NOTES TOWARD AN ALGEBRA OF PARTITIONS

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Abstract. These "notes to self" combine multipartitions (multisets whose elements are integer partitions) and ideas from algebra with the author's previous work on a multiplicative theory of additive partitions. We define a number of notational conventions and partition operations, with the eventual aim of establishing a *purely partition-theoretic* algebra of partitions (i.e., a theory relying only on maps between partitions, as opposed to mapping from partitions to $\mathbb{N}$ or $\mathbb{C}$, or appealing to algebraic properties of the integer parts) in harmony with the conventions of abstract algebra as well as the techniques of partition theory.

1. Usual arithmetic holds for partitions

Recall that a *multiset* is a generalization of a set (finite or infinite) whose different elements are permitted finitely many repetitions. Let $\mathcal{P}$ denote the set of *integer partitions*, i.e., all finite multisets of natural numbers $\mathbb{N} := \{1, 2, 3, 4, ...\}$, and let $\lambda = (\lambda_1, \lambda_2, ..., \lambda_r)$ denote a generic partition, where $\lambda_1 \geq \lambda_2 \geq ... \geq \lambda_r \geq 1$, with $\emptyset$ the *empty partition*. We recall *partition multiplication* (multiset union of the parts) and *partition division* (deleting a subpartition) as defined in [8, 9], as well as the multiplicative theory of partition analogs of classical arithmetic functions, Cauchy products, zeta functions, etc. developed in [5, 6, 7, 8, 9, 10] and other works, which fits naturally into the theoretical framework sketched here[1].

Let $M(S)$ denote the set of all multisets with elements from some countable set S. In particular, if S is the set $\mathcal{P}$ then $\mathcal{A} \in M(\mathcal{P})$ means

$$\mathcal{A} = \{\alpha, \beta, \gamma, ...\}, \quad \alpha, \beta, \gamma, ... \in \mathcal{P},$$

with partitions $\alpha, \beta, \gamma, ...$, not necessarily distinct. If there are $k < \infty$ components α, β, γ, etc., then we will call the multiset $\mathcal{A}$ a *multipartition*. We define $\mathbf{0}$ to be the *empty multipartition*, noting $\mathbf{0} \neq \emptyset$ (the empty partition is a *nonempty* multipartition in this setting). Ordered multipartitions with k components are usually visualized as k-colored partitions. Differing from the usual multipartition conventions, for full formal generality, here we allow multipartitions to be unordered, and either finite or infinite in cardinality. Then $\mathcal{P}$ itself is a multipartition (an infinite one), as is any subset $\mathcal{P}' \subseteq \mathcal{P}$. For notational ease, we will identify a multipartition consisting of a single partition with the partition itself:

$$\{\lambda\} = \lambda.$$

Let us define an addition "+" between multipartitions by their multiset union, viz.

$$\{\alpha, \beta\} + \{\gamma\} := \{\alpha, \beta, \gamma\},$$

[1]This treatment is strongly inspired by conversations with George E. Andrews, Philip Engel, Matthew Just, Ken Ono and Andrew V. Sills, and by work with Ian Wagner summarized in [9], Appendix B.3.

and allow this operation (and most of the following algebraic considerations) to extend to multisets in general. Thus addition is commutative and associative, and $\mathcal{A} + \mathbf{0} = \mathcal{A}$ for all $\mathcal{A} \in M(\mathcal{P})$ ($\mathbf{0}$ is the additive identity). We note by our convention $\{\lambda\} = \lambda$ that the "+" operation then extends formally to partitions, e.g.

$$\lambda + \gamma := \{\lambda\} + \{\gamma\} = \{\lambda, \gamma\}.$$

So we may feel justified in writing, for example,

$$\lambda + \mathbf{0} = \lambda, \qquad \lambda + \lambda = 2\lambda, \qquad \sum_{\lambda \in \mathcal{A}} \lambda = \mathcal{A},$$

keeping in mind the sums represent multipartitions, where one might think of the sum on the right as *convergent* if no partition is repeated infinitely often (that is, the sum strictly represents a multiset), and *divergent* otherwise — or perhaps a better definition of convergence is in order. In any event, partitions enjoy something like the usual arithmetic.

Moreover, we use the usual multipartition valuation for finite $\mathcal{A} \in M(\mathcal{P})$:

$$|\mathcal{A}| = \left| \sum_{\lambda \in \mathcal{A}} \lambda \right| := \sum_{\lambda \in \mathcal{A}} |\lambda| \in \mathbb{N},$$

where $|\lambda|$ is the *size* (sum of parts) of the partition.

Let us define a *negative multipartition* $-\mathcal{A}$ as the additive inverse of $\mathcal{A} \in M(\mathcal{P})$:

$$\mathcal{A} - \mathcal{A} := \mathcal{A} + (-\mathcal{A}) = \mathbf{0}.$$

Then it also makes sense to write down a "negative partition", viz. $-\lambda := -\{\lambda\}$; we will compute the product of negative partitions using the rule $(-\emptyset)(-\emptyset) := \emptyset$. We may also admit $\mathcal{A} - \mathcal{B}$ as a "formal multipartition" for $\mathcal{A}, \mathcal{B} \in M(\mathcal{P})$.

We define multiplication of multipartitions as in the construction of free algebras:

$$\{\alpha, \beta\}\{\lambda, \gamma\} := \{\alpha\lambda, \alpha\gamma, \beta\lambda, \beta\gamma\},$$

where the right-hand elements represent products of partitions as in [8, 9], with $\emptyset = \{\emptyset\}$ being the identity, and setting $\mathbf{0} \cdot \mathcal{A} := \mathbf{0}$ for any $\mathcal{A} \in M(\mathcal{P})$. Multiset brackets behave well under this product, in terms of partition multiplication:

$$\{\lambda\}\{\gamma\} = \{\lambda\gamma\} = \lambda\gamma.$$

We define the *reciprocal multipartition* $\mathcal{A}^{-1}$ of $\mathcal{A}$ so that every nonempty multipartition has a multiplicative inverse, e.g. $\mathcal{A}\mathcal{A}^{-1} = \emptyset$ if $\mathcal{A} \neq \mathbf{0}$, which we can extend to "reciprocal partitions", viz. $\lambda^{-1} := \{\lambda\}^{-1}$. For consistency, we will take size $|\lambda^{-1}| := -|\lambda|$, length $\ell(\lambda^{-1}) := -\ell(\lambda)$, and norm $N(\lambda^{-1}) := 1/N(\lambda)$. If λ is a subpartition of partition γ then we identify the product $\gamma\lambda^{-1} = \lambda^{-1}\gamma = \{\lambda\}^{-1}\{\gamma\}$ with the partition γ/λ formed by deleting the parts of λ from γ. We may admit $\mathcal{A}\mathcal{B}^{-1}$ as a "formal multipartition" for $\mathcal{A}, \mathcal{B} \in M(\mathcal{P}), \mathcal{B} \neq \mathbf{0}$.

Clearly this multipartition product induces a semiring theory in the set $M(\mathcal{P})$ and a ring theory if we admit negative multipartitions, as the product is defined to yield distributivity, viz.

$$\lambda(\alpha + \beta) = \{\lambda\}\{\alpha, \beta\} = \{\alpha\lambda, \beta\lambda\} = \alpha\lambda + \beta\lambda.$$

If we denote by $\widetilde{M} = \widetilde{M}(\mathcal{P})$ the set of *formal multipartitions* consisting of $M(\mathcal{P})$ adjoined with the negative and reciprocal multipartitions and all formal combinations of these elements, then $\widetilde{M}$ is a field.

2. Functions on partitions extend to multipartitions

Consider now a function $f : \mathcal{P} \to T$ defined on partitions, with a target set T in which we shall assume addition, multiplication, and inverse operations make sense, when needed. As we did with $|\mathcal{A}|$ above, we wish to extend *all* functions on partitions to functions $f : M(\mathcal{P}) \to M(T)$ on multipartitions (as well as "formal multipartitions" $\widetilde{M}(\mathcal{P})$) by applying them to each constituent partition:

$$f(\mathcal{A}) = f\left(\{\alpha, \beta, \gamma, ...\}\right) := \{f(\alpha), f(\beta), f(\gamma), ...\} \in M(T),$$

or in summation form,

$$f\left(\sum_{\lambda \in \mathcal{A}} \lambda\right) = \sum_{\lambda \in \mathcal{A}} f(\lambda),$$

which (unless $T \subseteq M(\mathcal{P})$) now represents a multiset of elements from T on the right-hand side (or perhaps a sum in T if addition is defined there) instead of a multipartition.

If, furthermore, the function $f : \mathcal{P} \to T$ is completely multiplicative, i.e., $f(\lambda\gamma) = f(\lambda)f(\gamma)$ with $f(\emptyset)$ mapped to the multiplicative identity in T, a quick calculation shows that f is also multiplicative as a function on $M(\mathcal{P})$, e.g. for $\mathcal{A}, \mathcal{B} \in M(\mathcal{P})$ we have

$$f(\mathcal{A}\mathcal{B}) = f(\mathcal{A})f(\mathcal{B}).$$

These formal ideas are in harmony with the theory of product-sum generating functions. For $(n) \in \mathcal{P}$ a partition of length one, $n \geq 1$, let us define a multipartition that captures the essence of Eulerian partition generating functions.

Definition 1. *We define an "empty" Pochhammer symbol as follows:*

$$(\ \cdot\ )_\infty := \ \{\emptyset, -(1)\}\,\{\emptyset, -(2)\}\,\{\emptyset, -(3)\}\cdots \ = \ \prod_{n=1}^{\infty} \left(\emptyset - (n)\right).$$

Theorem 2. *The reciprocal of this "empty" Pochhammer symbol is the set of partitions:*

$$(\ \cdot\ )_\infty^{-1} \ = \ \mathcal{P}.$$

Proof. If we formally extend geometric series to this multipartition setting, and further extend the action of partition functions to formal multipartitions, then by the notations above and standard ideas about generating functions we write

$$\begin{aligned}(\ \cdot\ )_\infty^{-1} \ &= \ \prod_{n=1}^{\infty} \left(\emptyset - (n)\right)^{-1} \ = \ \prod_{n=1}^{\infty} \left(\emptyset + (n) + (n)^2 + (n)^3 + ...\right) \\ &= \sum_{\lambda \in \mathcal{P}} \prod_{n=1}^{\infty} (n)^{m_n(\lambda)} \ = \ \sum_{\lambda \in \mathcal{P}} \lambda \ = \ \mathcal{P},\end{aligned}$$

where $m_n(\lambda) \geq 0$ denotes the *frequency* (or *multiplicity*) of n as a part of λ. □

Along similar lines, if $f : \mathcal{P} \to T$ is completely multiplicative we have

$$f\left((\,\cdot\,)_\infty^{-1}\right) = \prod_{n=1}^{\infty} f\left((\emptyset - (n))^{-1}\right) = \prod_{n=1}^{\infty} \left(f(\emptyset) - f\left((n)\right)\right)^{-1}$$
$$= \sum_{\lambda\in\mathcal{P}} \prod_{n=1}^{\infty} f\left((n)\right)^{m_n(\lambda)} = \sum_{\lambda\in\mathcal{P}} f(\lambda) = f\left(\sum_{\lambda\in\mathcal{P}} \lambda\right) = f\left(\mathcal{P}\right).$$

If $T \subseteq \mathbb{C}$ (thus $f(\emptyset) = 1$), we can define a convenient analytic object $f : \mathcal{P} \to \mathbb{C}$ by

$$f_{z,q}(\lambda) := f(\lambda) z^{\ell(\lambda)} q^{|\lambda|}$$

for appropriate $z, q \in \mathbb{C}$ so as to ensure convergence, where $\ell(\lambda), |\lambda|$ denote the *length* and *size* of λ, respectively, giving connections to classical q-series such as in the next identity.

Theorem 3. *Let $I(\lambda) \equiv 1$ for all partitions λ; thus $I_{z,q}(\lambda) = z^{\ell(\lambda)} q^{|\lambda|}$. Then for $|q| < 1$:*

$$I_{z,q}(\mathcal{P}) = (1 - z) \sum_{n=0}^{\infty} \frac{z^n}{(q;q)_n},$$

where $(q;q)_n := \prod_{i=1}^{n} (1 - q^i)$ is the usual q-Pochhammer symbol.

Proof. From the preceding considerations, noting $I_{z,q}\left((n)\right) := z^{\ell((n))} q^{|(n)|} = zq^n$ is completely multiplicative in the partition sense, set $|q| < 1$ for convergence to yield

$$I_{z,q}(\mathcal{P}) = I_{z,q}((\,\cdot\,)_\infty^{-1}) = \prod_{n=1}^{\infty} (1 - zq^n)^{-1},$$

where the product is taken in $\mathbb{C}$, and apply the q-binomial theorem. $\square$

If for $\mathbb{X} \subseteq \mathbb{N}$ we more generally define

$$(\,\cdot\,)_{\mathbb{X}} := \prod_{n\in\mathbb{X}} (\emptyset - (n)),$$

then if $\mathcal{P}_{\mathbb{X}} \subseteq \mathcal{P}$ denotes partitions whose parts are elements of $\mathbb{X}$, we also have that

$$(\,\cdot\,)_{\mathbb{X}}^{-1} = \mathcal{P}_{\mathbb{X}}, \qquad f\left((\,\cdot\,)_{\mathbb{X}}^{-1}\right) = f(\mathcal{P}_{\mathbb{X}}),$$

for f completely multiplicative in the partition sense. For example, let $N(\lambda) := \lambda_1 \lambda_2 \cdots \lambda_r$ denote the *norm* of the partition with $N(\emptyset) := 1$, and $N^{-s}(\lambda) := 1/N(\lambda)^s$ for $s \in \mathbb{C}$.

Theorem 4. *For* $\mathrm{Re}(s) > 1$ *we have that*

$$N^{-s}\left(\mathcal{P}_{\mathbb{P}}\right) = \zeta(s).$$

Proof. The proof proceeds much as with Theorem 3, noting $N^{-s}\left(\mathcal{P}_{\mathbb{P}}\right) = N^{-s}\left((\,\cdot\,)_{\mathbb{P}}^{-1}\right)$. $\square$

3. Frequencies represent logarithms

We now turn our attention to structures within the set of frequencies (i.e., multiplicities) of the parts of partitions. Letting $m_i(\lambda) \geq 0$ denote the frequency of i as a part of $\lambda \in \mathcal{P}$ (or just m_i if the partition λ is clear), recall the usual "part-frequency notation":

$$\lambda = (1^{m_1}\ 2^{m_2}\ 3^{m_3}\ \ldots),$$

where only finitely many m_i are nonzero. It is just a slight abuse of this notation to allow infinitely many nonzero frequencies, and we will identify these more general multisets

$\{1^{m_1}2^{m_2}...\ i^{m_i}...\}$ with infinite multisets of natural numbers (or subsets of $\mathbb{N}$ if all m_i equal 0 or 1). If we also allow m_i to take negative values, then we will assume a part i appearing $-m_i$ times means it belongs to a "reciprocal" partition, e.g.

$$\lambda^{-1} = (1^{-m_1}\ 2^{-m_2}\ 3^{-m_3}\ ...),$$

or as a component of a "formal multipartition" of the shape $\gamma\lambda^{-1}$.

We wish to abstract the structures of partitions' frequencies. Let us define a *frequency operator* $(m_1, m_2, m_3, ...)$ from the sequence of frequencies associated to partition λ or another multiset of natural numbers (allowing infinitely or finitely many nonzero $m_i \in \mathbb{Z}$), the entries of which we let act by "exponentiation" on the natural numbers as follows:

$$\mathbb{N}^{(m_1,m_2,m_3,...)} = \{1, 2, 3, ...\}^{(m_1,m_2,m_3,...)} := \{1^{m_1}\ 2^{m_2}\ 3^{m_3}...\}.$$

If there are only finitely many nonzero m_i the object is a (formal) partition, and we write

$$\mathbb{N}^{(m_1,m_2,m_3,...)} = (1^{m_1}\ 2^{m_2}\ 3^{m_3}...) = \lambda.$$

Thus the frequency operators (central players in Andrews's theory of partition ideals [2]) are analogous to logarithms in this setting. It is natural then to define

$$\log_{\mathbb{N}} \lambda := (m_1, m_2, m_3, ...),$$

with $\log_{\mathbb{N}} \emptyset = (0, 0, 0, ...)$. Direct computation of partition products and quotients shows frequency operators enjoy addition "$\oplus$", subtraction "$\ominus$" and multiplication by a positive integral constant:

$$(m_1, m_2, m_3, ...) \oplus (n_1, n_2, n_3, ...) := (m_1 + n_1, m_2 + n_2, m_3 + n_3, ...),$$
$$(m_1, m_2, m_3, ...) \ominus (n_1, n_2, n_3, ...) := (m_1 - n_1, m_2 - n_2, m_3 - n_3, ...),$$
$$k \cdot (m_1, m_2, m_3, ...) := (k \cdot m_1, k \cdot m_2, k \cdot m_3, ...),$$

much as with standard exponents (thus $(0, 0, 0, ...)$ is the additive identity), such that

$$\log_{\mathbb{N}}(\lambda\gamma) = \log_{\mathbb{N}} \lambda \oplus \log_{\mathbb{N}} \gamma, \quad \log_{\mathbb{N}}(\lambda/\gamma) = \log_{\mathbb{N}} \lambda \ominus \log_{\mathbb{N}} \gamma, \quad \log_{\mathbb{N}}(\lambda^k) = k \log_{\mathbb{N}} \lambda.$$

We can allow the frequency operator to act on a *multiset* of natural numbers as well by turning each of the m_i occurrences of i in the initial multiset, into n_i copies of itself in the resulting multiset. Then by noting the effect of applying two frequency operators to the set $\mathbb{N}$ one after the other, we can also define multiplication "$\otimes$" by the operation

$$(m_1, m_2, m_3, ...) \otimes (n_1, n_2, n_3, ...) := (m_1n_1, m_2n_2, m_3n_3, ...).$$

Endowed with the $\oplus$ and $\otimes$ operations, frequency operators form a semiring with identity $(1, 1, 1, ...)$; if we also allow negative frequencies, these operators form a ring.

4. Partitions induce other multisets and bijections

Likewise, one may abstract the frequency sequence $(m_1, m_2, m_3, ...)$ from a partition λ and apply it to any ordered set $S = \{s_1, s_2, s_3, ...\}$ to induce a multiset in $M(S)$ indexed by λ:

$$S^{(\log_{\mathbb{N}} \lambda)} = S^{((m_1,m_2,m_3,...))} = \{s_1^{m_1}\ s_2^{m_2}\ s_3^{m_3}...\},$$

where we use additional parentheses in the exponent when $S \neq \mathbb{N}$ to distinguish cases where the frequencies act on the indices (and, conversely, every finite ordered multiset is associated to a partition by abstracting the frequencies in order). For instance, we have

$$\sum_{\lambda \in \mathcal{P}} \mathbb{P}^{(\log_{\mathbb{N}} \lambda)} = \sum_{\lambda \in \mathcal{P}} (2^{m_1(\lambda)}\ 3^{m_2(\lambda)}\ 5^{m_3(\lambda)}\ ...\ p_i^{m_i(\lambda)} ...) = \mathcal{P}_{\mathbb{P}},$$

producing a bijective map, say $\pi : \mathcal{P} \to \mathcal{P}_{\mathbb{P}}$, between $\mathcal{P}$ and the set $\mathcal{P}_{\mathbb{P}} \subset \mathcal{P}$ of "prime partitions" whose parts are all prime numbers, which is studied in [3, 4]. As the prime factorizations of natural numbers are in bijective correspondence with prime partitions (see [1]), then if we set $m = N(\pi(\lambda)), n = N(\pi(\gamma))$, where N is the partition norm (product of parts), we recover integer arithmetic from this multipartition algebra, viz.

$$N(\pi\,(\lambda + \gamma)) = m + n, \qquad N(\pi(\lambda\gamma)) = mn.$$

Furthermore, well-known bijections between subsets of $\mathcal{P}$ reveal rather obscure bijections between subsets of $\mathbb{N}$. For instance, as is studied in [3], there is a natural one-to-one correspondence between natural numbers with k prime factors including repetition, and natural numbers whose largest prime factor is p_k (the kth prime), by the correspondence between partitions of length k and partitions with largest part k. Similarly, there is a natural bijection between the squarefree integers and integers having only odd-indexed prime factors (i.e., divisible only by $p_1 = 2, p_3 = 5, p_5 = 11$, etc.), by the correspondence between partitions into distinct parts and partitions into odd parts.

It is the author's hope that new partition bijections, congruences and interrelations can be identified by applying the isomorphism theorems and other tools from abstract algebra.

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