On unimodality of Kronecker coefficients

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(joint work with Damir Yeliussizov)
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Partitions

- Let d be odd.
- $\lambda \vdash m$ is a partition of m, i.e. $\lambda_1 \geq \lambda_2 \geq \ldots$ and $\sum \lambda_i = m$
- $a \times b = (b, \ldots, b) \vdash ab$ is a rectangular partition.
- λ' is conjugate partition (transposed diagram).
- $\boldsymbol{\lambda} = (\lambda^{(1)}, \ldots, \lambda^{(d)})$ is a d-tuple of partitions with $\lambda^{(i)} \vdash m$. for function f we write $f(\boldsymbol{\lambda}) = (f(\lambda^{(1)}), \ldots, f(\lambda^{(d)}))$, for instance $\boldsymbol{\lambda'} = ((\lambda^{(1)})', \ldots, (\lambda^{(d)})')$.

Kronecker coefficients

- by $[\lambda]$ we denote irreducible S_m-representation indexed with λ, a.k.a. *Specht module*.
- Define *d-dimensional Kronecker coefficient* $g(\boldsymbol{\lambda})$ as

$$g(\boldsymbol{\lambda}) = \text{ mult. of } [1 \times m] \text{ in } [\lambda^{(1)}] \otimes \ldots \otimes [\lambda^{(d)}].$$

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For odd d it is equivalent to

$$g(\boldsymbol{\lambda}) = \text{ mult. of } [m \times 1] \text{ in } [(\lambda^{(1)})'] \otimes \ldots \otimes [(\lambda^{(d)})'].$$

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Conjectures

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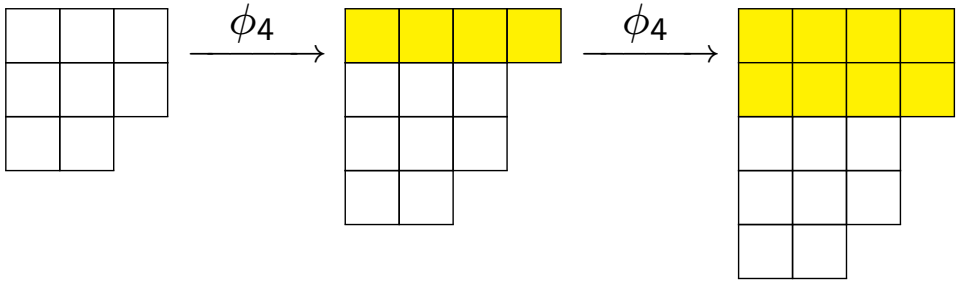
- $g_d(n,k) = g_d(k^{d-1} - n, k)$, i.e. sequence is symmetric,
- $g_d(0,k) = g(k^{d-1}, k) = 1$.

Generalized conjecture

Let us generalize this conjecture. Let $\phi_k \lambda$ be the partition with new part k inserted. For example:



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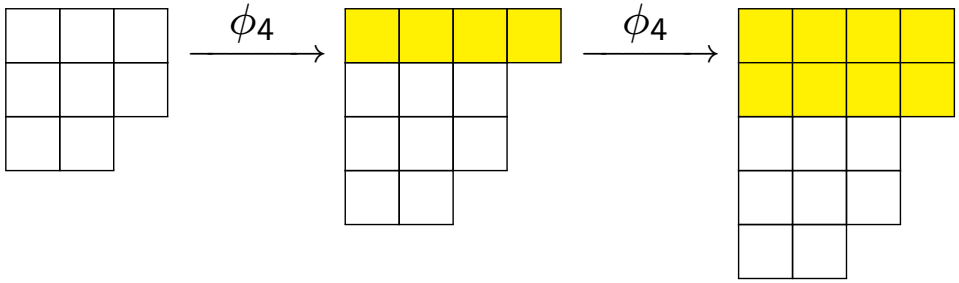
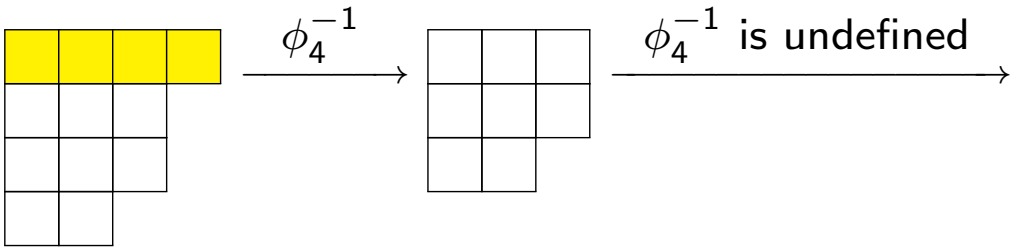


We write $\phi_k^n := \phi_k^{n-1}(\phi_k \lambda)$ and ϕ_k^{-1} is the removal of one part k:



Generalized conjecture

Conjecture

Let d be odd, k even and $\boldsymbol{\lambda}$ be d-tuple of partitions of m with parts $\leq k$. Denote $a = \max_i\{(\lambda^{(i)})'_1\}$ and $b = \min_i\{(\lambda^{(i)})'_k\}$. Then the sequence $\{g(\phi_k^n\boldsymbol{\lambda})\}_{n=-b,\ldots,k^{d-1}-a}$ is unimodal.

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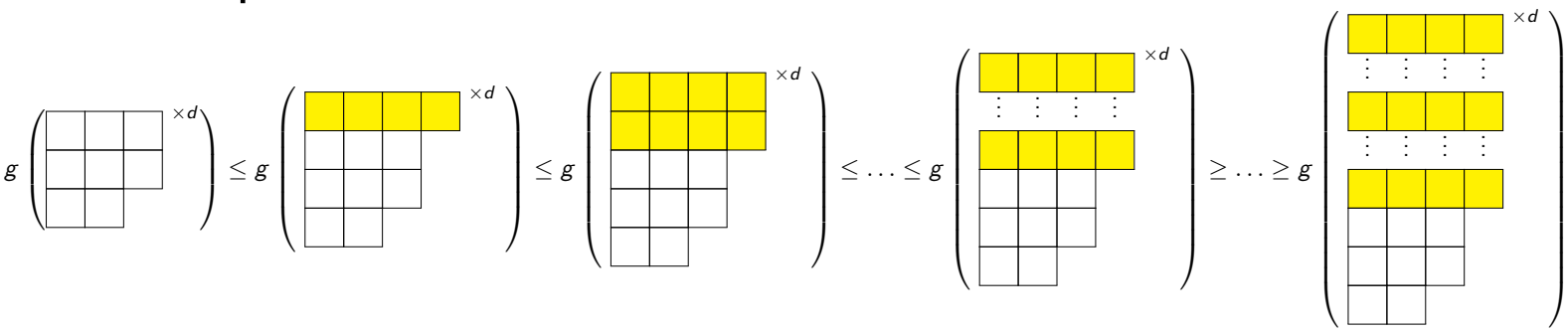

i.e. $g(\boldsymbol{\lambda}) \leq g(\phi_4\boldsymbol{\lambda}) \leq g(\phi_4^2\boldsymbol{\lambda}) \leq \ldots \leq g(\phi_4^{n_0}\boldsymbol{\lambda}) \geq \ldots \geq g(\phi_4^{4^{d-1}-3}\boldsymbol{\lambda})$.

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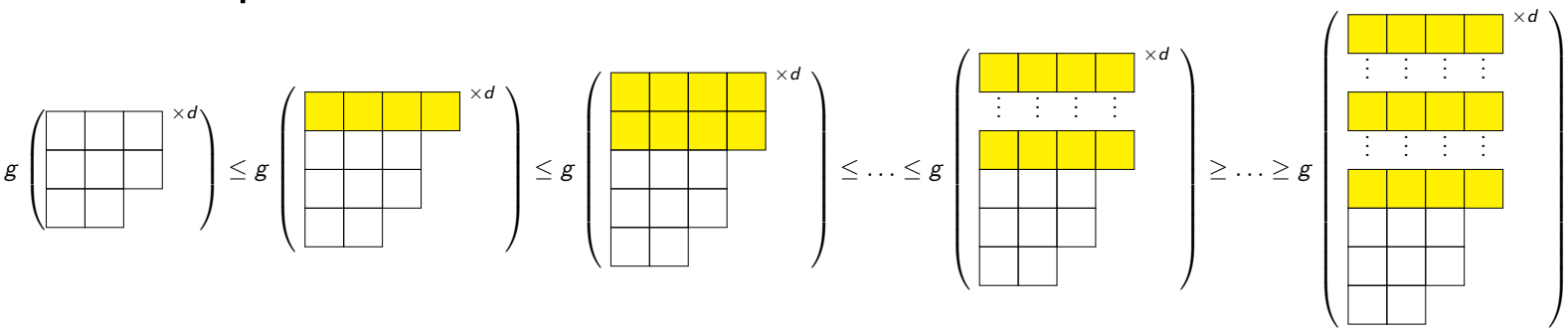

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Theorem (A.-Yeliussizov,23)

Conjecture is true for $k = 2$.

Note, that this Conjecture splits d-tuples of partitions in rectangle $k^{d-1} \times k$ into disjoint sequences.

Examples

Table: The table of $g_3(n,k) = g(n \times k, n \times k, n \times k)$ for $1 \leq k \leq 6$, $1 \leq n \leq 8$.

$k \backslash n$	1	2	3	4	5	6	7	8
1	1	0	0	0	0	0	0	0
2	1	1	1	1	0	0	0	0
3	1	0	1	1	1	1	0	1
4	1	1	2	5	6	13	14	18
5	1	0	1	4	21	158	1456	9854
6	1	1	3	16	216	9309	438744	17957625

Table: The table of $g_5(n,k)$ for $1 \leq k \leq 5$, $1 \leq n \leq 6$.

$k \backslash n$	1	2	3	4	5	6
1	1	0	0	0	0	0
2	1	5	11	35	52	112
3	1	1	385	44430	5942330	781763535
4	1	36	44522	381857353	5219755745322	87252488565829772
5	1	15	6008140	5220537438711	10916817688177999825	36929519748583464067841925

Examples

Few examples of sequences $\{g(\phi_k^n \boldsymbol{\lambda})\}$:

– *(symmetric and unimodal)* $\boldsymbol{\lambda} = (42, 222, 321)$, $k = 4$, for $n \in [0, 13]$:

$$1, 15, 128, 728, 2684, 6395, 9884, 9884, 6395, 2684, 728, 128, 15, 1$$

– *(only unimodal)* $\boldsymbol{\lambda} = (3, 21, 21)$, $k = 4$, for $n \in [0, 14]$:

$$1, 4, 18, 88, 342, 956, 1848, 2441, 2183, 1326, 552, 159, 34, 6, 1$$

– *(only unimodal)* $\boldsymbol{\lambda} = (32, 221, 41)$, $k = 4$ and $n \in [0, 13]$:

$$1, 8, 54, 281, 1027, 2531, 4179, 4584, 3331, 1613, 521, 114, 18, 2$$

– *(odd k)* $\boldsymbol{\lambda} = (32, 221, 311)$, $k = 3$ and for $n \in [0, 6]$:

$$1, 4, 7, 7, 5, 3, 1$$

Content

Tensor space

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- The group $\mathrm{GL}(k)^{\times d}$ (over $\mathbb{C}$) acts of $V = (\mathbb{C}^k)^{\otimes d}$ multi-linearly:

$$(G_1, \ldots, G_d) \cdot v_1 \otimes \ldots v_d = G_1 v_1 \otimes \ldots \otimes G_d v_d$$

where $G_i \in \mathrm{GL}(k)$ and $v_i \in \mathbb{C}^k$.

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  where $G_i \in \mathrm{GL}(k)$ and $v_i \in \mathbb{C}^k$.
- this induces the diagonal action on $\bigotimes^m (\mathbb{C}^k)^{\otimes d}$, i.e.

$$G \cdot T_1 \otimes \ldots \otimes T_m = GT_1 \otimes \ldots \otimes GT_m.$$

  for $G \in \mathrm{GL}(k)^{\times d}$.

Highest weight vectors

$T(k) \subset U(k) \subset \mathrm{GL}(k)$ be subgroups of diagonal and unitriangular matrices.

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– Vector v is *the weight vector* (w.r.t. G) of weight $\boldsymbol{\lambda}$ if for

$$T = (T^1, \ldots, T^d) \in T(n)^{\times d} : T \cdot v = (t^{(1)})^{\lambda^{(1)}} \cdots (t^{(d)})^{\lambda^{(d)}} v,$$

where $t^{(i)} = t_j^{(i)}$ is the j-th diagonal entry of $T^{(i)}$. Here $\lambda^{(i)}$ are not necessarily partitions.

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– Vector v is *the highest weight vector* (w.r.t. G) of weight $\boldsymbol{\lambda}$ if for

$$U \in U(k)^{\times d} : U \cdot v = v$$

the highest weight vectors are indexed with d-tuple of partitions $\boldsymbol{\lambda}$ of m.

Kronecker coefficients

Schur-Weyl duality states the decomposition with respect to the action of the group $\mathrm{GL}(n) \times S_m$:

$$\bigotimes^m \mathbb{C}^k = \bigoplus_{\lambda \vdash m, \ell(\lambda) \leq n} [\lambda] \otimes V(\lambda).$$

From this we obtain

$$g(\boldsymbol{\lambda}) = \dim \mathrm{HWV}_{\boldsymbol{\lambda}}^{\mathrm{GL}(n)^d} \mathrm{Sym}^m(\mathbb{C}^n)^{\otimes d}$$
$$g(\boldsymbol{\lambda}) = \dim \mathrm{HWV}_{\boldsymbol{\lambda}'}^{\mathrm{GL}(k)^d} \mathrm{Alt}^m(\mathbb{C}^k)^{\otimes d}$$

where $\mathrm{Sym}^m V \cong \mathbb{C}[V]_m$ and $\mathrm{Alt}^m V = \bigwedge^m V$ are symmetric and alternating subspaces of $\bigotimes^m V$.

Wedge space

$$g(\boldsymbol{\lambda}) = \dim \mathrm{HWV}_{\boldsymbol{\lambda}'} \bigwedge^{m} (\mathbb{C}^k)^{\otimes d}$$

For instance reactangular partitions:

$\mathrm{HWV}_{\emptyset^d} \bigwedge^0 V$	$\cdots$	$\mathrm{HWV}_{(k\times n)^d} \bigwedge^{nk} V$	$\cdots$	$\mathrm{HWV}_{(k\times k^{d-1})^d} \bigwedge^{k^d} V$
$\dim = g_d(0,k)$	$\cdots$	$\dim = g_d(n,k)$	$\cdots$	$\dim = g_d(k^{d-1},k)$

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$$\dim = g_d(0,k) \quad \cdots \quad \dim = g_d(n,k) \quad \cdots \quad \dim = g_d(k^{d-1},k)$$

or more generally for d-tuple $\boldsymbol{\lambda} \vdash m$

$$\mathrm{HWV}_{(\boldsymbol{\lambda})'} \bigwedge^{m} V \quad \cdots \quad \mathrm{HWV}_{(\phi_k^n \boldsymbol{\lambda})'} \bigwedge^{m+nk} V \quad \cdots \quad \mathrm{HWV}_{(\phi_k^{k^{d-1}-a} \boldsymbol{\lambda})'} \bigwedge^{m+k^d-ak} V$$

$$\dim = g(\boldsymbol{\lambda}) \quad \cdots \quad \dim = g(\phi_k^n \boldsymbol{\lambda}) \quad \cdots \quad \dim = g(\phi_k^{k^{d-1}-a} \boldsymbol{\lambda})$$

Cayley form

For $\boldsymbol{\lambda} = (1 \times k)^d$ (1-rows), there is a unique highest weight vector

$$\omega_{d,k} := \omega$$

of weight $\boldsymbol{\lambda}'$ (1-columns) defined as

$$\omega := \sum_{\pi_2,\ldots,\pi_d \in S_k} \operatorname{sgn}(\pi_2 \ldots \pi_d) \bigwedge_{i=1}^{k} e_i \otimes e_{\pi_2(i)} \otimes \ldots \otimes e_{\pi_d(i)},$$

$$\omega \in \mathrm{HWV}_{(k\times 1)^d} \bigwedge^{k} (\mathbb{C}^k)^{\otimes d}.$$

that we call *Cayley form*. It is also $\mathrm{SL}(k)^d$-invariant. It is essentially a Cayley's first hyperderminant.

Content

Lefschetz properties

Define the map

$$L : \bigwedge(\mathbb{C}^k)^{\otimes d} \to \bigwedge(\mathbb{C}^k)^{\otimes d}, \quad L : v \mapsto v \wedge \omega$$

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Define the following Lefschetz property.

$(\mathrm{LP}_{\boldsymbol{\lambda},k})$ *For some* $n_0 \in [-b, k^{d-1} - a]$ *the map*

$$L : \mathrm{HWV}_{(\phi_k^n \boldsymbol{\lambda})'} \bigwedge V \longrightarrow \mathrm{HWV}_{(\phi_k^{n+1} \boldsymbol{\lambda})'} \bigwedge V$$

is injective for each $n \in [-b, n_0)$ *and is surjective for each* $n \in [n_0, k^{d-1} - a)$.

Lefschetz operator action

For instance, for rectangular d-tuple of partitions:

$$\begin{array}{ccccc}
\mathrm{HWV}_{\emptyset^d} \bigwedge^0 V & \xrightarrow{L} \cdots \xrightarrow{L} & \mathrm{HWV}_{(k\times n)^d} \bigwedge^{nk} V & \xrightarrow{L} \cdots \xrightarrow{L} & \mathrm{HWV}_{(k\times k^{d-1})^d} \bigwedge^{k^d} V \\
\dim = g_d(0,k) & \cdots & \dim = g_d(n,k) & \cdots & \dim = g_d(k^{d-1},k)
\end{array}$$

or more generally for d-tuple $\boldsymbol{\lambda} \vdash m$

$$\begin{array}{ccccc}
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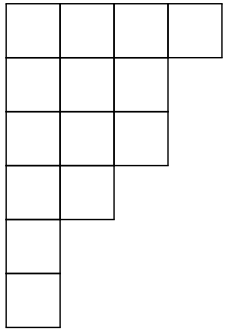
\dim = g(\boldsymbol{\lambda}) & \cdots & \dim = g(\phi_k^n\boldsymbol{\lambda}) & \cdots & \dim = g(\phi_k^{k^{d-1}-a}\boldsymbol{\lambda})
\end{array}$$

Rectangular complementarity

Let $\boldsymbol{\lambda} \subseteq k^{d-1} \times k$. We say $\boldsymbol{\lambda}$ is k-complementary if for some n each $\lambda \in \boldsymbol{\lambda}$ satisfy

$$\phi_k^n \lambda = (k^{d-1} - \lambda_{k^{d-1}}, \ldots, k^{d-1} - \lambda_1) = k^{d-1} \times k - \lambda.$$

For instance, $k = 4$, $d = 3$

For $\lambda =$  $\implies \phi_4^9 \lambda = (4^9, \lambda) = (4^{16}) - \lambda$

Lemma

For k-complementary $\boldsymbol{\lambda}$ we have $g(\boldsymbol{\lambda}) = g(k^{d-1} \times k - \boldsymbol{\lambda})$.

Hard Lefschetz property

For k-complementary $\boldsymbol{\lambda}$ we also define the following statement.

$(\mathrm{HLP}_{\boldsymbol{\lambda},k})$ *For each* $n \in [-b, (k^{d-1} - a + b)/2]$ *the map*

$$L^{k^{d-1}-a-2n} : \mathrm{HWV}_{(\phi_k^n \boldsymbol{\lambda})'} \bigwedge V \longrightarrow \mathrm{HWV}_{(\phi_k^{k^{d-1}-2m/k-2n} \boldsymbol{\lambda})'} \bigwedge V$$

is an isomorphism.

The underlying sequence of Kronecker coefficients is already symmetric.

More general LP

Why highest weight vectors? Let's simplify!

$(\mathrm{LP}_{d,k})$ *For each* $n \in [0, k^d - k]$ *the map*

$$L : \bigwedge^{n} (\mathbb{C}^k)^{\otimes d} \longrightarrow \bigwedge^{n+k} (\mathbb{C}^k)^{\otimes d}$$

has full rank.

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has full rank.

Turns out, $\mathrm{LP}_{d,k}$ is false for $k > 2$ for any d. But $\mathrm{LP}_{d,2}$ is true.

Content

Implication 1

Lefschetz properties has two important implications.

Proposition (A.-Yeliussizov,23)

Let $d \geq 3$ be odd and k be even. Then

$$\mathrm{LP}_{\boldsymbol{\lambda},k} \implies \text{the sequence } \{g(\phi_k^n \boldsymbol{\lambda})\}_{n\in[-b,k^{d-1}-a]} \text{ is unimodal.}$$

Implication 2

Definition

Latin hypercube L is the map $L : [k]^d \to [k^{d-1}]$ so that each slice contains a permutation of length k^{d-1} (slice = set of points with one fixed coordinate). For instance,

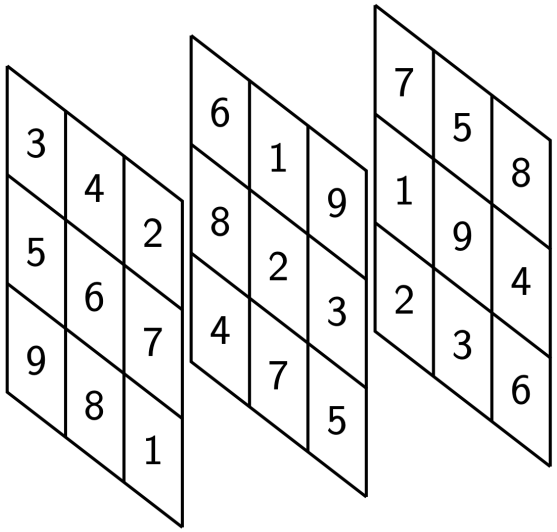

Figure: Latin cube for $d = 3$ and $k = 3$

The sign $\mathrm{sgn}(L)$ of a Latin hypercube L is the product of signs over all slice permutations.

For $d = 2$ we get Latin squares.

Implication 2

The d-dimensional Alon-Tarsi number is

$$\mathrm{AT}_d(k) = \sum_L \mathrm{sgn}(L)$$

where sum runs over all d-dimensional Latin hypercubes of length k. For $d = 2$, *Alon-Tarsi conjecture* states that $\mathrm{AT}_2(k) \neq 0$ for even k.

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Proof.

$L^{k^{d-1}}(1) = \omega^{k^{d-1}} = \pm \mathrm{AT}_d(k)\mathrm{vol} \neq 0.$ $\square$

Content

Proof of $\mathrm{LP}_{d,2}$: step 1

The proof is quite classical. The key is present ω in a suitable way.

Lemma

For $k = 2$ and odd d the Cayley vector $\omega = \omega_{d,2}$ and its dual ω^* can be written as follows:

$$\omega = \sum_{\mathbf{i} \in I^+} e_{\mathbf{i}} \wedge e_{\bar{\mathbf{i}}}, \qquad \omega^* = \sum_{\mathbf{i} \in I^+} e_{\mathbf{i}}^* \wedge e_{\bar{\mathbf{i}}}^*.$$

where $I^+ \subseteq [2]^d$ is the set of elements with odd sum of coordinates and for $\mathbf{i} \in [2]^d$ we write $e_{\mathbf{i}} = e_{i_1} \otimes \ldots \otimes e_{i_d}$.

Proof of $\mathrm{LP}_{d,2}$: step 2

Define operators raising, lowering and counting operators $X, Y, H : \bigwedge V \to \bigwedge V$ by

$$X : v \mapsto \omega \wedge v, \quad Y : v \mapsto \omega^* \wedge v,$$

and H that reduces to multiplication by scalar $(\ell - 2^{d-1})$ on $\bigwedge^\ell V$.

Lemma

Operators X, Y, H satisfy

$$[X, Y] = H, \quad [H, Y] = -2Y, \quad [H, X] = 2X. \tag{1}$$

In other words, $\bigwedge V$ is $\mathfrak{sl}(2)$ representation with triplet (X, Y, H).

Proof of $\mathrm{LP}_{d,2}$: step 3

Lemma

If $v \in \mathrm{HWV} \bigwedge V$ then also $X(v), Y(v), H(v) \in \mathrm{HWV} \bigwedge V$.

Hence $\mathrm{HWV} \bigwedge V$ is also $\mathfrak{sl}(2)$ representation.
For $\boldsymbol{\lambda} \subseteq (2^{d-1} \times 2)^{\times d}$ we pick

$$U_n := \mathrm{HWV}_{(\phi_2^n \boldsymbol{\lambda})'} \bigwedge (\mathbb{C}^2)^{\otimes d}, \quad U := \bigoplus_{n \in [0, 2^{d-1} - a]} U_n$$

with $\dim U_n = g(\phi_2^n \boldsymbol{\lambda})$.

Theorem

The space U is $\mathfrak{sl}(2)$ representation.

Proof of $\mathrm{LP}_{d,2}$: remark

For $k > 2$ the commutation relations no longer hold.

Thank you for your attention!