On New Modulo 8 Cylindric Partition Identities

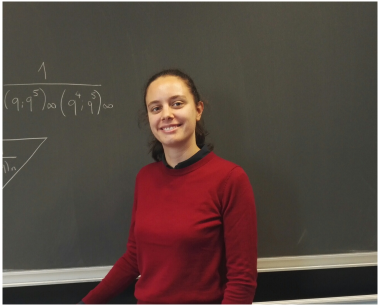
Ali Kemal Uncu (aku21@bath.ac.uk)

Specialty Seminar in Partition Theory, q-Series and Related Topics

The speaker is partially supported by the EPSRC grant EP/T015713/1 and partially by the FWF grant P-34501N.

Two Amazing Mathematicians

Sylvie Corteel



Jehanne Dousse

Cylindric partitions and some new A_2 Rogers-Ramanujan identities

Accepted in Proc. Amer. Math. Soc. 2021
https://doi.org/10.1090/proc/15570

Partitions

A *partition* π is a finite sequence of *non-increasing* positive integers $(\lambda_1, \lambda_2, \dots, \lambda_{\#(\pi)})$.

For a given partition $\pi = (\lambda_1, \lambda_2, \dots, \lambda_{\#(\pi)})$ the sum $\lambda_1 + \lambda_2 + \dots + \lambda_{\#(\pi)}$ is the *size* of the partition π and it is denoted by $|\pi|$.

<u>Ex:</u>

- $\pi = (5, 1, 1)$ is a partition of $|\pi| = 7$.
- $\pi = \emptyset$ is the unique partition of 0.

Generating Functions

Basics
Partitions
Generating Functions
Rogers–Ramanujan Identities
Plane Partitions
Cylindric Partitions
Future Work

For a sequence $\{a_n\}_{n=0}^{\infty}$, the series

$$\sum_{n\geq 0} a_n q^n$$

is called a *generating function*.

Let $\mathcal{D}$ be the set of all partitions into non-repeating parts.

$$\sum_{\pi\in\mathcal{D}} q^{|\pi|} = 1 + q + q^2 + 2q^3 + 2q^4 + 3q^5 + 4q^6 + 5q^7 + 6q^8 + 8q^9 \dots .$$

$\emptyset$	$(2,1),\ (3)$	$(3,2,1),\ (5,1),\ (4,2),\ (6)$
(1)	$(3,1),\ (4)$	$(4,2,1),\ (6,1),\ (5,2),\ (4,3),\ (7)$
(2)	$(4,1),\ (3,2),\ (5)$	$\dots$

q-Pochhammer Symbol

Basics
Partitions
Generating Functions
Rogers–Ramanujan Identities
Plane Partitions
Cylindric Partitions
Future Work

$$(a;q)_L := \prod_{i=0}^{L-1}(1-aq^i), \text{ and } (a;q)_\infty := \lim_{L\to\infty}(a;q)_L.$$

q-Pochhammer Symbol

Basics
- Partitions
- Generating Functions
- Rogers–Ramanujan Identities
- Plane Partitions

Cylindric Partitions

Future Work

$$(a;q)_L := \prod_{i=0}^{L-1}(1-aq^i), \text{ and } (a;q)_\infty := \lim_{L\to\infty}(a;q)_L.$$

$$(-q;q)_\infty = (1+q^1)(1+q^2)(1+q^3)\ldots$$

Basics
Partitions
Generating Functions
Rogers–Ramanujan Identities
Plane Partitions
Cylindric Partitions
Future Work

q-Pochhammer Symbol

$$(a;q)_L := \prod_{i=0}^{L-1}(1-aq^i), \text{ and } (a;q)_\infty := \lim_{L\to\infty}(a;q)_L.$$

$$\begin{aligned}(-q;q)_\infty &= (1+q^1)(1+q^2)(1+q^3)\dots\\ &= 1+q^1+q^2+(q^{1+2}+q^3)+\dots\end{aligned}$$

q-Pochhammer Symbol

Basics
- Partitions
- Generating Functions
- Rogers–Ramanujan Identities
- Plane Partitions

Cylindric Partitions

Future Work

$$(a;q)_L := \prod_{i=0}^{L-1}(1-aq^i), \text{ and } (a;q)_\infty := \lim_{L\to\infty}(a;q)_L.$$

$$\begin{aligned}(-q;q)_\infty &= (1+q^1)(1+q^2)(1+q^3)\dots\\ &= 1+q^1+q^2+(q^{1+2}+q^3)+\dots\\ &= \sum_{\pi\in\mathcal{D}} q^{|\pi|},\end{aligned}$$

where $\mathcal{D}$ is the set of all partitions into non-repeating parts.

q-Pochhammer Symbol

Basics
- Partitions
- Generating Functions
- Rogers–Ramanujan Identities
- Plane Partitions

Cylindric Partitions

Future Work

$$(a;q)_L := \prod_{i=0}^{L-1}(1-aq^i), \text{ and } (a;q)_\infty := \lim_{L\to\infty}(a;q)_L.$$

$$\begin{aligned}(-q;q)_\infty &= (1+q^1)(1+q^2)(1+q^3)\dots\\ &= 1+q^1+q^2+(q^{1+2}+q^3)+\dots\\ &= \sum_{\pi\in\mathcal{D}} q^{|\pi|},\end{aligned}$$

where $\mathcal{D}$ is the set of all partitions into non-repeating parts. Similarly,

$$\frac{1}{(q;q)_\infty} = \sum_{\pi\in\mathcal{U}} q^{|\pi|},$$

where $\mathcal{U}$ is the set of partitions.

q-Binomial Coefficients

$$(a;q)_L := \prod_{i=0}^{L-1}(1-aq^i), \text{ and } (a;q)_\infty := \lim_{L\to\infty}(a;q)_L.$$

Basics
Partitions
Generating Functions
Rogers–Ramanujan Identities
Plane Partitions
Cylindric Partitions
Future Work

q-Binomial Coefficients

Basics
- Partitions
- Generating Functions
- Rogers–Ramanujan Identities
- Plane Partitions

Cylindric Partitions

Future Work

$$(a;q)_L := \prod_{i=0}^{L-1}(1-aq^i), \text{ and } (a;q)_\infty := \lim_{L\to\infty}(a;q)_L.$$

$$(a_1,a_2,\ldots,a_k;q)_L := (a_1;q)_L(a_2;q)_L\ldots(a_k;q)_L.$$

q-Binomial Coefficients

$$(a;q)_L := \prod_{i=0}^{L-1}(1-aq^i), \text{ and } (a;q)_\infty := \lim_{L\to\infty}(a;q)_L.$$

$$(a_1,a_2,\dots,a_k;q)_L := (a_1;q)_L(a_2;q)_L\dots(a_k;q)_L.$$

We define the q-binomial coefficients as

$$\begin{bmatrix} m+n \\ m \end{bmatrix}_q := \begin{cases} \frac{(q;q)_{m+n}}{(q;q)_m(q;q)_n}, & \text{for } m,n\geq 0,\\ 0, & \text{otherwise,}\end{cases}$$

q-Binomial Coefficients

Basics
- Partitions
- Generating Functions
- Rogers–Ramanujan Identities
- Plane Partitions

Cylindric Partitions

Future Work

$$(a;q)_L := \prod_{i=0}^{L-1}(1-aq^i), \text{ and } (a;q)_\infty := \lim_{L\to\infty}(a;q)_L.$$

$$(a_1,a_2,\dots,a_k;q)_L := (a_1;q)_L(a_2;q)_L\dots(a_k;q)_L.$$

We define the q-binomial coefficients as

$$\begin{bmatrix} m+n \\ m \end{bmatrix}_q := \begin{cases} \frac{(q;q)_{m+n}}{(q;q)_m(q;q)_n}, & \text{for } m,n\geq 0, \\ 0, & \text{otherwise,} \end{cases}$$

This q-binomial coefficient is the generating function for partitions in an $m\times n$-box.

Basics
Partitions
Generating Functions
Rogers–Ramanujan Identities
Plane Partitions
Cylindric Partitions
Future Work

Partition Identities

Theorem (Rogers–Ramanujan Identities)

For $m = 1, 2$ and $n \in \mathbb{Z}_{\geq 0}$, the number of partitions of n with gaps between parts ≥ 2, all $\geq m$

$=$

the number of partitions of n into $\pm m$ mod 5 parts.

Partition Identities

Basics
Partitions
Generating Functions
Rogers–Ramanujan Identities
Plane Partitions
Cylindric Partitions
Future Work

Theorem (Rogers–Ramanujan Identities)

For $m = 1, 2$ and $n \in \mathbb{Z}_{\geq 0}$, the number of partitions of n with gaps between parts ≥ 2, all $\geq m$

$=$

the number of partitions of n into $\pm m$ mod 5 parts.

Theorem (Rogers–Ramanujan Identities)

For $m = 1, 2$, we have

$$\sum_{n\geq 0} \frac{q^{n^2+(m-1)n}}{(q;q)_n} = \frac{1}{(q^m, q^{5-m}; q^5)_\infty}.$$

G. E. Andrews, *The Theory of Partitions*, Cambridge Mathematical Library, Cambridge University Press, Cambridge, 1998. Reprint of the 1976 original.

Partition Identities

Basics
- Partitions
- Generating Functions
- Rogers–Ramanujan Identities
- Plane Partitions

Cylindric Partitions

Future Work

Theorem (The First Rogers–Ramanujan Identity)

For any $n \in \mathbb{Z}_{\geq 0}$, the number of partitions of n with gaps between parts ≥ 2

$=$

the number of partitions of n into ± 1 mod 5 parts.

Partition Identities

Theorem (The First Rogers–Ramanujan Identity)

For any $n \in \mathbb{Z}_{\geq 0}$, the number of partitions of n with gaps between parts ≥ 2
$=$
the number of partitions of n into ± 1 mod 5 parts.

Example: $n = 10$

(10)	$(9,1)$
$(9,1)$	$(6,4)$
$(8,2)$	$(6,1,1,1,1)$
$(7,3)$	$(4,4,1,1)$
$(6,4)$	$(4,1,1,1,1,1,1)$
$(6,3,1)$	$(1,1,1,1,1,1,1,1,1,1)$

Basics
Partitions
Generating Functions
Rogers–Ramanujan Identities
Plane Partitions
Cylindric Partitions
Future Work

Plane Partitions

A vector of partitions $\pi = (\pi_1, \pi_2, \dots)$ (each $\pi_i = (\lambda_{i,1}, \lambda_{i,2}, \dots, \lambda_{i,\#(\pi_i)})$ is called a *plane partition* if for all i and j, we have

$$\lambda_{i,j} \geq \lambda_{i+1,j}. \text{ and } \lambda_{i,j} \geq \lambda_{i,j+1},$$

Plane Partitions

Basics
Partitions
Generating Functions
Rogers–Ramanujan Identities
Plane Partitions
Cylindric Partitions
Future Work

A vector of partitions $\pi = (\pi_1, \pi_2, \dots)$ (each $\pi_i = (\lambda_{i,1}, \lambda_{i,2}, \dots, \lambda_{i,\#(\pi_i)})$ is called a *plane partition* if for all i and j, we have

$$\lambda_{i,j} \geq \lambda_{i+1,j}. \text{ and } \lambda_{i,j} \geq \lambda_{i,j+1},$$

Example: $\pi = (\ (4,3,3,2,1,1), (3,3,2,2,1), (3,1,1,1), (2)\ )$

$$4 \quad 3 \quad 3 \quad 2 \quad 1 \quad 1$$

Basics
Partitions
Generating Functions
Rogers–Ramanujan Identities
Plane Partitions
Cylindric Partitions
Future Work

Plane Partitions

A vector of partitions $\pi = (\pi_1, \pi_2, \dots)$ (each $\pi_i = (\lambda_{i,1}, \lambda_{i,2}, \dots, \lambda_{i,\#(\pi_i)})$ is called a *plane partition* if for all i and j, we have

$$\lambda_{i,j} \geq \lambda_{i+1,j}.\text{ and } \lambda_{i,j} \geq \lambda_{i,j+1},$$

Example: $\pi = (\ (4,3,3,2,1,1), (3,3,2,2,1), (3,1,1,1), (2)\ )$

$$\begin{matrix} 4 & 3 & 3 & 2 & 1 & 1 \\ 3 & 3 & 2 & 2 & 1 & \end{matrix}$$

Plane Partitions

A vector of partitions $\pi = (\pi_1, \pi_2, \dots)$ (each $\pi_i = (\lambda_{i,1}, \lambda_{i,2}, \dots, \lambda_{i,\#(\pi_i)})$ is called a *plane partition* if for all i and j, we have

$$\lambda_{i,j} \geq \lambda_{i+1,j}.\text{ and } \lambda_{i,j} \geq \lambda_{i,j+1},$$

Example: $\pi = (\ (4,3,3,2,1,1),(3,3,2,2,1),(3,1,1,1),(2)\ )$

$$\begin{matrix} 4 & 3 & 3 & 2 & 1 & 1 \\ 3 & 3 & 2 & 2 & 1 & \\ 3 & 1 & 1 & 1 & & \\ 2 & & & & & \end{matrix}$$

Basics
Cylindric Partitions
Borodin's Theorem
Corteel and Welsh's Lemma
Rogers–Ramanujan Again
The New Mod 8 Identities
Future Work

Plane Partitions

A vector of partitions $\pi = (\pi_1, \pi_2, \dots)$ (each $\pi_i = (\lambda_{i,1}, \lambda_{i,2}, \dots, \lambda_{i,\#(\pi_i)})$ is called a *plane partition* if for all i and j, we have

$$\lambda_{i,j} \geq \lambda_{i+1,j}. \text{ and } \lambda_{i,j} \geq \lambda_{i,j+1},$$

Basics
Cylindric Partitions
Borodin's Theorem
Corteel and Welsh's Lemma
Rogers–Ramanujan Again
The New Mod 8 Identities
Future Work

Plane Partitions

A vector of partitions $\pi = (\pi_1, \pi_2, \dots)$ (each $\pi_i = (\lambda_{i,1}, \lambda_{i,2}, \dots, \lambda_{i,\#(\pi_i)})$ is called a *plane partition* if for all i and j, we have

$$\lambda_{i,j} \geq \lambda_{i+1,j}. \text{ and } \lambda_{i,j} \geq \lambda_{i,j+1},$$

Generating function for the plane partitions is

$$\prod_{i\geq 1} \frac{1}{(q^i;q)_\infty}.$$

Basics
Cylindric Partitions
Borodin's Theorem
Corteel and Welsh's Lemma
Rogers–Ramanujan Again
The New Mod 8 Identities
Future Work

Plane Partitions

A vector of partitions $\pi = (\pi_1, \pi_2, \dots)$ (each $\pi_i = (\lambda_{i,1}, \lambda_{i,2}, \dots, \lambda_{i,\#(\pi_i)})$ is called a *plane partition* if for all i and j, we have

$$\lambda_{i,j} \geq \lambda_{i+1,j}. \text{ and } \lambda_{i,j} \geq \lambda_{i,j+1},$$

Generating function for the plane partitions is

$$\prod_{i\geq 1} \frac{1}{(q^i;q)_\infty}.$$

Compare it with the generating function for ordinary partitions

$$\frac{1}{(q;q)_\infty}.$$

Cylindric Partitions

Introduced by Gessel and Krattenthaler, cylindric partitions go as follows:

A vector of k partitions $\pi = (\pi_1, \pi_2, \ldots, \pi_k)$ for a *profile* $c = (c_1, c_2, \ldots, c_k)$ ($c_i \in \mathbb{Z}_{\geq 0}$) is called a *cylindric partition* if for every $i \in \{1, 2, \ldots, k-1\}$

$$\lambda_{i,j} \geq \lambda_{i+1,j+c_{i+1}}. \text{ and } \lambda_{k,j} \geq \lambda_{1,j+c_1},$$

where $\lambda_{i,j}$ is the j-th element of the i-th partition π_i.

I. Gessel and C. Krattenthaler, *Cylindric partitions*, Trans. Amer. Math. Soc. **349** (1997), no. 2, 429-479.

An Example

$$\pi = (\ (3,2,2,1),(4,3,3,1,1),(4,1,1,1)\ )$$

Basics
Cylindric Partitions
Borodin's Theorem
Corteel and Welsh's Lemma
Rogers–Ramanujan Again
The New Mod 8 Identities
Future Work

An Example

$\pi = (\ (3,2,2,1),(4,3,3,1,1),(4,1,1,1)\ )$ is a cylindric partition of profile $(2,1,2)$ with largest part size 4 and total size 27.

Basics
Cylindric Partitions
Borodin's Theorem
Corteel and Welsh's Lemma
Rogers–Ramanujan Again
The New Mod 8 Identities
Future Work

An Example

$\pi = (\ (3,2,2,1),(4,3,3,1,1),(4,1,1,1)\ )$ is a cylindric partition of profile $(2,1,2)$ with largest part size 4 and total size 27.

$$3 \quad 2 \quad 2 \quad 1$$

An Example

Basics
Cylindric Partitions
Borodin's Theorem
Corteel and Welsh's Lemma
Rogers–Ramanujan Again
The New Mod 8 Identities
Future Work

$\pi = (\ (3,2,2,1),(4,3,3,1,1),(4,1,1,1)\ )$ is a cylindric partition of profile $(2,1,2)$ with largest part size 4 and total size 27.

$$\begin{array}{cccccc} & & 3 & 2 & 2 & 1 \\ 4 & 3 & 3 & 1 & 1 & \end{array}$$

An Example

Basics
Cylindric Partitions
Borodin's Theorem
Corteel and Welsh's Lemma
Rogers–Ramanujan Again
The New Mod 8 Identities
Future Work

$\pi = (\ (3,2,2,1),(4,3,3,1,1),(4,1,1,1)\ )$ is a cylindric partition of profile $(2,1,2)$ with largest part size 4 and total size 27.

$$
\begin{array}{cccccccc}
 & & & 3 & 2 & 2 & 1 \\
 & 4 & 3 & 3 & 1 & 1 & \\
4 & 1 & 1 & 1 & & &
\end{array}
$$

Basics
Cylindric Partitions
Borodin's Theorem
Corteel and Welsh's Lemma
Rogers–Ramanujan Again
The New Mod 8 Identities
Future Work

An Example

$\pi = (\ (3,2,2,1),(4,3,3,1,1),(4,1,1,1)\ )$ is a cylindric partition of profile $(2,1,2)$ with largest part size 4 and total size 27.

$$
\begin{array}{cccccccccc}
 & & & & & 3 & 2 & 2 & 1 \\
 & & & 4 & 3 & 3 & 1 & 1 & \\
 & & 4 & 1 & 1 & 1 & & & \\
\hline
3 & 2 & 2 & 1 & & & & &
\end{array}
$$

An Example

$\pi = (\ (3,2,2,1),(4,3,3,1,1),(4,1,1,1)\ )$ is a cylindric partition of profile $(2,1,2)$ with largest part size 4 and total size 27.

$$\begin{array}{cccccccccc}
 & & & & & 3 & 2 & 2 & 1 \\
 & & & 4 & 3 & 3 & 1 & 1 & \\
 & & 4 & 1 & 1 & 1 & & & \\
\hline
3 & 2 & 2 & 1 & & & & &
\end{array}$$

Also there is a one-to-one correspondence between cylindric partitions of profile $(2,1,2)$ and $(2,2,1)$, where π is matched with $\pi^* = (\ (4,1,1,1),(3,2,2,1),(4,3,3,1,1)\ )$.

An Example

$\pi = (\ (3,2,2,1),(4,3,3,1,1),(4,1,1,1)\ )$ is a cylindric partition of profile $(2,1,2)$ with largest part size 4 and total size 27.

$$
\begin{array}{cccccccccc}
 & & & & & 3 & 2 & 2 & 1 \\
 & & & 4 & 3 & 3 & 1 & 1 & \\
 & & 4 & 1 & 1 & 1 & & & \\
\hline
3 & 2 & 2 & 1 & & & & &
\end{array}
$$

Also there is a one-to-one correspondence between cylindric partitions of profile $(2,1,2)$ and $(2,2,1)$, where π is matched with $\pi^* = (\ (4,1,1,1),(3,2,2,1),(4,3,3,1,1)\ )$.
A self-note: Don't forget to talk about removing a largest part.

Cylindric Partitions
Borodin's Theorem

Basics
Cylindric Partitions
Borodin's Theorem
Corteel and Welsh's Lemma
Rogers–Ramanujan Again
The New Mod 8 Identities
Future Work

Let P_c be the set of all cylindric partitions with profile c. Let the generating function

$$F_c(y,q) := \sum_{\pi \in P_c} y^{max(\pi)} q^{|\pi|}.$$

Cylindric Partitions

Borodin's Theorem

Basics

Cylindric Partitions

Borodin's Theorem

Corteel and Welsh's Lemma

Rogers–Ramanujan Again

The New Mod 8 Identities

Future Work

Let P_c be the set of all cylindric partitions with profile c. Let the generating function

$$F_c(y,q) := \sum_{\pi \in P_c} y^{max(\pi)} q^{|\pi|}.$$

Theorem (Borodin, 2007)

For a given profile $c = (c_1, \ldots, c_k)$

$$F_c(1,q) = \frac{1}{(q^t;q^t)_\infty} \prod_{i,j=1}^{m} \prod_{m=1}^{c_i} \frac{1}{(q^{m+d_{i+1,j}+j-i}, q^{t-(m+d_{j,i-1})+i-j}; q^t)_\infty},$$

where $d_{i,j} := c_i + c_{i+1} + \ldots c_j$, and 0 if $j > i$, and $t = k + d_{1,k}$.

A. Borodin, *Periodic Schur process and cylindric partitions*, Duke Math. J. **140** (2007), no. 3, 391-468.

Cylindric Partitions

Basics
Cylindric Partitions
Borodin's Theorem
Corteel and Welsh's Lemma
Rogers–Ramanujan Again
The New Mod 8 Identities
Future Work

Let

$$G_c(y,q) := (yq;q)_\infty F_c(y,q).$$

Cylindric Partitions

Basics
Cylindric Partitions
Borodin's Theorem
Corteel and Welsh's Lemma
Rogers–Ramanujan Again
The New Mod 8 Identities
Future Work

Let

$$G_c(y,q) := (yq;q)_\infty F_c(y,q).$$

Theorem (Corteel-Welsh, 2019)

For a given profile $c = (c_1, \ldots, c_k)$

$$G_c(y,q) = \sum_{J \subset I} (-1)^{|J|-1} (yq;q)_{|J|-1} G_{c(J)}(yq^{|J|}, q),$$

where this J is the affect of removing the largest part.

S. Corteel and T. Welsh *The A_2 Rogers–Ramanujan identities revisited*, published in Annals of Combinatorics in honor of the 80th birthday of George Andrews

Rogers-Ramanujan Again

Basics

Cylindric Partitions

Borodin's Theorem

Corteel and Welsh's Lemma

Rogers–Ramanujan Again

The New Mod 8 Identities

Future Work

Let's look at the cylindric partitions with the profile $(3, 0)$ and its adjacency.

Basics

Cylindric Partitions

Borodin's Theorem

Corteel and Welsh's Lemma

Rogers–Ramanujan Again

The New Mod 8 Identities

Future Work

Rogers-Ramanujan Again

Let's look at the cylindric partitions with the profile $(3,0)$ and its adjacency.

$$G_{(3,0)}(y,q) = G_{(2,1)}(yq,q),$$
$$G_{(2,1)}(y,q) = G_{(3,0)}(yq,q) + G_{(2,1)}(yq,q) - (1-yq)G_{(2,1)}(yq^2,q).$$

Rogers-Ramanujan Again

Basics
Cylindric Partitions
Borodin's Theorem
Corteel and Welsh's Lemma
Rogers–Ramanujan Again
The New Mod 8 Identities
Future Work

Let's look at the cylindric partitions with the profile $(3, 0)$ and its adjacency.

$$G_{(3,0)}(y, q) = G_{(2,1)}(yq, q),$$
$$G_{(2,1)}(y, q) = G_{(3,0)}(yq, q) + G_{(2,1)}(yq, q) - (1 - yq)G_{(2,1)}(yq^2, q).$$

Let

$$G_c(y, q) := \sum_{n \geq 0} g_c(n) y^n$$

Rogers-Ramanujan Again

Basics
Cylindric Partitions
Borodin's Theorem
Corteel and Welsh's Lemma
Rogers–Ramanujan Again
The New Mod 8 Identities
Future Work

Let's look at the cylindric partitions with the profile $(3,0)$ and its adjacency.

$$G_{(3,0)}(y,q) = G_{(2,1)}(yq,q),$$
$$G_{(2,1)}(y,q) = G_{(3,0)}(yq,q) + G_{(2,1)}(yq,q) - (1-yq)G_{(2,1)}(yq^2,q).$$

Let

$$G_c(y,q) := \sum_{n\geq 0} g_c(n) y^n,$$

where $g_c(0)=1$ and $g_c(<0)\equiv 0$.

Rogers-Ramanujan Again

Let's look at the cylindric partitions with the profile $(3,0)$ and its adjacency.

$$G_{(3,0)}(y,q) = G_{(2,1)}(yq,q),$$
$$G_{(2,1)}(y,q) = G_{(3,0)}(yq,q) + G_{(2,1)}(yq,q) - (1-yq)G_{(2,1)}(yq^2,q).$$

Let

$$G_c(y,q) := \sum_{n\geq 0} g_c(n)y^n,$$

where $g_c(0)=1$ and $g_c(<0)\equiv 0$. Then,

$$g_{(3,0)}(n) = q^n g_{(2,1)}(n),$$
$$(1-q^n+q^{2n})g_{(2,1)}(n) = q^n g_{(3,0)}(n) + q^{2n-1}g_{(2,1)}(n-1).$$

Rogers-Ramanujan Again

Basics
Cylindric Partitions
Borodin's Theorem
Corteel and Welsh's Lemma
Rogers–Ramanujan Again
The New Mod 8 Identities
Future Work

Let's look at the cylindric partitions with the profile $(3,0)$ and its adjacency.

$$\begin{aligned} G_{(3,0)}(y,q) &= G_{(2,1)}(yq,q), \\ G_{(2,1)}(y,q) &= G_{(3,0)}(yq,q) + G_{(2,1)}(yq,q) - (1-yq)G_{(2,1)}(yq^2,q). \end{aligned}$$

Let

$$G_c(y,q) := \sum_{n\geq 0} g_c(n) y^n$$

 Ali K. Uncu : *aku21@bath.ac.uk*

Rogers-Ramanujan Again

Basics
Cylindric Partitions
Borodin's Theorem
Corteel and Welsh's Lemma
Rogers–Ramanujan Again
The New Mod 8 Identities
Future Work

Let's look at the cylindric partitions with the profile $(3,0)$ and its adjacency.

$$\begin{aligned} G_{(3,0)}(y,q) &= G_{(2,1)}(yq,q), \\ G_{(2,1)}(y,q) &= G_{(3,0)}(yq,q) + G_{(2,1)}(yq,q) - (1-yq)G_{(2,1)}(yq^2,q). \end{aligned}$$

Let

$$G_c(y,q) := \sum_{n\geq 0} g_c(n) y^n,$$

where $g_c(0) = 1$ and $g_c(<0) \equiv 0$. Then,

$$g_{(3,0)}(n) = \frac{q^{2n}}{(1-q^n)} g_{(3,0)}(n-1), \text{ and } g_{(2,1)}(n) = \frac{q^{2n-1}}{(1-q^n)} g_{(2,1)}(n-1)$$

Rogers-Ramanujan Again

Basics

Cylindric Partitions

Borodin's Theorem

Corteel and Welsh's Lemma

Rogers–Ramanujan Again

The New Mod 8 Identities

Future Work

$$g_{(3,0)}(n) = \frac{q^{2n}}{(1-q^n)} g_{(3,0)}(n-1), \text{and } g_{(2,1)}(n) = \frac{q^{2n-1}}{(1-q^n)} g_{(2,1)}(n-1)$$

and the initial conditions $g_c(0) = 1$ and $g_c(< 0) \equiv 0$ is enough to find the formulas for the generating functions:

$$G_{(3,0)}(y,q) = \sum_{n\geq 0} \frac{q^{n^2+n} y^n}{(q;q)_n} \text{and } G_{(2,1)}(y,q) = \sum_{n\geq 0} \frac{q^{n^2} y^n}{(q;q)_n}.$$

Rogers-Ramanujan Again

Basics
Cylindric Partitions
Borodin's Theorem
Corteel and Welsh's Lemma
Rogers–Ramanujan Again
The New Mod 8 Identities
Future Work

$$g_{(3,0)}(n) = \frac{q^{2n}}{(1-q^n)} g_{(3,0)}(n-1), \text{ and } g_{(2,1)}(n) = \frac{q^{2n-1}}{(1-q^n)} g_{(2,1)}(n-1)$$

and the initial conditions $g_c(0) = 1$ and $g_c(<0) \equiv 0$ is enough to find the formulas for the generating functions:

$$G_{(3,0)}(y,q) = \sum_{n\geq 0} \frac{q^{n^2+n} y^n}{(q;q)_n} \text{ and } G_{(2,1)}(y,q) = \sum_{n\geq 0} \frac{q^{n^2} y^n}{(q;q)_n}.$$

Setting $y = 1$ and using Borodin's theorem yields the analytic version of the Rogers–Ramanujan identities: For $m = 1, 2$ we have

$$\sum_{n\geq 0} \frac{q^{n^2+(m-1)n}}{(q;q)_n} = \frac{1}{(q^m, q^{5-m}; q^5)_\infty}.$$

Basics
Cylindric Partitions
Borodin's Theorem
Corteel and Welsh's Lemma
Rogers–Ramanujan Again
The New Mod 8 Identities
Future Work

The (5,0,0) profile and the related system

The $(5,0,0)$ system has these essentially unique cylindric partition profiles:

$$(5,0,0),(4,1,0),(4,0,1),(3,2,0),(3,0,2),(3,1,1),(2,2,1).$$

The (5,0,0) profile and the related system

Basics

Cylindric Partitions

Borodin's Theorem

Corteel and Welsh's Lemma

Rogers–Ramanujan Again

The New Mod 8 Identities

Future Work

The $(5,0,0)$ system has these essentially unique cylindric partition profiles:

$$(5,0,0),(4,1,0),(4,0,1),(3,2,0),(3,0,2),(3,1,1),(2,2,1).$$

The related product generating functions:

$$G_{(5,0,0)}(1,q) = \frac{1}{(q^2,q^3,q^3,q^4,q^4,q^5,q^5,q^6;q^8)_\infty},$$

$$G_{(4,1,0)}(1,q) = G_{(4,0,1)}(1,q) = \frac{1}{(q,q^2,q^3,q^4,q^4,q^5,q^6,q^7;q^8)_\infty},$$

$$G_{(3,2,0)}(1,q) = G_{(3,0,2)}(1,q) = \frac{1}{(q,q^2,q^2,q^3,q^5,q^6,q^6,q^7;q^8)_\infty},$$

$$G_{(3,1,1)}(1,q) = \frac{1}{(q,q,q^3,q^3,q^5,q^5,q^7,q^7;q^8)_\infty},$$

$$G_{(2,2,1)}(1,q) = \frac{1}{(q,q,q^2,q^4,q^4,q^6,q^7,q^7;q^8)_\infty}.$$

Basics
Cylindric Partitions
Borodin's Theorem
Corteel and Welsh's Lemma
Rogers–Ramanujan Again
The New Mod 8 Identities
Future Work

One extra symmetry

We were able to show that for 3 element profiles we have

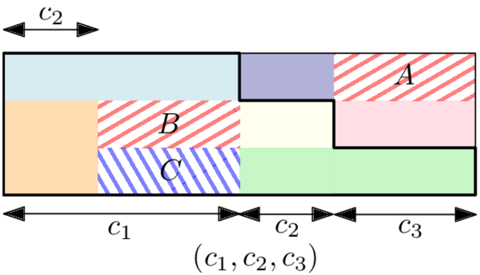
$$G_{(c_1,c_2,c_3)}(1,q) = G_{(c_2,c_1,c_3)}(1,q),$$

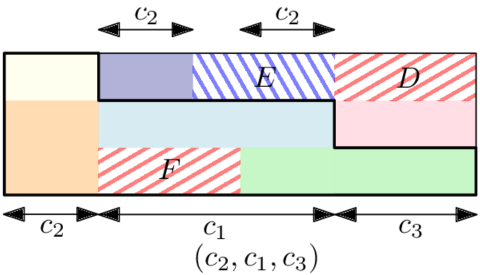
$$(q;q)_\infty F_{(c_1,c_2,c_3)}(1,q) = (q;q)_\infty F_{(c_2,c_1,c_3)}(1,q).$$

One extra symmetry

We were able to show that for 3 element profiles we have

$$F_{(c_1,c_2,c_3)}(1,q) = F_{(c_2,c_1,c_3)}(1,q).$$





The (5,0,0) profile and the related system

Basics

Cylindric Partitions

- Borodin's Theorem
- Corteel and Welsh's Lemma
- Rogers–Ramanujan Again
- The New Mod 8 Identities

Future Work

The $(5,0,0)$ system has these essentially unique cylindric partition profiles:

$$(5,0,0),(4,1,0),(4,0,1),(3,2,0),(3,0,2),(3,1,1),(2,2,1).$$

The related product generating functions:

$$G_{(5,0,0)}(1,q) = \frac{1}{(q^2,q^3,q^3,q^4,q^4,q^5,q^5,q^6;q^8)_\infty},$$

$$G_{(4,1,0)}(1,q) = G_{(4,0,1)}(1,q) = \frac{1}{(q,q^2,q^3,q^4,q^4,q^5,q^6,q^7;q^8)_\infty},$$

$$G_{(3,2,0)}(1,q) = G_{(3,0,2)}(1,q) = \frac{1}{(q,q^2,q^2,q^3,q^5,q^6,q^6,q^7;q^8)_\infty},$$

$$G_{(3,1,1)}(1,q) = \frac{1}{(q,q,q^3,q^3,q^5,q^5,q^7,q^7;q^8)_\infty},$$

$$G_{(2,2,1)}(1,q) = \frac{1}{(q,q,q^2,q^4,q^4,q^6,q^7,q^7;q^8)_\infty}.$$

The (5,0,0) profile and the related system

Basics
Cylindric Partitions
Borodin's Theorem
Corteel and Welsh's Lemma
Rogers–Ramanujan Again
The New Mod 8 Identities
Future Work

$$
\begin{aligned}
G_{(5,0,0)}(y,q) &= G_{(4,1,0)}(yq,q),\\
G_{(4,1,0)}(y,q) &= G_{(4,0,1)}(yq,q) + G_{(3,2,0)}(yq,q) - (1-yq)G_{(3,1,1)}(yq^2,q),\\
G_{(4,0,1)}(y,q) &= G_{(5,0,0)}(yq,q) + G_{(3,1,1)}(yq,q) - (1-yq)G_{(4,1,0)}(yq^2,q),\\
G_{(3,2,0)}(y,q) &= G_{(3,1,1)}(yq,q) + G_{(3,0,2)}(yq,q) - (1-yq)G_{(2,2,1)}(yq^2,q),\\
G_{(3,1,1)}(y,q) &= G_{(4,1,0)}(yq,q) + G_{(3,0,2)}(yq,q) + G_{(2,2,1)}(yq,q)\\
&\quad - (1-yq)\big(G_{(4,0,1)}(yq^2,q) + G_{(3,2,0)}(yq^2,q) + G_{(2,2,1)}(yq^2,q)\big)\\
&\quad + (1-yq)(1-yq^2)G_{(3,1,1)}(yq^3,q),\\
G_{(3,0,2)}(y,q) &= G_{(4,0,1)}(yq,q) + G_{(2,2,1)}(yq,q) - (1-yq)G_{(3,1,1)}(yq^2,q),\\
G_{(2,2,1)}(y,q) &= G_{(3,2,0)}(yq,q) + G_{(3,1,1)}(yq,q) + G_{(2,2,1)}(yq,q)\\
&\quad - (1-yq)\big(G_{(3,1,1)}(yq^2,q) + G_{(3,0,2)}(yq^2,q) + G_{(2,2,1)}(yq^2,q)\big)\\
&\quad + (1-yq)(1-yq^2)G_{(2,2,1)}(yq^3,q).
\end{aligned}
$$

The (5,0,0) profile and the related system

Basics
Cylindric Partitions
Borodin's Theorem
Corteel and Welsh's Lemma
Rogers–Ramanujan Again
The New Mod 8 Identities
Future Work

This is clearly where we should stop doing work by hand.

The (5,0,0) profile and the related system

Basics
Cylindric Partitions
Borodin's Theorem
Corteel and Welsh's Lemma
Rogers–Ramanujan Again
The New Mod 8 Identities
Future Work

$$G_{(5,0,0)}(y,q) = \sum_{n_1,n_2,n_3,n_4\geq 0} \frac{y^{n_1} q^{n_1^2+n_2^2+n_3^2+n_4^2+n_1+n_2+n_3+n_4-n_1n_2+n_2n_4}}{(q;q)_{n_1}} \begin{bmatrix} n_1 \\ n_2 \end{bmatrix}_q \begin{bmatrix} n_1 \\ n_4 \end{bmatrix}_q \begin{bmatrix} n_2 \\ n_3 \end{bmatrix}_q,$$

$$G_{(4,1,0)}(y,q) = \sum_{n_1,n_2,n_3,n_4\geq 0} \frac{y^{n_1} q^{n_1^2+n_2^2+n_3^2+n_4^2+n_2+n_3+n_4-n_1n_2+n_2n_4}}{(q;q)_{n_1}} \begin{bmatrix} n_1 \\ n_2 \end{bmatrix}_q \begin{bmatrix} n_1 \\ n_4 \end{bmatrix}_q \begin{bmatrix} n_2 \\ n_3 \end{bmatrix}_q,$$

$$G_{(4,0,1)}(y,q) = \sum_{n_1,n_2,n_3,n_4\geq 0} \frac{y^{n_1} q^{n_1^2+n_2^2+n_3^2+n_4^2+n_1+n_3-n_1n_2+n_2n_4}\left(1+yq^{n_1+n_2+n_4+1}\right)}{(q;q)_{n_1}} \begin{bmatrix} n_1 \\ n_2 \end{bmatrix}_q \begin{bmatrix} n_1 \\ n_4 \end{bmatrix}_q \begin{bmatrix} n_2 \\ n_3 \end{bmatrix}_q$$

$$G_{(3,0,2)}(y,q) = \sum_{n_1,n_2,n_3,n_4\geq 0} \frac{y^{n_1} q^{n_1^2+n_2^2+n_3^2+n_4^2+n_1-n_1n_2+n_2n_4}\left(1+yq^{n_1+n_3+1}+yq^{2n_1+n_2+n_3+n_4+2}\right)}{(q;q)_{n_1}} \begin{bmatrix} n_1 \\ n_2 \end{bmatrix}_q \begin{bmatrix} n_1 \\ n_4 \end{bmatrix}_q \begin{bmatrix} n_2 \\ n_3 \end{bmatrix}_q,$$

$$G_{(3,2,0)}(y,q) = \sum_{n_1,n_2,n_3,n_4\geq 0} \frac{y^{n_1} q^{n_1^2+n_2^2+n_3^2+n_4^2+n_1-n_1n_2+n_2n_4}\left(q^{n_3}+yq^{n_1+1}+yq^{2n_1+n_3+2}+yq^{3n_1+n_2+n_3+n_4+3}\right)}{(q;q)_{n_1}} \begin{bmatrix} n_1 \\ n_2 \end{bmatrix}_q \begin{bmatrix} n_1 \\ n_4 \end{bmatrix}_q \begin{bmatrix} n_2 \\ n_3 \end{bmatrix}_q,$$

$$G_{(3,1,1)}(y,q) = \sum_{n_1,n_2,n_3,n_4\geq 0} \frac{y^{n_1} q^{n_1^2+n_2^2+n_3^2+n_4^2+n_3-n_1n_2+n_2n_4}}{(q;q)_{n_1}} \begin{bmatrix} n_1 \\ n_2 \end{bmatrix}_q \begin{bmatrix} n_1 \\ n_4 \end{bmatrix}_q \begin{bmatrix} n_2 \\ n_3 \end{bmatrix}_q,$$

$$G_{(2,2,1)}(y,q) = \sum_{n_1,n_2,n_3,n_4\geq 0} \frac{y^{n_1} q^{n_1^2+n_2^2+n_3^2+n_4^2-n_1n_2+n_2n_4}}{(q;q)_{n_1}} \begin{bmatrix} n_1 \\ n_2 \end{bmatrix}_q \begin{bmatrix} n_1 \\ n_4 \end{bmatrix}_q \begin{bmatrix} n_2 \\ n_3 \end{bmatrix}_q.$$

The (5,0,0) profile and the related system

Basics
Cylindric Partitions
Borodin's Theorem
Corteel and Welsh's Lemma
Rogers–Ramanujan Again
The New Mod 8 Identities
Future Work

Theorem (Corteel-Dousse-U)

$$G_{(5,0,0)}(1,q) = \sum_{n_1,n_2,n_3,n_4\geq 0} \frac{q^{n_1^2+n_2^2+n_3^2+n_4^2+n_1+n_2+n_3+n_4-n_1n_2+n_2n_4}}{(q;q)_{n_1}} \begin{bmatrix} n_1 \\ n_2 \end{bmatrix}_q \begin{bmatrix} n_1 \\ n_4 \end{bmatrix}_q \begin{bmatrix} n_2 \\ n_3 \end{bmatrix}_q$$

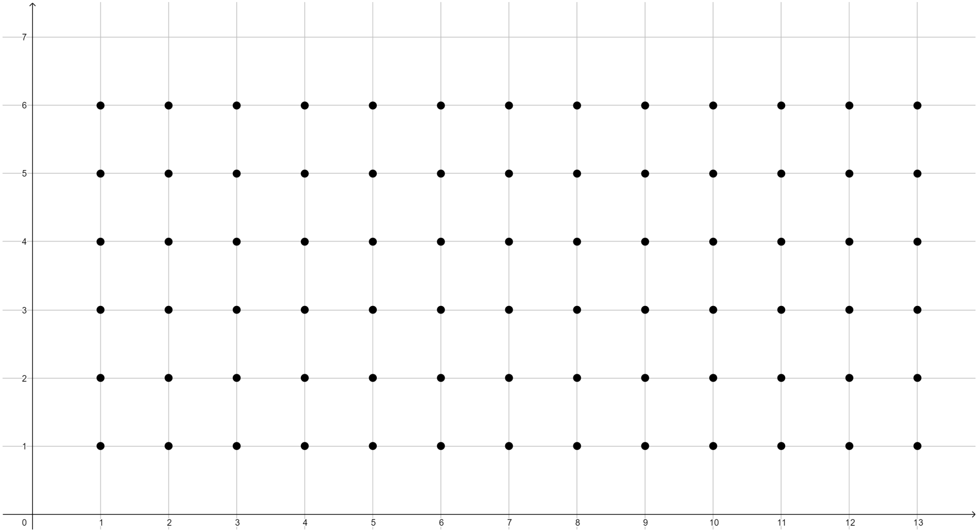
$$= \frac{1}{(q^2,q^3,q^3,q^4,q^4,q^5,q^5,q^6;q^8)_\infty},$$

where

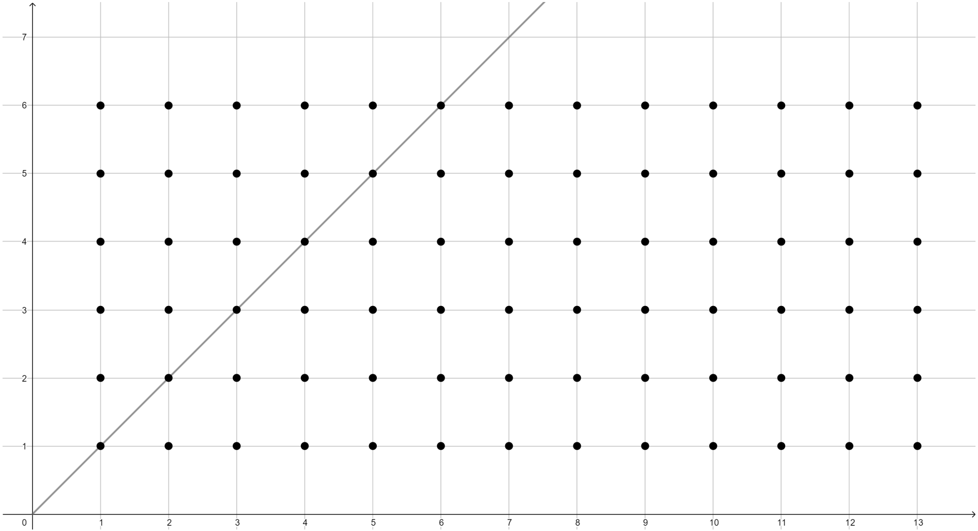
$$\begin{bmatrix} m+n \\ m \end{bmatrix}_q := \begin{cases} \frac{(q;q)_{m+n}}{(q;q)_m(q;q)_n}, & \text{for } m,n\geq 0, \\ 0, & \text{otherwise.} \end{cases}$$

S. Corteel, J. Dousse, and A.K.U. *Cylindric partitions and some new A_2 Rogers-Ramanujan identities*, Accepted in Proc. Amer. Math. Soc. 2021, https://doi.org/10.1090/proc/15570

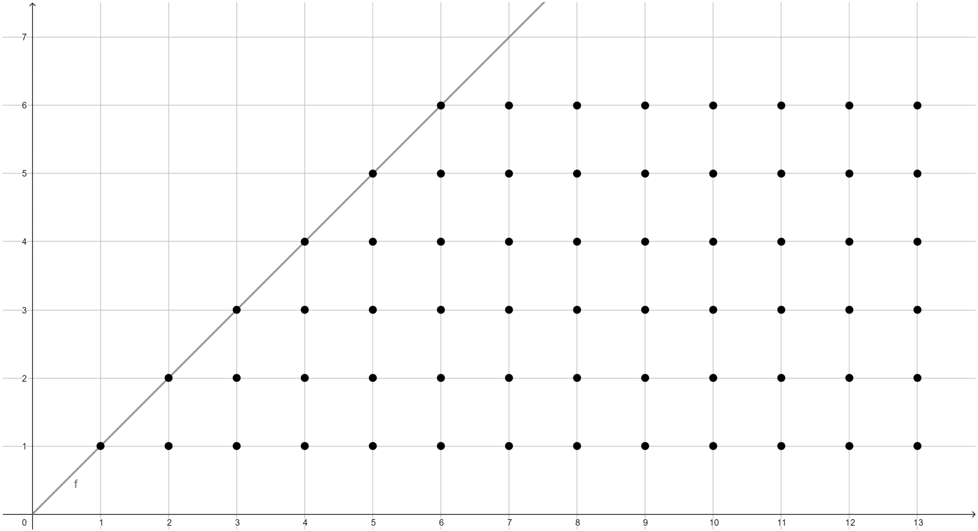
What else is there?



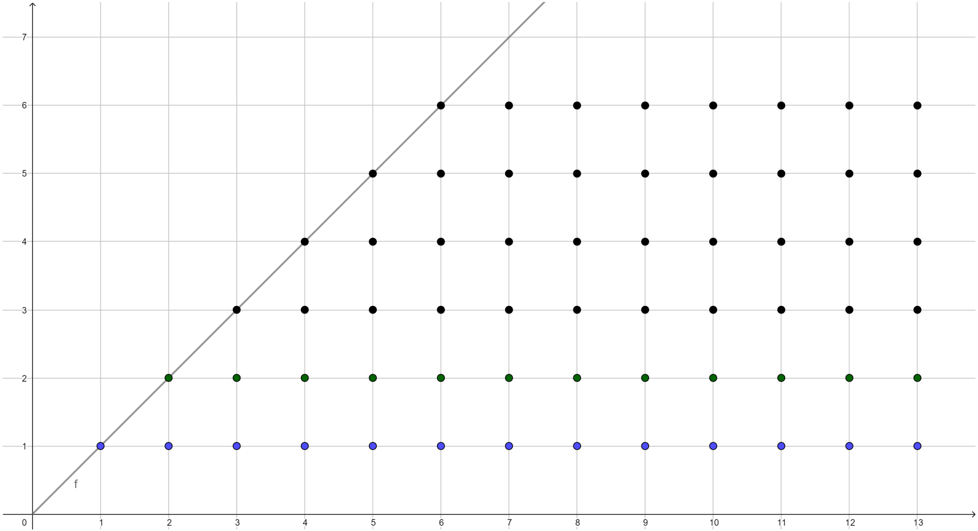
What else is there?



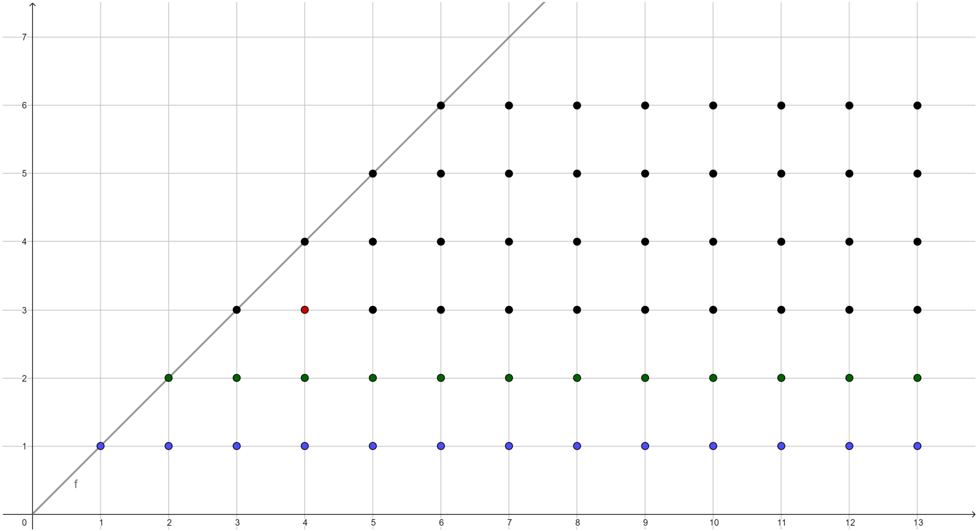
What else is there?



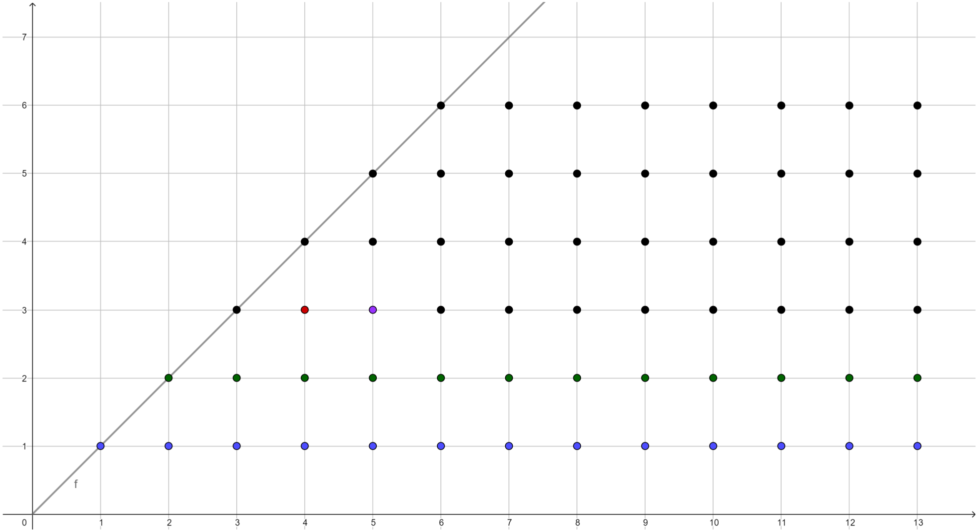
What else is there?



What else is there?



What else is there?

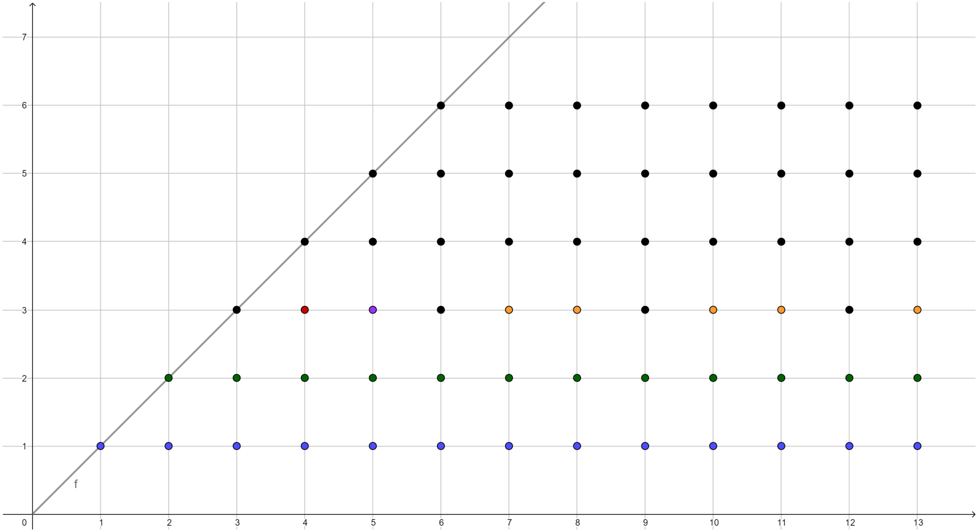


What else is there?

Basics

Cylindric Partitions

Future Work



Wait! There is more:

We can also look at symmetric cylindric partitions and their properties.

Wait! There is more:

We can also look at symmetric cylindric partitions and their properties.

This is exactly what we are doing with Walter Bridges now.

Wait! There is more:

Basics
Cylindric Partitions
Future Work

We can also look at symmetric cylindric partitions and their properties.

This is exactly what we are doing with Walter Bridges now.

Advertisement: Don't miss next week's talk. He will be giving an account on what we found so far.

Thank you for your time

On New Modulo 8 Cylindric Partition Identities

Ali Kemal Uncu (aku21@bath.ac.uk)

Specialty Seminar in Partition Theory, q-Series and Related Topics

The speaker is partially supported by the EPSRC grant EP/T015713/1 and partially by the FWF grant P-34501N.