

New Infinite Hierarchies Of Polynomial Identities Related To Capparelli's Identities

joint work with Alexander Berkovich

Ali Kemal Uncu (akuncu@ricam.oeaw.ac.at)



UNIVERSITY OF
BATH

ÖAW

AUSTRIAN
ACADEMY OF
SCIENCES

RICAM
JOHANN RADON INSTITUTE
FOR COMPUTATIONAL AND APPLIED MATHEMATICS

Michigan Technological University

Oct 5, 2023



Partitions

Everything we need to know at the moment

Introduction

Partitions

Generating Functions

Capparelli's Partition Theorems

Finitizations

A *partition* π is a finite sequence of *non-decreasing* positive integers $(\lambda_1, \lambda_2, \dots, \lambda_{\#(\pi)})$.

For a given partition $\pi = (\lambda_1, \lambda_2, \dots, \lambda_{\#(\pi)})$ the sum $\lambda_1 + \lambda_2 + \dots + \lambda_{\#(\pi)}$ is called the *size* of the partition π and is denoted by $|\pi|$.



Partitions

Everything we need to know at the moment

Introduction

Partitions

Generating Functions

Capparelli's Partition Theorems

Finitizations

A *partition* π is a finite sequence of *non-decreasing* positive integers $(\lambda_1, \lambda_2, \dots, \lambda_{\#(\pi)})$.

For a given partition $\pi = (\lambda_1, \lambda_2, \dots, \lambda_{\#(\pi)})$ the sum $\lambda_1 + \lambda_2 + \dots + \lambda_{\#(\pi)}$ is called the *size* of the partition π and is denoted by $|\pi|$.

Ex:

- $\pi = (1, 1, 5)$ is a partition of $|\pi| = 7$.
- $\pi = \emptyset$ is the unique partition of 0.



Definitions and Notations

Young Diagrams

Introduction

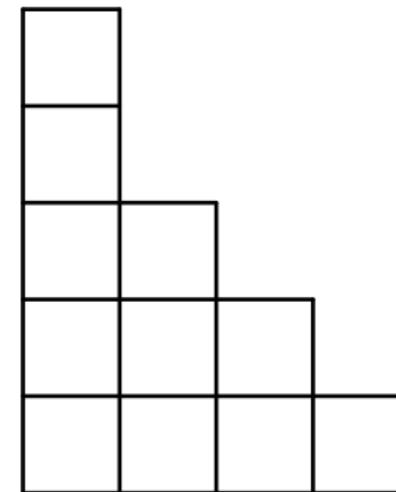
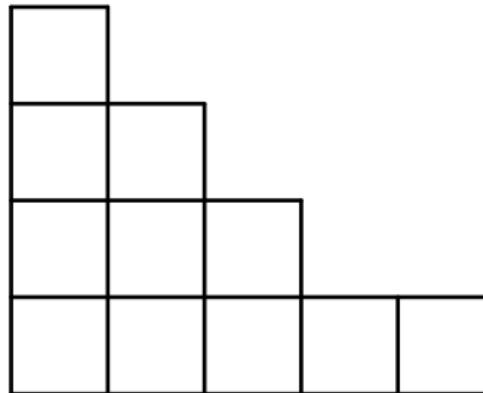
Partitions

Generating Functions

Capparelli's Partition Theorems

Finitizations

Ex: $\pi = (1, 2, 3, 5)$ and $\pi' = (1, 1, 2, 3, 4)$





Definitions and Notations

Generating Functions

For a sequence $\{a_n\}_{n=0}^{\infty}$, the series

$$\sum_{n \geq 0} a_n q^n$$

is called a *generating function*.

Let \mathcal{U} be the set of all partitions.

$$\sum_{\pi \in \mathcal{U}} q^{|\pi|} = \sum_{n \geq 0} p(n) q^n = 1 + q + 2q^2 + 3q^3 + 5q^4 + \dots,$$

where $p(n)$ is the number of partitions of n .

\emptyset

(1)

(1, 1), (2)

(1, 1, 1), (1, 2), (3)



Definitions and Notations

Generating Functions

Introduction

Partitions

Generating
Functions

Capparelli's
Partition
Theorems

Finitizations

$$(a)_L := (a; q)_L := \prod_{i=0}^{L-1} (1 - aq^i),$$

$$(a; q)_{\infty} := \lim_{L \rightarrow \infty} (a; q)_L \text{ for } |q| < 1,$$

$$(a_1, a_2, \dots, a_k; q)_L := \prod_{i=1}^k (a_i; q)_L \text{ for } k \in \mathbb{Z}_{\geq 1}.$$

$$\begin{bmatrix} m+n \\ m \end{bmatrix}_q := \begin{cases} \frac{(q)_{m+n}}{(q)_m(q)_n} & \text{for } m, n \geq 0, \\ 0 & \text{otherwise.} \end{cases}$$



Definitions and Notations

Generating Functions

Introduction
Partitions
Generating Functions

Capparelli's Partition Theorems

Finitizations

$$(a)_L := (a; q)_L := \prod_{i=0}^{L-1} (1 - aq^i),$$

$$(a; q)_{\infty} := \lim_{L \rightarrow \infty} (a; q)_L \text{ for } |q| < 1,$$

$$(a_1, a_2, \dots, a_k; q)_L := \prod_{i=1}^k (a_i; q)_L \text{ for } k \in \mathbb{Z}_{\geq 1}.$$

$$\begin{bmatrix} m+n \\ m \end{bmatrix}_q := \begin{cases} \frac{(q)_{m+n}}{(q)_m(q)_n} & \text{for } m, n \geq 0, \\ 0 & \text{otherwise.} \end{cases}$$

$$\lim_{N \rightarrow \infty} \begin{bmatrix} N \\ m \end{bmatrix}_q = \frac{1}{(q; q)_m}$$



First Partition Theorem

Difference-Congruence Identity

Introduction

Capparelli's
Partition
Theorems

Original
Statements

Refinement
Attempts

Capparelli's
Companion

A breakthrough...

Finitizations

Theorem (Capparelli's First Partition Theorem)

For any integer n , the number of partitions of n into parts, not equal to 1, where the minimal difference between consecutive parts is 4, except if the consecutive parts are

- $3k$ and $3k + 3$ (yielding a difference of 3), or
- $3k - 1$ and $3k + 1$ (yielding a difference of 2) for some $k \in \mathbb{N}$,

=

the number of partitions of n into distinct parts where no part is congruent to ± 1 modulo 6.

G. E. Andrews, *Schur's theorem. Capparelli's conjecture and q-trinomial coefficients*,
Contemporary Mathematics **166** (1994), 141-154.



Second Partition Theorem

Difference-Congruence Identity

Introduction

Capparelli's
Partition
Theorems

Original
Statements

Refinement
Attempts

Capparelli's
Companion

A breakthrough...

Finitizations

Theorem (Capparelli's Second Partition Theorem)

For any integer n , the number of partitions of n into parts, not equal to 2, where the minimal difference between consecutive parts is 4, except if the consecutive parts are

- $3k$ and $3k + 3$ (yielding a difference of 3), or
- $3k - 1$ and $3k + 1$ (yielding a difference of 2) for some $k \in \mathbb{N}$,

=

the number of partitions of n into distinct parts where no part is congruent to ± 2 modulo 6.

S. Capparelli, A combinatorial proof of a partition identity related to the level 3 representation of twisted affine Lie algebra, Communications in Algebra **23** (1995), no. 8, 2959-2969..



Sum-Product identity?

Back to Capparelli's Identity

Introduction

Capparelli's
Partition
Theorems

Original
Statements

Refinement
Attempts

Capparelli's
Companion

A breakthrough...

Finitizations

A Difference-Congruence identity usually gets accompanied with a Sum-Product identity.



Sum-Product identity?

Back to Capparelli's Identity

Introduction
Capparelli's Partition Theorems

Original Statements

Refinement Attempts

Capparelli's Companion

A breakthrough...

Finitizations

A Difference-Congruence identity usually gets accompanied with a Sum-Product identity.

Not in Capparelli's partition theorems case



Sum-Product identity?

Back to Capparelli's Identity

Introduction

Capparelli's
Partition
Theorems

Original
Statements

Refinement
Attempts

Capparelli's
Companion

A breakthrough...

Finitizations

A Difference-Congruence identity usually gets accompanied with a Sum-Product identity.

Not in Capparelli's partition theorems case, or at least not in the 20th Century.



Sum-Product identity?

Back to Capparelli's Identity

Introduction
Capparelli's Partition Theorems

Original Statements

Refinement Attempts

Capparelli's Companion

A breakthrough...

Finitizations

A Difference-Congruence identity usually gets accompanied with a Sum-Product identity.

Not in Capparelli's partition theorems case, or at least not in the 20th Century.

Note that the product side of the Capparelli's identities are easy:

$$(-q^{3-m}, -q^{3+m}; q^6)_{\infty} (-q^3; q^3)_{\infty},$$

for $m = 1, 2$.



Refinements of Capparelli's Theorem

Introduction
Capparelli's Partition Theorems
Original Statements
Refinement Attempts
Capparelli's Companion
A breakthrough...
Finitziations

Let $G_{m,N}(a, b, q)$ be the generating function for the m -th Capparelli's partition theorem with the extra bound that all parts are $\leq N$, and the exponents of a and b count the number of parts $3 \mp m$ modulo 6.

Theorem (Alladi, Andrews, Gordon, 1995)

$$G_{1,3N+1}(a, b, q) = \sum_{l=0}^{\lfloor (N+1)/2 \rfloor} q^{3\binom{N-2l+1}{2}} \begin{bmatrix} N+1 \\ 2l \end{bmatrix}_{q^3} (-aq^2, -bq^4; q^6)_l,$$

where $\lfloor x \rfloor$ denotes the greatest integer $\leq x$.

K. Alladi, G. E. Andrews, and B. Gordon, *Refinements and Generalizations of Capparelli's Conjecture on Partitions*, Journal of Algebra **174** (1995), no. 2, 636–658.



Refinements of Capparelli's Theorem

Introduction

Capparelli's
Partition
Theorems

Original
Statements

Refinement
Attempts

Capparelli's
Companion
A breakthrough...

Finitizations

Let $G_{m,N}(a, b, q)$ be the generating function for the m -th Capparelli's partition theorem with the extra bound that all parts are $\leq N$, and the exponents of a and b count the number of parts $3 \mp m$ modulo 6.

Theorem (Alladi, Andrews, Gordon, 1995)

$$G_{1,3N+1}(a, b, q) = \sum_{l=0}^{\lfloor (N+1)/2 \rfloor} q^{3\binom{N-2l+1}{2}} \begin{bmatrix} N+1 \\ 2l \end{bmatrix}_{q^3} (-aq^2, -bq^4; q^6)_l,$$

where $\lfloor x \rfloor$ denotes the greatest integer $\leq x$.

K. Alladi, G. E. Andrews, and B. Gordon, *Refinements and Generalizations of Capparelli's Conjecture on Partitions*, Journal of Algebra **174** (1995), no. 2, 636–658.



Refinements of Capparelli's Theorem

Introduction
Capparelli's Partition Theorems
Original Statements
Refinement Attempts
Capparelli's Companion
A breakthrough...
Finitziations

Let $G_{m,N}(a, b, q)$ be the generating function for the m -th Capparelli's partition theorem with the extra bound that all parts are $\leq N$, and the exponents of a and b count the number of parts $3 \mp m$ modulo 6.

Theorem (Berkovich-U, 2015)

$$G_{2,3N+1}(a, b, q) = \sum_{l=0}^{\lfloor N/2 \rfloor} q^{3\binom{N-2l}{2}} \begin{bmatrix} N+1 \\ 2l+1 \end{bmatrix}_{q^3} (-aq^5; q^6)_l (-bq; q^6)_{l+1}.$$

A. Berkovich and A. K. Uncu, *A new companion to Capparelli's Identities*, Adv. in Appl. Math. 71 (2015), 125-137.



Refinements of Capparelli's Theorem

Introduction

Capparelli's
Partition
Theorems

Original
Statements

Refinement
Attempts

Capparelli's
Companion
A breakthrough...

Finitizations

Theorem (Berkovich-U, 2015)

$$G_{1,3N}(a, b, q) = S(a, b, q, N) + aq^{3N-1}S(a, b, q, N-1),$$

$$G_{2,3N}(a, b, q) = T(a, b, q, N) + bq^{3N-2}T(a, b, q, N-1),$$

where

$$S(a, b, q, N) := \sum_{l=0}^{\lfloor N/2 \rfloor} q^{3(\binom{N-2l}{2})} \begin{bmatrix} N+1 \\ 2l+1 \end{bmatrix}_{q^3} (-aq^2, -bq^4; q^6)_l,$$

$$T(a, b, q, N) := \sum_{l=0}^{\lfloor N/2 \rfloor} q^{3(\binom{N-2l}{2})} \begin{bmatrix} N+1 \\ 2l+1 \end{bmatrix}_{q^3} (-aq^5, -bq; q^6)_l.$$



How did we get here?

Introduction

Capparelli's
Partition
Theorems

Original
Statements
Refinement
Attempts

Capparelli's
Companion
A breakthrough...

Finitizations

Theorem (Berkovich-U, 2015)

Let n be any integer, the partitions $\pi = (\lambda_1, \lambda_2, \dots, \lambda_k)$ of n where

- $\lambda_{2i+1} \not\equiv m \pmod{3}$, $\lambda_{2i} \not\equiv 3 - m \pmod{3}$,
- $3k + 1$ and $3k + 2$ does not appear together in the partition,

=

the number of partitions of n into distinct parts where no part is congruent to $\pm m$ modulo 6.

A. Berkovich and A. K. Uncu, *A new companion to Capparelli's Identities*, Adv. in Appl. Math. 71 (2015), 125-137.



Sum-Product Identities Revealed

Capparelli's Partition Theorems

Introduction

Capparelli's
Partition
Theorems

Original
Statements
Refinement
Attempts

Capparelli's
Companion
A breakthrough...

Finitizations

Theorem (Kanade–Russell, 2019)

Let $Q(m, n) = 2m^2 + 6mn + 6n^2$,

$$\begin{aligned} \sum_{m,n \geq 0} \frac{q^{Q(m,n)}}{(q; q)_m (q^3; q^3)_n} &= (-q^2, -q^4; q^6)_\infty (-q^3; q^3)_\infty, \\ \sum_{m,n \geq 0} \frac{q^{Q(m,n)+m}}{(q; q)_m (q^3; q^3)_n} + \sum_{m,n \geq 0} \frac{q^{Q(m,n)+4m+6n+1}}{(q; q)_m (q^3; q^3)_n} \\ &= (-q, -q^5; q^6)_\infty (-q^3; q^3)_\infty. \end{aligned}$$

S. Kanade and M. Russell, *Staircases to analytic sum-sides for many new integer partition identities of Rogers–Ramanujan type*, Electron. J. Combin. 26 (2019), no. 1, Paper 1.6.



Sum-Product Identities Revealed

Capparelli's Partition Theorems

Introduction

Capparelli's
Partition
Theorems

Original
Statements

Refinement
Attempts

Capparelli's
Companion

A breakthrough...

Finitizations

Theorem (Kurşungöz , 2019)

Let $Q(m, n) = 2m^2 + 6mn + 6n^2$,

$$\begin{aligned} \sum_{m,n \geq 0} \frac{q^{Q(m,n)}}{(q;q)_m (q^3;q^3)_n} &= (-q^2, -q^4; q^6)_\infty (-q^3; q^3)_\infty, \\ \sum_{m,n \geq 0} \frac{q^{Q(m,n)+m+3n}}{(q;q)_m (q^3;q^3)_n} + \sum_{m,n \geq 0} \frac{q^{Q(m,n)+3m+6n+1}}{(q;q)_m (q^3;q^3)_n} \\ &= (-q, -q^5; q^6)_\infty (-q^3; q^3)_\infty. \end{aligned}$$

K. Kurşungöz, *Andrews–Gordon type series for Capparelli's and Göllnitz–Gordon identities*, J.Combin.Theory Ser.A 165 (2019), 117-138.



Sum-Product Identities Revealed

How did Kağan construct the analytic Sum side?

Introduction

Capparelli's
Partition
Theorems

Original
Statements

Refinement
Attempts

Capparelli's
Companion

A breakthrough...

Finitizations

A more complicated minimal configuration:

$$(\underbrace{2, 4, 8, 10, \dots,}_{3(2n-1)-1} \underbrace{3(2n-1)+1,}_{3(2n-1)+1+4,} 3(2n-1)+1+8, \dots, 3(2n-1)+1+4m).$$

This partition has n pairs with minimal Capparelli distance and m singletons.



Sum-Product Identities Revealed

How did Kağan construct the analytic Sum side?

Introduction

Capparelli's
Partition
Theorems

Original
Statements

Refinement
Attempts

Capparelli's
Companion

A breakthrough...

Finitizations

A more complicated minimal configuration:

$$(\underbrace{2, 4, 8, 10, \dots,}_{\text{n pairs}} \underbrace{3(2n-1)-1, 3(2n-1)+1,}_{\text{n pairs}} \\ 3(2n-1)+1+4, 3(2n-1)+1+8, \dots, 3(2n-1)+1+4m).$$

This partition has n pairs with minimal Capparelli distance and m singletons. The size of the partition is

$$Q(m, n) := 2m^2 + 6mn + 6n^2.$$



Sum-Product Identities Revealed

How did Kağan construct the analytic Sum side?

Introduction

Capparelli's
Partition
Theorems

Original
Statements

Refinement
Attempts

Capparelli's
Companion

A breakthrough...

Finitizations

A more complicated minimal configuration:

$$(\underbrace{2, 4, 8, 10, \dots,}_{\text{n pairs}} \underbrace{3(2n-1)-1, 3(2n-1)+1,}_{\text{n pairs}} \\ 3(2n-1)+1+4, 3(2n-1)+1+8, \dots, 3(2n-1)+1+4m).$$

This partition has n pairs with minimal Capparelli distance and m singletons. The size of the partition is

$$Q(m, n) := 2m^2 + 6mn + 6n^2.$$

The singletons can move freely as before and the related generating function is $1/(q; q)_m$.



Sum-Product Identities Revealed

How did Kağan construct the analytic Sum side?

Introduction

Capparelli's
Partition
Theorems

Original
Statements

Refinement
Attempts

Capparelli's
Companion

A breakthrough...

Finitizations

A more complicated minimal configuration:

$$(\underbrace{2, 4, 8, 10, \dots,}_{\text{n pairs}} \underbrace{3(2n-1)-1, 3(2n-1)+1,}_{\text{m singletons}} \\ 3(2n-1)+1+4, 3(2n-1)+1+8, \dots, 3(2n-1)+1+4m).$$

This partition has n pairs with minimal Capparelli distance and m singletons. The size of the partition is

$$Q(m, n) := 2m^2 + 6mn + 6n^2.$$

The singletons can move freely as before and the related generating function is $1/(q; q)_m$.

The motions of the pairs are more complicated.



Sum-Product Identities Revealed

How did Kağan construct the analytic Sum side?

After the motion of the singletons we have:

$$(\underbrace{2, 4, 8, 10, \dots}_{}, \underbrace{3(2n-1)-1, 3(2n-1)+1, \dots}_{}, \dots).$$

Introduction
Capparelli's Partition Theorems

Original Statements

Refinement Attempts

Capparelli's Companion

A breakthrough...

Finitizations



Sum-Product Identities Revealed

How did Kağan construct the analytic Sum side?

After the motion of the singletons we have:

$$(\underbrace{2, 4, 8, 10, \dots}_{}, \underbrace{3(2n-1)-1, 3(2n-1)+1, \dots}_{}).$$

A pair needs to move to the next available pair.

Introduction

Capparelli's Partition

Theorems

Original Statements

Refinement Attempts

Capparelli's Companion

A breakthrough...

Finitizations



Sum-Product Identities Revealed

How did Kağan construct the analytic Sum side?

After the motion of the singletons we have:

$$(\underbrace{2, 4, 8, 10, \dots}, \underbrace{3(2n-1)-1, 3(2n-1)+1, \dots}).$$

A pair needs to move to the next available pair. Two possibilities of the free motions:

$$\underbrace{3k-1, 3k+1}_{\text{3 pairs}} \mapsto \underbrace{3k, 3k+3}_{\text{2 pairs}},$$

$$\underbrace{3k, 3k+3}_{\text{2 pairs}} \mapsto \underbrace{3(k+1)-1, 3(k+1)+1}_{\text{3 pairs}}$$

Introduction
Capparelli's Partition Theorems
Original Statements
Refinement Attempts
Capparelli's Companion
A breakthrough...

Finitizations



Sum-Product Identities Revealed

How did Kağan construct the analytic Sum side?

Introduction
Capparelli's Partition Theorems
Original Statements
Refinement Attempts
Capparelli's Companion
A breakthrough...

Finitizations

After the motion of the singletons we have:

$$(\underbrace{2, 4, 8, 10, \dots}, \underbrace{3(2n-1)-1, 3(2n-1)+1, \dots}).$$

A pair needs to move to the next available pair. Two possibilities of the free motions:

$$\underbrace{3k-1, 3k+1}_{\text{3 pairs}} \mapsto \underbrace{3k, 3k+3}_{\text{2 pairs}},$$

$$\underbrace{3k, 3k+3}_{\text{2 pairs}} \mapsto \underbrace{3(k+1)-1, 3(k+1)+1}_{\text{3 pairs}}$$

Pairs might need to cross singletons:

$$\underbrace{3k-1, 3k+1}_{\text{3 pairs}}, \underbrace{3(k+2)-1}_{\text{1 singleton}} \mapsto \underbrace{3k-1}_{\text{1 singleton}}, \underbrace{3(k+1), 3(k+2)}_{\text{3 pairs}},$$

$$\underbrace{3k, 3(k+1)}_{\text{2 pairs}}, \underbrace{3(k+2)-1}_{\text{1 singleton}} \mapsto \underbrace{3k}_{\text{1 singleton}}, \underbrace{3(k+2)-1, 3(k+2)+1}_{\text{3 pairs}}$$

$$\underbrace{3k, 3(k+1)}_{\text{2 pairs}}, \underbrace{3(k+2)+1}_{\text{1 singleton}} \mapsto \underbrace{3k+1}_{\text{1 singleton}}, \underbrace{3(k+2)-1, 3(k+2)+1}_{\text{3 pairs}}$$



Sum-Product Identities Revealed

How did Kağan construct the analytic Sum side?

Introduction
Capparelli's Partition Theorems
Original Statements
Refinement Attempts
Capparelli's Companion
A breakthrough...

Finitizations

After the motion of the singletons we have:

$$(\underbrace{2, 4, 8, 10, \dots}, \underbrace{3(2n-1)-1, 3(2n-1)+1, \dots}).$$

A pair needs to move to the next available pair. Two possibilities of the free motions:

$$\underbrace{3k-1, 3k+1}_{\text{3 pairs}} \mapsto \underbrace{3k, 3k+3}_{\text{2 pairs}},$$

$$\underbrace{3k, 3k+3}_{\text{2 pairs}} \mapsto \underbrace{3(k+1)-1, 3(k+1)+1}_{\text{3 pairs}}$$

Pairs might need to cross singletons:

$$\underbrace{3k-1, 3k+1}_{\text{2 pairs}}, \underbrace{3(k+2)-1}_{\text{1 singleton}} \mapsto \underbrace{3k-1}_{\text{1 singleton}}, \underbrace{3(k+1), 3(k+2)}_{\text{3 pairs}},$$

$$\underbrace{3k, 3(k+1)}_{\text{2 pairs}}, \underbrace{3(k+2)-1}_{\text{1 singleton}} \mapsto \underbrace{3k}_{\text{1 singleton}}, \underbrace{3(k+2)-1, 3(k+2)+1}_{\text{3 pairs}}$$

$$\underbrace{3k, 3(k+1)}_{\text{2 pairs}}, \underbrace{3(k+2)+1}_{\text{1 singleton}} \mapsto \underbrace{3k+1}_{\text{1 singleton}}, \underbrace{3(k+2)-1, 3(k+2)+1}_{\text{3 pairs}}$$

Every motion adds 3 to the size and the related generating function is $1/(q^3, q^3)_n$.



Sum-Product Identities Revealed

Capparelli's Partition Theorems

Introduction

Capparelli's
Partition
Theorems

Original
Statements

Refinement
Attempts

Capparelli's
Companion

A breakthrough...

Finitizations

Theorem (Kurşungöz , 2019)

Let $Q(m, n) = 2m^2 + 6mn + 6n^2$,

$$\begin{aligned} \sum_{m,n \geq 0} \frac{q^{Q(m,n)}}{(q; q)_m (q^3; q^3)_n} &= (-q^2, -q^4; q^6)_\infty (-q^3; q^3)_\infty, \\ \sum_{m,n \geq 0} \frac{q^{Q(m,n)+m+3n}}{(q; q)_m (q^3; q^3)_n} + \sum_{m,n \geq 0} \frac{q^{Q(m,n)+3m+6n+1}}{(q; q)_m (q^3; q^3)_n} \\ &= (-q, -q^5; q^6)_\infty (-q^3; q^3)_\infty. \end{aligned}$$



Finite versions

First Group of Polynomial Identities that imply Capparelli's Theorems

Introduction
Capparelli's
Partition
Theorems

Finitizations

First Finite
Version

Second Finite
Version

A Curious Result

Two Infinite
Families

Third Finite
Version

More Infinite
Families

Now that we know the motions, we can put a bound on the largest part N and make sure that we do not pass this point N .



Finite versions

First Group of Polynomial Identities that imply Capparelli's Theorems

Introduction
Capparelli's
Partition
Theorems

Finitizations

First Finite
Version

Second Finite
Version

A Curious Result

Two Infinite
Families

Third Finite
Version

More Infinite
Families

Now that we know the motions, we can put a bound on the largest part N and make sure that we do not pass this point N .

This will make the q -Pochhammers to be exchanged with some q -binomial coefficients.



Finite versions

First Group of Polynomial Identities that imply Capparelli's Theorems

Introduction
Capparelli's
Partition
Theorems

Finitizations

First Finite
Version

Second Finite
Version

A Curious Result

Two Infinite
Families

Third Finite
Version

More Infinite
Families

Now that we know the motions, we can put a bound on the largest part N and make sure that we do not pass this point N .

This will make the q -Pochhammers to be exchanged with some q -binomial coefficients.

Recall that we know the finite version of the product side from our earlier constructions.



Finite versions

First Group of Polynomial Identities that imply Capparelli's Theorems

Introduction

Capparelli's
Partition
Theorems

Finitizations

First Finite
Version

Second Finite
Version

A Curious Result

Two Infinite
Families

Third Finite
Version

More Infinite
Families

Theorem (Berkovich-U, 2019)

Let $Q(m, n) = 2m^2 + 6mn + 6n^2$,

$$\sum_{m,n \geq 0} q^{Q(m,n)} \begin{bmatrix} 3(N-2n-m) \\ m \end{bmatrix}_q \begin{bmatrix} 2(N-2n-m)+n \\ n \end{bmatrix}_{q^3} = \sum_{l=0}^N q^{3\binom{N-2l}{2}} \begin{bmatrix} N \\ 2l \end{bmatrix}_{q^3} (-q^2, -q^4; q^6)_l,$$
$$\sum_{m,n \geq 0} q^{Q(m,n)+m+3n} \begin{bmatrix} 3(N-2n-m)+2 \\ m \end{bmatrix}_q \begin{bmatrix} 2(N-2n-m)+n+1 \\ n \end{bmatrix}_{q^3}$$
$$+ \sum_{m,n \geq 0} q^{Q(m,n)+3m+6n+1} \begin{bmatrix} 3(N-2n-m) \\ m \end{bmatrix}_q \begin{bmatrix} 2(N-2n-m)+n \\ n \end{bmatrix}_{q^3}$$
$$= \sum_{l=0}^N q^{3\binom{N-2l}{2}} \begin{bmatrix} N+1 \\ 2l+1 \end{bmatrix}_{q^3} (-q; q^6)_{l+1} (-q^5; q^6)_l.$$

A. Berkovich and A. K. Uncu, *Polynomial identities implying Capparelli's partition theorems*, J. Number Theory 201 (2019), 77-107.



Finite Versions

Second Group of Polynomial Identities that imply Capparelli's Theorems

Introduction

Capparelli's
Partition
Theorems

Finitizations

First Finite
Version

Second Finite
Version

A Curious Result

Two Infinite
Families

Third Finite
Version

More Infinite
Families

Theorem (Berkovich-U, 2019)

Let $Q(m, n) = 2m^2 + 6mn + 6n^2$,

$$\begin{aligned} \sum_{m,n \geq 0} \frac{q^{Q(m,n)}(q^3; q^3)_M}{(q; q)_m(q^3; q^3)_n(q^3; q^3)_{M-2n-m}} &= \sum_{j=-M}^M q^{3j^2+j} \left[\begin{matrix} 2M \\ M+j \end{matrix} \right]_{q^3}, \\ \sum_{m,n \geq 0} \frac{q^{Q(m,n)+m+3n}(q^3; q^3)_M}{(q; q)_m(q^3; q^3)_n(q^3; q^3)_{M-2n-m}} + \sum_{m,n \geq 0} \frac{q^{Q(m,n)+3m+6n+1}(q^3; q^3)_M}{(q; q)_m(q^3; q^3)_n(q^3; q^3)_{M-2n-m}} \\ &= \sum_{j=-M-1}^M q^{3j^2+2j} \left[\begin{matrix} 2M+1 \\ M-j \end{matrix} \right]_{q^3}. \end{aligned}$$

A. Berkovich and A. K. Uncu, *Elementary polynomial identities involving q -trinomial coefficients*, Ann. Comb. 23 (2019), no. 3-4, 549-560.



What lies underneath

q -Trinomial Coefficients

Introduction

Capparelli's
Partition
Theorems

Finitizations

First Finite
Version

Second Finite
Version

A Curious Result

Two Infinite
Families

Third Finite
Version

More Infinite
Families

$$\left(\begin{array}{c} L, b \\ a \end{array}; q \right)_2 := \sum_{n \geq 0} q^{n(n+b)} \frac{(q; q)_L}{(q; q)_n (q; q)_{n+a} (q; q)_{L-2n-a}},$$

$$T_n \left(\begin{array}{c} L \\ a \end{array}; q \right) := q^{(L(L-n)-a(a-n))/2} \left(\begin{array}{c} L, a-n \\ a \end{array}; \frac{1}{q} \right)_2.$$

We have

$$\sum_{a=-L}^L \left(\begin{array}{c} L, b \\ a \end{array}; 1 \right)_2 t^a = \left(t + 1 + \frac{1}{t} \right)^L$$



What lies underneath

q -Trinomial Coefficients

Introduction
Capparelli's Partition Theorems

Finitizations
First Finite Version

Second Finite Version

A Curious Result

Two Infinite Families

Third Finite Version

More Infinite Families

$$\sum_{a=-L}^L \binom{L, b}{a}_2 t^a = \left(t + 1 + \frac{1}{t} \right)^L$$

the trinomial coefficients are the coefficients on the generalized Pascal's Triangle:

					1						
					1	1	1				
					1	2	3	2	1		
					1	3	6	7	6	3	1
					1	4	10	16	19	16	10
				



What lies underneath

q -Trinomial Coefficients

Introduction
Capparelli's Partition Theorems
Finitizations
First Finite Version
Second Finite Version
A Curious Result
Two Infinite Families
Third Finite Version
More Infinite Families

Theorem (Berkovich-U, 2019)

$$\begin{aligned} \sum_{n \geq 0} q^{\binom{L-2n}{2}} \frac{(q^3; q^3)_L}{(q; q)_{L-2n} (q^3; q^3)_n} + q^{L+1} \sum_{n \geq 0} q^{\binom{L-2n+1}{2}+n} \frac{(q^3; q^3)_L}{(q; q)_{L-2n} (q^3; q^3)_n} \\ = \sum_{j=-L}^L q^{\frac{3j^2+i}{2}} \left\{ T_{-1} \left(\begin{array}{c} L \\ j \end{array}; q^3 \right) + T_{-1} \left(\begin{array}{c} L \\ j+1 \end{array}; q^3 \right) \right\}, \\ \sum_{n \geq 0} q^{\binom{L-2n}{2}} \frac{(q^3; q^3)_L}{(q; q)_{L-2n} (q^3; q^3)_n} = \sum_{j=-L}^L q^{\frac{3j^2-i}{2}} T_1 \left(\begin{array}{c} L \\ j \end{array}; q^3 \right), \\ \sum_{n \geq 0} q^{\frac{(L-2n)^2}{2}} \frac{(q^3; q^3)_L}{(q; q)_{L-2n} (q^3; q^3)_n} = \sum_{j=-L}^L q^{\frac{3j^2+2j}{2}} T_0 \left(\begin{array}{c} L \\ j \end{array}; q^3 \right). \end{aligned}$$



Finite Versions

Introduction

Capparelli's
Partition
Theorems

Finitizations

First Finite
Version

Second Finite
Version

A Curious Result

Two Infinite
Families

Third Finite
Version

More Infinite
Families

Theorem (Warnaar, 1999)

$$\sum_{i \geq 0} q^{\frac{i^2}{2}} \left[\begin{matrix} L \\ i \end{matrix} \right]_q T_0 \left(\begin{matrix} i \\ a \end{matrix} ; q \right) = q^{\frac{a^2}{2}} \left[\begin{matrix} 2L \\ L-a \end{matrix} \right]_q,$$

$$\sum_{i \geq 0} q^{\binom{i}{2}} (1 + q^L) \left[\begin{matrix} L \\ i \end{matrix} \right]_q T_1 \left(\begin{matrix} i \\ a \end{matrix} ; q \right) = (1 + q^a) q^{\binom{a}{2}} \left[\begin{matrix} 2L \\ L-a \end{matrix} \right]_q.$$

Theorem (Berkovich-U, 2019)

$$\sum_{i \geq 0} q^{\binom{i+1}{2}} \left[\begin{matrix} L \\ i \end{matrix} \right]_q \left\{ T_{-1} \left(\begin{matrix} i \\ a \end{matrix} ; q \right) + T_{-1} \left(\begin{matrix} i \\ a+1 \end{matrix} ; q \right) \right\} = q^{\binom{a+1}{2}} \left[\begin{matrix} 2L+1 \\ L-a \end{matrix} \right]_q.$$



Sum of the Capparelli's Products

Introduction

Capparelli's
Partition
Theorems

Finitizations

First Finite
Version

Second Finite
Version

A Curious Result

Two Infinite
Families

Third Finite
Version

More Infinite
Families

Theorem (Berkovich-U, 2019)

$$\sum_{m,n \geq 0} \frac{q^{2m^2+6mn+6n^2-2m-3n}(q^3; q^3)_M}{(q; q)_m(q^3; q^3)_n(q^3; q^3)_{M-2n-m}} (1+q^{3M}) = \sum_{j=-M}^M q^{3j^2-2j} (1+q^{3j}) \begin{bmatrix} 2M \\ M-j \end{bmatrix}_{q^3}.$$

Theorem (Berkovich-U, 2019)

Let $Q(m, n) = 2m^2 + 6mn + 6n^2$,

$$\begin{aligned} & \sum_{m,n \geq 0} \frac{q^{Q(m,n)-2m-3n}}{(q; q)_m(q^3; q^3)_n} \\ &= (-q^2, -q^4; q^6)_\infty (-q^3; q^3)_\infty + (-q, -q^5; q^6)_\infty (-q^3; q^3)_\infty. \end{aligned}$$



An Infinite Hierarchy

The core of the hierarchies

Introduction
Capparelli's Partition Theorems
Finitizations
First Finite Version
Second Finite Version
A Curious Result
Two Infinite Families
Third Finite Version
More Infinite Families

Theorem (Berkovich-U, 2019)

For L and M being non-negative integers, we have

$$\sum_{\substack{m \geq 0, \\ L \equiv m \pmod{2}}} q^{m^2} \begin{bmatrix} 3M \\ m \end{bmatrix}_{q^2} \begin{bmatrix} 2M + \frac{L-m}{2} \\ 2M \end{bmatrix}_{q^6} = \sum_{j=-\infty}^{\infty} q^{3j^2+2j} \mathcal{T} \left(\begin{matrix} L, M \\ j, j \end{matrix}; q^6 \right),$$

where

$$\mathcal{T} \left(\begin{matrix} L, M \\ a, b \end{matrix}; q \right) := \sum_{\substack{n \geq 0, \\ L-a \equiv n \pmod{2}}} q^{\frac{n^2}{2}} \begin{bmatrix} M \\ n \end{bmatrix}_q \begin{bmatrix} M+b+\frac{L-a-n}{2} \\ M+b \end{bmatrix}_q \begin{bmatrix} M-b+\frac{L+a-n}{2} \\ M-b \end{bmatrix}_q$$



An Infinite Hierarchy

The core of the hierarchies

Introduction
Capparelli's Partition Theorems
Finitizations
First Finite Version
Second Finite Version
A Curious Result
Two Infinite Families
Third Finite Version
More Infinite Families

Theorem (Berkovich-U, 2019)

For L and M being non-negative integers, we have

$$\sum_{\substack{m \geq 0, \\ L \equiv m \pmod{2}}} q^{m^2} \begin{bmatrix} 3M \\ m \end{bmatrix}_{q^2} \begin{bmatrix} 2M + \frac{L-m}{2} \\ 2M \end{bmatrix}_{q^6} = \sum_{j=-\infty}^{\infty} q^{3j^2+2j} \mathcal{T} \left(\begin{matrix} L, M \\ j, j \end{matrix}; q^6 \right),$$

where

$$\mathcal{T} \left(\begin{matrix} L, M \\ a, b \end{matrix}; q \right) := \sum_{\substack{n \geq 0, \\ L-a \equiv n \pmod{2}}} q^{\frac{n^2}{2}} \begin{bmatrix} M \\ n \end{bmatrix}_q \begin{bmatrix} M+b+\frac{L-a-n}{2} \\ M+b \end{bmatrix}_q \begin{bmatrix} M-b+\frac{L+a-n}{2} \\ M-b \end{bmatrix}_q$$

$$\lim_{M \rightarrow \infty} \mathcal{T} \left(\begin{matrix} L, M \\ a, b \end{matrix}; q \right) = \frac{1}{(q; q)_L} T_0 \left(\begin{matrix} L \\ j \end{matrix}; q \right).$$



Trinomial style Bailey Lemma

Introduction

Capparelli's
Partition
Theorems

Finitizations

First Finite
Version

Second Finite
Version

A Curious Result

Two Infinite
Families

Third Finite
Version

More Infinite
Families

Theorem (Warnaar, 1999)

For $L, M, a, b \in \mathbb{Z}$ and $ab \geq 0$

$$\sum_{i=0}^M q^{\frac{i^2}{2}} \left[L + M - i \atop L \right]_q \mathcal{T} \left(L - i, i; a, b; q \right) = q^{\frac{b^2}{2}} \mathcal{T} \left(L, M; a+b, b; q \right).$$

For $L, M, a, b \in \mathbb{Z}$ with $ab \geq 0$, and $|a| \leq M$ if $|b| \leq M$ and $|a+b| \leq L$, then

$$\sum_{i=0}^M q^{\frac{i^2}{2}} \left[L + M - i \atop L \right]_q \mathcal{T} \left(i, L - i; b, a; q \right) = q^{\frac{b^2}{2}} \mathcal{S} \left(L, M; a+b, b; q \right),$$

where

$$\mathcal{S} \left(L, M; a, b; q \right) := \sum_{n>0} q^{n(n+a)} \left[M + L - a - 2n \atop M \right]_q \left[M - a + b \atop n \right]_q \left[M + a - b \atop n + a \right]_q.$$



Unified Finite Capparelli's First Identity

Introduction

Capparelli's Partition Theorems

Finitizations

First Finite Version

Second Finite Version

A Curious Result

Two Infinite Families

Third Finite Version

More Infinite Families

Theorem (Berkovich-U, 2019)

$$\sum_{\substack{i, m \geq 0, \\ i+m \equiv 0 \pmod{2}}} q^{\frac{m^2+3i^2}{2}} \left[\begin{matrix} L+M-i \\ L \end{matrix} \right]_{q^3} \left[\begin{matrix} 3(L-i) \\ m \end{matrix} \right]_q \left[\begin{matrix} 2(L-i) + \frac{i-m}{2} \\ 2(L-i) \end{matrix} \right]_{q^3} = \sum_{j=-\infty}^{\infty} q^{3j^2+j} S \left(\begin{matrix} L, M \\ 2j, j \end{matrix}; q^3 \right),$$



Unified Finite Capparelli's First Identity

Introduction

Capparelli's Partition Theorems

Finitizations

First Finite Version

Second Finite Version

A Curious Result

Two Infinite Families

Third Finite Version

More Infinite Families

Theorem (Berkovich-U, 2019)

$$\sum_{\substack{i, m \geq 0, \\ i+m \equiv 0 \pmod{2}}} q^{\frac{m^2+3i^2}{2}} \left[\begin{matrix} L+M-i \\ L \end{matrix} \right]_{q^3} \left[\begin{matrix} 3(L-i) \\ m \end{matrix} \right]_q \left[\begin{matrix} 2(L-i) + \frac{i-m}{2} \\ 2(L-i) \end{matrix} \right]_{q^3} = \sum_{j=-\infty}^{\infty} q^{3j^2+j} S \left(\begin{matrix} L, M \\ 2j, j \end{matrix}; q^3 \right),$$

$$\sum_{m, n \geq 0} q^{Q(m, n)} \left[\begin{matrix} 3(N-2n-m) \\ m \end{matrix} \right]_q \left[\begin{matrix} 2(N-2n-m)+n \\ n \end{matrix} \right]_{q^3} = \sum_{l=0}^N q^{3 \binom{N-2l}{2}} \left[\begin{matrix} N \\ 2l \end{matrix} \right]_{q^3} (-q^2, -q^4; q^6)_l,$$



Unified Finite Capparelli's First Identity

Introduction

Capparelli's Partition Theorems

Finitizations
First Finite Version

Second Finite Version
A Curious Result

Two Infinite Families
Third Finite Version

More Infinite Families

Theorem (Berkovich-U, 2019)

$$\sum_{\substack{i, m \geq 0, \\ i+m \equiv 0 \pmod{2}}} q^{\frac{m^2+3i^2}{2}} \left[\begin{matrix} L+M-i \\ L \end{matrix} \right]_{q^3} \left[\begin{matrix} 3(L-i) \\ m \end{matrix} \right]_q \left[\begin{matrix} 2(L-i) + \frac{i-m}{2} \\ 2(L-i) \end{matrix} \right]_{q^3} = \sum_{j=-\infty}^{\infty} q^{3j^2+j} S \left(\begin{matrix} L, M \\ 2j, j \end{matrix}; q^3 \right),$$

$$\sum_{m, n \geq 0} q^{Q(m, n)} \left[\begin{matrix} 3(N-2n-m) \\ m \end{matrix} \right]_q \left[\begin{matrix} 2(N-2n-m)+n \\ n \end{matrix} \right]_{q^3} = \sum_{l=0}^N q^{3 \binom{N-2l}{2}} \left[\begin{matrix} N \\ 2l \end{matrix} \right]_{q^3} (-q^2, -q^4; q^6)_l,$$

$$\sum_{m, n \geq 0} \frac{q^{Q(m, n)} (q^3; q^3)_M}{(q; q)_m (q^3; q^3)_n (q^3; q^3)_{M-2n-m}} = \sum_{j=-M}^M q^{3j^2+j} \left[\begin{matrix} 2M \\ M+j \end{matrix} \right]_{q^3}.$$



First Infinite Hierarchy

Polynomial Version

Introduction
Capparelli's Partition Theorems

Finitizations
First Finite Version
Second Finite Version

A Curious Result
Two Infinite Families
Third Finite Version

More Infinite Families

Theorem (Berkovich-U, 2019)

Let ν be a non-negative integer, and let $N_k = n_k + n_{k+1} + \cdots + n_\nu$, for $k = 1, 2, \dots, \nu$. Then,

$$\begin{aligned} & \sum_{\substack{m, n_1, n_2, \dots, n_\nu \geq 0, \\ L+m \equiv N_1+N_2+\dots+N_\nu \pmod{2}}} q^{m^2+3(N_1^2+N_2^2+\dots+N_\nu^2)} \left[\begin{matrix} L+M-N_1 \\ L \end{matrix} \right]_{q^6} \left[\begin{matrix} 3n_\nu \\ m \end{matrix} \right]_{q^2} \\ & \quad \times \left[\begin{matrix} 2n_\nu + \frac{L-m-N_1-N_2-\dots-N_\nu}{2} \\ 2n_\nu \end{matrix} \right]_{q^6} \prod_{j=1}^{\nu-1} \left[\begin{matrix} L - \sum_{l=1}^j N_l + n_j \\ n_j \end{matrix} \right]_{q^6} \\ & = \sum_{j=-\infty}^{\infty} q^{3(\nu+1)j^2+2j} \mathcal{T} \left(\begin{matrix} L, M \\ (\nu+1)j, j \end{matrix} \right)_{q^6}. \end{aligned}$$

A. Berkovich and A. K. Uncu, Refined q-Trinomial Coefficients and Two Infinite Hierarchies of q-Series Identities, Algorithmic Combinatorics, Springer Cham.



First Infinite Hierarchy

Introduction

Capparelli's
Partition
Theorems

Finitizations

First Finite
Version

Second Finite
Version

A Curious Result

Two Infinite
Families

Third Finite
Version

More Infinite
Families

Theorem (Berkovich-U, 2019)

For non-negative integer ν , we have

$$\begin{aligned} & \sum_{n_1, n_2, \dots, n_\nu \geq 0} \frac{q^{3(N_1^2 + N_2^2 + \dots + N_\nu^2)} (-q; q^2)_{3n_\nu}}{(q^6; q^6)_{n_1} (q^6; q^6)_{n_2} \dots (q^6; q^6)_{n_{\nu-1}} (q^6; q^6)_{2n_\nu}} \\ &= \frac{(-q^3; q^3)_\infty}{(q^{12}; q^{12})_\infty} (q^{6(\nu+1)}, -q^{3\nu+1}, -q^{3\nu+5}; q^{6(\nu+1)})_\infty, \end{aligned}$$

where $N_i := n_i + n_{i+1} + \dots + n_\nu$ for $i = 1, 2, \dots, \nu$.



Second Infinite Hierarchy

Polynomial Version

Introduction

Capparelli's
Partition
Theorems

Finitizations

First Finite
Version

Second Finite
Version

A Curious Result

Two Infinite
Families

Third Finite
Version

More Infinite
Families

Theorem (Berkovich-U, 2019)

Let ν be a non-negative integer, and let $N_k = n_k + n_{k+1} + \cdots + n_\nu$, for $k = 1, 2, \dots, \nu$. Then,

$$\begin{aligned} & \sum_{\substack{i, m, n_1, n_2, \dots, n_\nu \geq 0, \\ i+m \equiv N_1 + N_2 + \cdots + N_\nu \pmod{2}}} q^{\frac{m^2 + 3(i^2 + N_1^2 + N_2^2 + \cdots + N_\nu^2)}{2}} \left[\begin{matrix} L + M - i \\ L \end{matrix} \right]_{q^3} \left[\begin{matrix} L - N_1 \\ i \end{matrix} \right]_{q^3} \left[\begin{matrix} 3n_\nu \\ m \end{matrix} \right]_q \\ & \quad \times \left[\begin{matrix} 2n_\nu + \frac{i-m-N_1-N_2-\cdots-N_\nu}{2} \\ 2n_\nu \end{matrix} \right]_{q^3} \prod_{j=1}^{\nu-1} \left[\begin{matrix} i - \sum_{l=1}^j N_l + n_j \\ n_j \end{matrix} \right]_{q^3} \\ & = \sum_{j=-\infty}^{\infty} q^{3(\frac{\nu+2}{2})j^2+j} \mathcal{S} \left(\begin{matrix} L, M \\ (\nu+2)j, (\nu+1)j \end{matrix}; q^3 \right). \end{aligned}$$



Second Infinite Hierarchy

Capparelli's first identity is included in this family

Theorem (Berkovich-U, 2019)

For non-negative integer ν , we have

$$\begin{aligned} & \sum_{\substack{i, m, n_1, n_2, \dots, n_\nu \geq 0, \\ i+m \equiv N_1 + N_2 + \dots + N_\nu \pmod{2}}} \frac{q^{\frac{m^2+3(i^2+N_1^2+N_2^2+\dots+N_\nu^2)}{2}}}{(q^3; q^3)_i} \left[\begin{matrix} 3n_\nu \\ m \end{matrix} \right]_q \\ & \times \left[\begin{matrix} 2n_\nu + \frac{i-N_1-N_2-\dots-N_\nu-m}{2} \\ 2n_\nu \end{matrix} \right]_{q^3} \prod_{j=1}^{\nu-1} \left[\begin{matrix} i - \sum_{k=1}^j N_k + n_j \\ n_j \end{matrix} \right]_{q^3} \\ & = \frac{(q^{6(\frac{\nu+2}{2})}, -q^{3(\frac{\nu+2}{2})+1}, -q^{3(\frac{\nu+2}{2})-1}; q^{6(\frac{\nu+2}{2})})_\infty}{(q^3; q^3)_\infty}. \end{aligned}$$

where $N_i := n_i + n_{i+1} + \dots + n_\nu$ for $i = 1, 2, \dots, \nu$.



A Third Set of Finite Identities that Imply Capparelli's theorems

Introduction

Capparelli's Partition Theorems

Finitizations

First Finite Version

Second Finite Version

A Curious Result

Two Infinite Families

Third Finite Version

More Infinite Families

Theorem (Berkovich-U, 2022)

Let $L \in \mathbb{Z}_{\geq 0}$, then

$$\begin{aligned} \sum_{m,n \geq 0} \frac{q^{2m^2+6mn+6n^2}(q;q)_L}{(q;q)_{L-3n-2m}(q;q)_m(q^3;q^3)_n} &= \sum_{j=-L}^L \left(\frac{j+1}{3}\right) q^{j^2} \begin{bmatrix} 2L \\ L-j \end{bmatrix}_q, \\ \sum_{m,n \geq 0} \frac{q^{2m^2+6mn+6n^2+m+3n}(q;q)_L}{(q;q)_{L-3n-2m}(q;q)_m(q^3;q^3)_n} + q \sum_{m,n \geq 0} \frac{q^{2m^2+6mn+6n^2+3m+6n}(q;q)_L}{(q;q)_{L-3n-2m-1}(q;q)_m(q^3;q^3)_n} \\ &= \sum_{j=-L}^L \left(\frac{j+1}{3}\right) q^{j(j+1)} \begin{bmatrix} 2L \\ L-j \end{bmatrix}_q, \end{aligned}$$

where $\left(\frac{\cdot}{\cdot}\right)$ is the Jacobi symbol.



What we saw so far

Introduction

Capparelli's
Partition
Theorems

Finitizations

First Finite
Version

Second Finite
Version

A Curious Result

Two Infinite
Families

Third Finite
Version

More Infinite
Families

Theorem

$$\sum_{m,n \geq 0} \frac{q^{2m^2+6mn+6n^2}(q^3;q^3)_M}{(q;q)_m(q^3;q^3)_n(q^3;q^3)_{M-2n-m}} = \sum_{j=-M}^M q^{3j^2+j} \begin{bmatrix} 2M \\ M-j \end{bmatrix}_{q^3},$$

$$\sum_{m,n \geq 0} \frac{q^{2m^2+6mn+6n^2+m+3n}(q^3;q^3)_M}{(q;q)_m(q^3;q^3)_n(q^3;q^3)_{M-2n-m}} + q \sum_{m,n \geq 0} \frac{q^{2m^2+6mn+6n^2+3m+6n}(q^3;q^3)_M}{(q;q)_m(q^3;q^3)_n(q^3;q^3)_{M-2n-m}}$$
$$= \sum_{j=-M-1}^M q^{3j^2+2j} \begin{bmatrix} 2M+1 \\ M-j \end{bmatrix}_{q^3},$$

$$\sum_{m,n \geq 0} \frac{q^{2m^2+6mn+6n^2-2m-3n}(q^3;q^3)_M}{(q;q)_m(q^3;q^3)_n(q^3;q^3)_{M-2n-m}} (1 + q^{3M}) = \sum_{j=-M}^M q^{3j^2-2j} (1 + q^{3j}) \begin{bmatrix} 2M \\ M-j \end{bmatrix}_{q^3}.$$



A Special case of Bailey Lemma

Introduction

Capparelli's
Partition
Theorems

Finitizations

First Finite
Version

Second Finite
Version

A Curious Result

Two Infinite
Families

Third Finite
Version

More Infinite
Families

Theorem

For $a = 0, 1$, if

$$F_a(L, q) = \sum_{j=-\infty}^{\infty} \alpha_j(q) \begin{bmatrix} 2L + a \\ L - j \end{bmatrix}_q$$

then

$$\sum_{r \geq 0} \frac{q^{r^2+ar}(q; q)_{2L+a}}{(q; q)_{L-r}(q; q)_{2r+a}} F_a(r, q) = \sum_{j=-\infty}^{\infty} \alpha_j(q) q^{j^2+aj} \begin{bmatrix} 2L + a \\ L - j \end{bmatrix}_q.$$



Infinite Hierarchy I

Polynomial Version

Introduction

Capparelli's
Partition
Theorems

Finitizations

First Finite
Version

Second Finite
Version

A Curious Result

Two Infinite
Families

Third Finite
Version

More Infinite
Families

Theorem (Berkovich-U, 2022)

Let $L \in \mathbb{Z}_{\geq 0}$, $f \in \mathbb{N}$ and $N_i := n_i + n_{i+1} + \dots + n_f$ with $i = 1, 2, \dots, f$, then

$$\begin{aligned} & \sum_{m, n, n_1, n_2, \dots, n_f \geq 0} \frac{q^{2m^2 + 6mn + 6n^2 + 3(N_1^2 + N_2^2 + \dots + N_f^2)} (q^3; q^3)_{2L} (q^3; q^3)_{n_f}}{(q; q)_m (q^3; q^3)_n (q^3; q^3)_{L-N_1} (q^3; q^3)_{n_f - 2n - m} (q^3; q^3)_{n_1} (q^3; q^3)_{n_2} \dots (q^3; q^3)_{n_f - 1} (q^3; q^3)_{2n_f}} \\ &= \sum_{j=-L}^L q^{3(f+1)j^2 + j} \begin{bmatrix} 2L \\ L-j \end{bmatrix}_{q^3}. \end{aligned}$$

A. Berkovich and A. K. Uncu, New infinite hierarchies of polynomial identities related to the Capparelli partition theorems, J. Math. Anal. Appl. 506 (2022), no. 2, Paper No. 125678, 18 pp.



Infinite Hierarchy I

Introduction

Capparelli's
Partition
Theorems

Finitizations

First Finite
Version

Second Finite
Version

A Curious Result

Two Infinite
Families

Third Finite
Version

More Infinite
Families

Theorem (Berkovich-U, 2022)

Let $f \in \mathbb{N}$ and $N_i := n_i + n_{i+1} + \cdots + n_f$ with $i = 1, 2, \dots, f$, then

$$\begin{aligned} & \sum_{m, n, n_1, n_2, \dots, n_f \geq 0} \frac{q^{2m^2+6mn+6n^2+3(N_1^2+N_2^2+\dots+N_f^2)}(q^3; q^3)_{n_f}}{(q; q)_m(q^3; q^3)_n(q^3; q^3)_{n_f-2n-m}(q^3; q^3)_{n_1}(q^3; q^3)_{n_2}\dots(q^3; q^3)_{n_f-1}(q^3; q^3)_{2n_f}} \\ &= \frac{(q^{6f+6}, -q^{3f+2}, -q^{3f+4}; q^{6f+6})_\infty}{(q^3; q^3)_\infty}. \end{aligned}$$



Infinite Hierarchy II

Polynomial Version

Introduction

Capparelli's
Partition
Theorems

Finitizations

First Finite
Version

Second Finite
Version

A Curious Result

Two Infinite
Families

Third Finite
Version

More Infinite
Families

Theorem (Berkovich-U, 2022)

Let $L \in \mathbb{Z}_{\geq 0}$, $f \in \mathbb{N}$ and $N_i := n_i + n_{i+1} + \dots + n_f$ with $i = 1, 2, \dots, f$, then

$$\begin{aligned} & \sum_{m, n, n_1, n_2, \dots, n_f \geq 0} \frac{q^{2m^2+6mn+6n^2+m+3n+3(N_1^2+N_2^2+\dots+N_f^2+N_1+N_2+\dots+N_f)}(1+q^{1+2m+3n})(q^3;q^3)_{2L+1}(q^3;q^3)_{n_f}}{(q;q)_m(q^3;q^3)_n(q^3;q^3)_{L-N_1}(q^3;q^3)_{n_f-2n-m}(q^3;q^3)_{n_1}(q^3;q^3)_{n_2}\dots(q^3;q^3)_{n_f-1}(q^3;q^3)_{2n_f+1}} \\ &= \sum_{j=-L-1}^{L+1} q^{3(f+1)j^2+(3f+2)j} \begin{bmatrix} 2L+1 \\ L-j \end{bmatrix}_{q^3}. \end{aligned}$$



Infinite Hierarchy II

Introduction

Capparelli's
Partition
Theorems

Finitizations

First Finite
Version

Second Finite
Version

A Curious Result

Two Infinite
Families

Third Finite
Version

More Infinite
Families

Theorem (Berkovich-U, 2022)

Let $f \in \mathbb{N}$ and $N_i := n_i + n_{i+1} + \cdots + n_f$ with $i = 1, 2, \dots, f$, then

$$\begin{aligned} & \sum_{m, n, n_1, n_2, \dots, n_f \geq 0} \frac{q^{2m^2+6mn+6n^2+m+3n+3(N_1^2+N_2^2+\cdots+N_f^2+N_1+N_2+\cdots+N_f)}(1+q^{1+2m+3n})(q^3; q^3)_{n_f}}{(q; q)_m(q^3; q^3)_n(q^3; q^3)_{n_f-2n-m}(q^3; q^3)_{n_1}(q^3; q^3)_{n_2} \dots (q^3; q^3)_{n_f-1}(q^3; q^3)_{2n_f+1}} \\ &= \frac{1}{(q^3; q^3)_{\infty}} \sum_{j=-\infty}^{\infty} q^{3(f+1)j^2+(3f+2)j} \\ &= \frac{(q^{6f+6}, -q, -q^{6f+5}; q^{6f+6})_{\infty}}{(q^3; q^3)_{\infty}}. \end{aligned}$$



Infinite Hierarchy III

Polynomial Version

Introduction

Capparelli's
Partition
Theorems

Finitizations

First Finite
Version

Second Finite
Version

A Curious Result

Two Infinite
Families

Third Finite
Version

More Infinite
Families

Theorem (Berkovich-U, 2022)

Let $L \in \mathbb{Z}_{\geq 0}$, $f \in \mathbb{N}$ and $N_i := n_i + n_{i+1} + \cdots + n_f$ with $i = 1, 2, \dots, f$, then

$$\begin{aligned} & \sum_{m, n, n_1, n_2, \dots, n_f \geq 0} \frac{q^{2m^2 + 6mn + 6n^2 - 2m - 3n + 3(N_1^2 + N_2^2 + \dots + N_f^2)} (q^3; q^3)_{2L} (q^3; q^3)_{n_f} (1 + q^{3n_f})}{(q; q)_m (q^3; q^3)_n (q^3; q^3)_{L - N_1} (q^3; q^3)_{n_f - 2n - m} (q^3; q^3)_{n_1} (q^3; q^3)_{n_2} \dots (q^3; q^3)_{n_f - 1} (q^3; q^3)_{2n_f}} \\ &= \sum_{j=-L-1}^{L+1} q^{3(f+1)j^2 - 2j} (1 + q^{3j}) \left[\begin{matrix} 2L \\ L-j \end{matrix} \right]_{q^3}. \end{aligned}$$



Infinite Hierarchy III

Introduction

Capparelli's
Partition
Theorems

Finitizations

First Finite
Version

Second Finite
Version

A Curious Result

Two Infinite
Families

Third Finite
Version

More Infinite
Families

Theorem (Berkovich-U, 2022)

Let $f \in \mathbb{N}$ and $N_i := n_i + n_{i+1} + \cdots + n_f$ with $i = 1, 2, \dots, f$, then

$$\begin{aligned} & \sum_{m, n, n_1, n_2, \dots, n_f \geq 0} \frac{q^{2m^2+6mn+6n^2-2m-3n+3(N_1^2+N_2^2+\cdots+N_f^2)}(q^3; q^3)_{n_f}(1+q^{3n_f})}{(q; q)_m(q^3; q^3)_n(q^3; q^3)_{n_f-2n-m}(q^3; q^3)_{n_1}(q^3; q^3)_{n_2} \cdots (q^3; q^3)_{n_f-1}(q^3; q^3)_{2n_f}} \\ &= \frac{(q^{6(f+1)}; q^{6(f+1)})_\infty}{(q^3; q^3)_\infty} \left((-q^{3f+1}, -q^{3f+5}; q^{6(f+1)})_\infty + (-q^{3f+2}, -q^{3f+4}; q^{6(f+1)})_\infty \right) \end{aligned}$$



Infinite Hierarchy IV

Polynomial Version

Introduction

Capparelli's
Partition
Theorems

Finitizations

First Finite
Version

Second Finite
Version

A Curious Result

Two Infinite
Families

Third Finite
Version

More Infinite
Families

Theorem (Berkovich-U, 2022)

Let $L \in \mathbb{Z}_{\geq 0}$, $f \in \mathbb{N}$ and $N_i := n_i + n_{i+1} + \cdots + n_f$ with $i = 1, 2, \dots, f$, then

$$\begin{aligned} & \sum_{m, n, n_1, n_2, \dots, n_f \geq 0} \frac{q^{2m^2 + 6mn + 6n^2 + N_1^2 + N_2^2 + \dots + N_f^2} (q; q)_{2L} (q; q)_{n_f}}{(q; q)_m (q^3; q^3)_n (q; q)_{L - N_1} (q; q)_{n_f - 3n - 2m} (q; q)_{n_1} (q; q)_{n_2} \dots (q; q)_{n_f - 1} (q; q)_{2n_f}} \\ &= \sum_{j=-L}^L \left(\frac{j+1}{3} \right) q^{(f+1)j^2} \begin{bmatrix} 2L \\ L-j \end{bmatrix}_q. \end{aligned}$$



Infinite Hierarchy IV

Introduction

Capparelli's
Partition
Theorems

Finitizations

First Finite
Version

Second Finite
Version

A Curious Result

Two Infinite
Families

Third Finite
Version

More Infinite
Families

Theorem (Berkovich-U, 2022)

Let $f \in \mathbb{N}$ and $N_i := n_i + n_{i+1} + \cdots + n_f$ with $i = 1, 2, \dots, f$, then

$$\begin{aligned} & \sum_{m, n, n_1, n_2, \dots, n_f \geq 0} \frac{q^{2m^2 + 6mn + 6n^2 + N_1^2 + N_2^2 + \cdots + N_f^2} (q; q)_{n_f}}{(q; q)_m (q^3; q^3)_n (q; q)_{n_f - 3n - 2m} (q; q)_{n_1} (q; q)_{n_2} \cdots (q; q)_{n_f - 1} (q; q)_{2n_f}} \\ &= \frac{1}{(q; q)_\infty} \sum_{j=-\infty}^{\infty} \left(\frac{j+1}{3} \right) q^{(f+1)j^2} \\ &= \frac{(q^{f+1}; q^{f+1})_\infty}{(q; q)_\infty} (-q^{3(f+1)}; q^{3(f+1)})_\infty (-q^{2(f+1)}, -q^{4(f+1)}; q^{6(f+1)})_\infty. \end{aligned}$$



Infinite Hierarchy V

Polynomial Version

Introduction

Capparelli's
Partition
Theorems

Finitizations

First Finite
Version

Second Finite
Version

A Curious Result

Two Infinite
Families

Third Finite
Version

More Infinite
Families

Theorem (Berkovich-U, 2022)

Let $L \in \mathbb{Z}_{\geq 0}$, $f \in \mathbb{N}$ and $N_i := n_i + n_{i+1} + \dots + n_f$ with $i = 1, 2, \dots, f$, then

$$\begin{aligned} & \sum_{m, n, n_1, n_2, \dots, n_f \geq 0} \frac{q^{2m^2 + 6mn + 6n^2 + m + 3n + N_1^2 + N_2^2 + \dots + N_f^2} (q; q)_{2L} (q; q)_{n_f}}{(q; q)_m (q^3; q^3)_n (q; q)_{L-N_1} (q; q)_{n_f-3n-2m} (q; q)_{n_1} (q; q)_{n_2} \dots (q; q)_{n_f-1} (q; q)_{2n_f}} \\ & + q \sum_{m, n, n_1, n_2, \dots, n_f \geq 0} \frac{q^{2m^2 + 6mn + 6n^2 + 3m + 6n + N_1^2 + N_2^2 + \dots + N_f^2} (q; q)_{2L} (q; q)_{n_f}}{(q; q)_m (q^3; q^3)_n (q; q)_{L-N_1} (q; q)_{n_f-3n-2m-1} (q; q)_{n_1} (q; q)_{n_2} \dots (q; q)_{n_f-1} (q; q)_{2n_f}} \\ & = \sum_{j=-L}^L \left(\frac{j+1}{3} \right) q^{(f+1)j^2 + j} \begin{bmatrix} 2L \\ L-j \end{bmatrix}_q. \end{aligned}$$



Infinite Hierarchy V

Introduction

Capparelli's
Partition
Theorems

Finitizations

First Finite
Version

Second Finite
Version

A Curious Result

Two Infinite
Families

Third Finite
Version

More Infinite
Families

Theorem (Berkovich-U, 2022)

Let $L \in \mathbb{Z}_{\geq 0}$, $f \in \mathbb{N}$ and $N_i := n_i + n_{i+1} + \dots + n_f$ with $i = 1, 2, \dots, f$, then

$$\begin{aligned} & \sum_{m, n, n_1, n_2, \dots, n_f \geq 0} \frac{q^{2m^2 + 6mn + 6n^2 + m + 3n + N_1^2 + N_2^2 + \dots + N_f^2} (q; q)_{n_f}}{(q; q)_m (q^3; q^3)_n (q; q)_{n_f - 3n - 2m} (q; q)_{n_1} (q; q)_{n_2} \dots (q; q)_{n_f - 1} (q; q)_{2n_f}} \\ & + q \sum_{m, n, n_1, n_2, \dots, n_f \geq 0} \frac{q^{2m^2 + 6mn + 6n^2 + 3m + 6n + N_1^2 + N_2^2 + \dots + N_f^2} (q; q)_{n_f}}{(q; q)_m (q^3; q^3)_n (q; q)_{n_f - 3n - 2m - 1} (q; q)_{n_1} (q; q)_{n_2} \dots (q; q)_{n_f - 1} (q; q)_{2n_f}} \\ & = \frac{(q^{f+2}, q^{5f+4}, q^{6f+6}; q^{6f+6})_\infty (q^{4f+2}, q^{8f+10}; q^{12f+12})_\infty}{(q; q)_\infty}. \end{aligned}$$



Fin

Introduction

**Capparelli's
Partition
Theorems**

Finitizations

First Finite
Version

Second Finite
Version

A Curious Result

Two Infinite
Families

Third Finite
Version

More Infinite
Families

Thank you for your time.

New Infinite Hierarchies Of Polynomial Identities Related To Capparelli's Identities

joint work with Alexander Berkovich

Ali Kemal Uncu (akuncu@ricam.oeaw.ac.at)



UNIVERSITY OF
BATH

ÖAW

AUSTRIAN
ACADEMY OF
SCIENCES

RICAM
JOHANN RADON INSTITUTE
FOR COMPUTATIONAL AND APPLIED MATHEMATICS

Michigan Technological University

Oct 5, 2023