{P_{r,i,J}}\right)_j q^j.$$

- We know that $P_{r,i,J}$ is generated by $x_{J+1}^i, x_{J+1}^i x_{J+2}^{r-i+1}, \ldots, x_{J+1} x_{J+2}^{r-1}$ and the monomials of the form $x_a^{r-t} x_{a+1}^t$, such that $a \geq J+2$ and $0 \leq t \leq r-1$.
- $\dim_{\mathbb{F}} \left(\frac{S_{J+1}}{P_{r,i,J}}\right)_j = B_{r,i,J}(j)$.

Therefore,

$$\mathrm{HP}_{\frac{S_{J+1}}{P_{r,i,J}}}(q) = \sum_{j\geq 0} B_{r,i,J}(j) q^j = \mathcal{B}_{r,i,J}(q). \tag{3}$$

The ideals P_k and P_k^ℓ

Let $a,\ t$, and ℓ be integers such that $1 \leq \ell \leq r$. We define the following two ideals of S_k for $k \geq J+1$:

$$P_k := \left(x_a^{r-t} x_{a+1}^{t} : a \geq k,\ 0 \leq t \leq r-1 \right)$$

The ideals P_k and P_k^ℓ

Let $a,\ t$, and ℓ be integers such that $1 \leq \ell \leq r$. We define the following two ideals of S_k for $k \geq J+1$:

$$P_k := \left(x_a^{r-t}x_{a+1}^t : a \geq k,\ 0 \leq t \leq r-1\right)$$

and

$$P_k^\ell := \left(x_k^\ell,\ x_k^{\ell-1}x_{k+1}^{r-\ell+1},\ x_k^{\ell-2}x_{k+1}^{r-\ell+2}, \ldots, x_k x_{k+1}^{r-1},\ P_{k+1}\right).$$

The ideals P_k and P_k^ℓ

Let $a,\ t$, and ℓ be integers such that $1 \leq \ell \leq r$. We define the following two ideals of S_k for $k \geq J+1$:

$$P_k := \left(x_a^{r-t}x_{a+1}^t : a \geq k,\ 0 \leq t \leq r-1\right)$$

and

$$P_k^\ell := \left(x_k^\ell,\ x_k^{\ell-1}x_{k+1}^{r-\ell+1},\ x_k^{\ell-2}x_{k+1}^{r-\ell+2}, \ldots, x_k x_{k+1}^{r-1},\ P_{k+1}\right).$$

- P_k and P_k^ℓ are homogeneous ideals of S_{J+1}.

The ideals P_k and P_k^ℓ

Let $a,\ t$, and ℓ be integers such that $1 \leq \ell \leq r$. We define the following two ideals of S_k for $k \geq J+1$:

$$P_k := \left(x_a^{r-t} x_{a+1}^{t} : a \geq k,\ 0 \leq t \leq r-1\right)$$

and

$$P_k^\ell := \left(x_k^\ell,\ x_k^{\ell-1} x_{k+1}^{r-\ell+1},\ x_k^{\ell-2} x_{k+1}^{r-\ell+2}, \ldots, x_k x_{k+1}^{r-1},\ P_{k+1}\right).$$

- P_k and P_k^ℓ are homogeneous ideals of S_{J+1}.
- $\frac{S_{J+1}}{P_k}$ and $\frac{S_{J+1}}{P_k^\ell}$ are graded algebras.

Notations and properties

- $\mathrm{HP}_{\frac{S_k}{P_k^{\ell}}}(q) := \mathrm{HP}_{\ell}^{k}$.
- $\mathrm{HP}_{\frac{A}{I}}(q) := \mathrm{HP}\left(\frac{A}{I}\right)$.

Notations and properties

- $\mathrm{HP}_{\frac{S_k}{P_k^{\ell}}}(q) := \mathrm{HP}_{\ell}^{k}$.
- $\mathrm{HP}_{\frac{A}{I}}(q) := \mathrm{HP}\left(\frac{A}{I}\right)$.
- $\mathrm{HP}\left(\frac{S_{J+1}}{P_{r,i,J}}\right) = \mathrm{HP}\left(\frac{S_{J+1}}{P_{J+1}^{i}}\right) = \mathrm{HP}_{i}^{J+1}$.

Notations and properties

- $\mathrm{HP}_{\frac{S_k}{P_k^\ell}}(q) := \mathrm{HP}_\ell^k$.
- $\mathrm{HP}_{\frac{A}{I}}(q) := \mathrm{HP}\left(\frac{A}{I}\right)$.
- $\mathrm{HP}\left(\frac{S_{J+1}}{P_{r,i,J}}\right) = \mathrm{HP}\left(\frac{S_{J+1}}{P_{J+1}^i}\right) = \mathrm{HP}_i^{J+1}$.
- $\mathrm{HP}_{\frac{S_{J+1}}{P_{r,i,J}}}(q) = \mathcal{B}_{r,i,J}(q)$.

Notations and properties

- $\mathrm{HP}_{\frac{S_k}{P_k^{\ell}}}(q) := \mathrm{HP}_{\ell}^{k}$.
- $\mathrm{HP}_{\frac{A}{I}}(q) := \mathrm{HP}\left(\frac{A}{I}\right)$.
- $\mathrm{HP}\left(\frac{S_{J+1}}{P_{r,i,J}}\right) = \mathrm{HP}\left(\frac{S_{J+1}}{P^{i}_{J+1}}\right) = \mathrm{HP}_{i}^{J+1}$.
- $\mathrm{HP}_{\frac{S_{J+1}}{P_{r,i,J}}}(q) = \mathcal{B}_{r,i,J}(q)$.

Hence, we have

$$\mathrm{HP}_{i}^{J+1} = \mathcal{B}_{r,i,J}(q).$$

Notations and properties

- $\mathrm{HP}_{\frac{S_k}{P_k^\ell}}(q) := \mathrm{HP}_\ell^k$.
- $\mathrm{HP}_{\frac{A}{I}}(q) := \mathrm{HP}\left(\frac{A}{I}\right)$.
- $\mathrm{HP}\left(\frac{S_{J+1}}{P_{r,i,J}}\right) = \mathrm{HP}\left(\frac{S_{J+1}}{P^i_{J+1}}\right) = \mathrm{HP}_i^{J+1}$.
- $\mathrm{HP}_{\frac{S_{J+1}}{P_{r,i,J}}}(q) = \mathcal{B}_{r,i,J}(q)$.

Hence, we have

$$\mathrm{HP}_i^{J+1} = \mathcal{B}_{r,i,J}(q).$$

To prove Theorem 33, i.e., $\mathcal{A}_{(r-1)J+\ell}(q) = \mathcal{B}_{r,i,J}(q)$, it is enough to prove that $\mathrm{HP}_i^{J+1} = \mathcal{A}_{(r-1)J+\ell}(q)$, where $\ell = r - i + 1$.

General recursion formula for HP_ℓ^k

Lemma 34 (Afsharijoo et al. [1]).

Let J be a nonnegative integer. Let $k,\ r$, and ℓ be positive integers with $r \geq 2$ and $1 \leq \ell \leq r$. Then for any $k \geq J+1$, we have

$$\mathrm{HP}_\ell^k = \sum_{j=1}^{\ell} q^{k(j-1)} \mathrm{HP}_{r-j+1}^{k+1}.$$

General recursion formula for HP_i^{J+1}

Lemma 35 (Ghosh-Barman (Submitted)).

Let J be a nonnegative integer, and r, i be integers with $r \geq 2,\ 1 \leq i \leq r$. Then we have the following recursion formula

$$\mathrm{HP}_i^{J+1} = \sum_{j=1}^{r} B^J_{i,j,(r-1)d+j} \mathrm{HP}^{d+1}_{r-j+1},$$

where $d \geq J+1$.

General recursion formula for HP_i^{J+1}

Lemma 35 (Ghosh-Barman (Submitted)).

Let J be a nonnegative integer, and r, i be integers with $r \geq 2,\ 1 \leq i \leq r$. Then we have the following recursion formula

$$\mathrm{HP}_i^{J+1} = \sum_{j=1}^{r} B^J_{i,j,(r-1)d+j} \mathrm{HP}^{d+1}_{r-j+1},$$

where $d \geq J+1$.Here, the coefficients $B^J_{i,j,(r-1)d+j} \in \mathbb{F}[[q]]$ satisfy the following recursion formula for $1 \leq j \leq r$

$$B^J_{i,j,(r-1)(d+1)+j} = q^{(d+1)(j-1)} \sum_{m=1}^{r-j+1} B^J_{i,m,(r-1)d+m}$$

General recursion formula for HP_i^{J+1}

Lemma 35 (Ghosh-Barman (Submitted)).

Let J *be a nonnegative integer, and* r, i *be integers with* $r \geq 2$, $1 \leq i \leq r$. *Then we have the following recursion formula*

$$\mathrm{HP}_i^{J+1} = \sum_{j=1}^{r} B^J_{i,j,(r-1)d+j} \mathrm{HP}^{d+1}_{r-j+1},$$

where $d \geq J+1$.*Here, the coefficients* $B^J_{i,j,(r-1)d+j} \in \mathbb{F}[[q]]$ *satisfy the following recursion formula for* $1 \leq j \leq r$

$$B^J_{i,j,(r-1)(d+1)+j} = q^{(d+1)(j-1)} \sum_{m=1}^{r-j+1} B^J_{i,m,(r-1)d+m}$$

with the following initial conditions

$$B^J_{i,j,(r-1)(J+1)+j} = \begin{cases} q^{(J+1)(j-1)} & \text{if } 1 \leq j \leq i; \\ 0 & \text{if } i+1 \leq j \leq r. \end{cases}$$

General recursion formula for $\mathcal{A}_{(r-1)J+\ell}$

Lemma 36 (Coulson et al., Ramanujan J. (2017)).

Let J *be a nonnegative integer and* $r,\ \ell$ *be integers with* $r \geq 2,\ 1 \leq \ell \leq r$. *Then for* $d \geq J+1$ *we have the following recursion formula*

$$\mathcal{A}_{(r-1)J+\ell} = \sum_{j=1}^{r} A^{J}_{\ell,j,(r-1)d+j}\mathcal{A}_{(r-1)d+j}.$$

General recursion formula for $\mathcal{A}_{(r-1)J+\ell}$

Lemma 36 (Coulson et al., Ramanujan J. (2017)).

Let J *be a nonnegative integer and* $r,\ \ell$ *be integers with* $r \geq 2,\ 1 \leq \ell \leq r$. *Then for* $d \geq J+1$ *we have the following recursion formula*

$$\mathcal{A}_{(r-1)J+\ell} = \sum_{j=1}^{r} A^{J}_{\ell,j,(r-1)d+j} \mathcal{A}_{(r-1)d+j}.$$

Here, the coefficients $A^{J}_{\ell,j,(r-1)d+j} \in \mathbb{F}[[q]]$ *satisfy the following recursion formula for* $1 \leq j \leq r$

$$A^{J}_{\ell,j,(r-1)(d+1)+j} = q^{(d+1)(j-1)} \sum_{m=1}^{r-j+1} A^{J}_{\ell,m,(r-1)d+m}$$

General recursion formula for $\mathcal{A}_{(r-1)J+\ell}$

Lemma 36 (Coulson et al., Ramanujan J. (2017)).

Let J be a nonnegative integer and $r,\ \ell$ be integers with $r \geq 2,\ 1 \leq \ell \leq r$. Then for $d \geq J+1$ we have the following recursion formula

$$\mathcal{A}_{(r-1)J+\ell} = \sum_{j=1}^{r} A^{J}_{\ell,j,(r-1)d+j}\mathcal{A}_{(r-1)d+j}.$$

Here, the coefficients $A^{J}_{\ell,j,(r-1)d+j} \in \mathbb{F}[[q]]$ satisfy the following recursion formula for $1 \leq j \leq r$

$$A^{J}_{\ell,j,(r-1)(d+1)+j} = q^{(d+1)(j-1)} \sum_{m=1}^{r-j+1} A^{J}_{\ell,m,(r-1)d+m}$$

with the following initial conditions

$$A^{J}_{\ell,j,(r-1)(J+1)+j} = \begin{cases} q^{2(J+1)(j-1)} & \text{if } 1 \leq j \leq r-\ell+1; \\ 0 & \text{if } r-\ell+2 \leq j \leq r. \end{cases}$$

Equality of coefficients $A^J_{\ell,v,(r-1)d+v}$ and $B^J_{i,v,(r-1)d+v}$

Lemma 37 (Ghosh-Barman (Submitted)).

For all $d \geq J+1$, $r \geq 2$, *and* $1 \leq v \leq r$, *we have*

$$A^J_{\ell,v,(r-1)d+v} = B^J_{i,v,(r-1)d+v},$$

where $\ell = r - i + 1$.

Equality of $\mathcal{A}_{(r-1)J+\ell}$ and HP_i^{J+1}

Theorem 38 (Ghosh-Barman (Submitted)).

For $r \geq 2$, $1 \leq i \leq r$, *and* $J \geq 0$, *we have*

$$\mathcal{A}_{(r-1)J+\ell} = \mathrm{HP}_i^{J+1},$$

where $\ell = r - i + 1$.

Outline

References

- P. Afsharijoo, *Looking for a new version of Gordon's identities*, Ann. Comb. 25 (2021) 543–571.
- G. E. Andrews, *A generalization of the Göllnitz-Gordon partition theorems*, Proc. Amer. Math. Soc. 18 (1967) 945–952.
- G. E. Andrews, *An analytic generalization of the Rogers-Ramanujan identities for odd moduli*, Proc. Natl. Acad. Sci. USA 71 (1974) 4082–4085.
- G. E. Andrews and B. C. Berndt, *Ramanujan's Lost Notebook, Part II*, Springer, New York, 2008.
- B. Coulson, S. Kanade, J. Lepowsky, R. McRae, F. Qi, M. C. Russell, and C. Sadowski, *A motivated proof of the Göllnitz-Gordon-Andrews identities*, Ramanujan J. 42 (2017) 97–129.

References

- H. Göllnitz, *Partitionen mit Differenzenbedingungen*, J. Reine Angew. Math. 225 (1967) 154–190.
- B. Gordon, *A combinatorial generalization of the Rogers-Ramanujan identities*, Amer. J. Math. 83 (1961) 393–399.
- B. Gordon, *Some continued fractions of the Rogers-Ramanujan type*, Duke Math. J. 31 (1965) 741–748
- P. A. MacMahon, *Combinatory Analysis, vol. II*, Chelsea Publishing Company, New York, 1984. Originally published in two volumes at Cambridge, 1918.
- J. Lepowsky and M. Zhu, *A motivated proof of Gordon's identities*, The Ramanujan Journal 29 (1-3) (2012) 199–211.

References

S. Ramanujan, *Collected Papers*, Cambridge University Press, Cambridge, 1927; reprinted by Chelsea, New York, 1962; reprinted by the American Mathematical Society, Providence, RI, 2000.

L. J. Rogers, *Second Memoir on the Expansion of certain Infinite Products*, Proceedings of the London Mathematical Society s1-25, Issue 1 (1894) 318–343.

I. Schur, *Ein Beitrag zur additiven Zahlentheorie und zur Theorie der Kettenbruchen*, Gesammelte Abhandlungen. Band II, Springer-Verlag, Berlin-New York, (1973) 117–136. Originally in: Sitzungsberichte der Preussischen Akadamie der Wissenschaften, Physikalisch-Mathematische Klasse (1917) 302–321.

Thank you