Andrews' partition ideals of order 1 and generalized Glaisher bijections

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PARTITION definition

- A *partition* λ of the integer n is a representation of n as an unordered sum of positive integers

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$$\lambda_1 + \lambda_2 + \cdots + \lambda_r = n.$$

- Each summand λ_i is called a *part* of the partition λ.
- Often a canonical ordering of parts is imposed:

$$\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_r.$$

The Partitions of 6

$$6 \qquad 5+1 \qquad 4+2 \qquad 4+1+1 \qquad 3+3 \qquad 3+2+1$$

$$3+1+1+1 \qquad 2+2+2 \qquad 2+2+1+1 \qquad 2+1+1+1+1$$

$$1+1+1+1+1+1$$

Euler's partition identity (1743)

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The number of partitions of n into odd parts equals the number of partitions of n into distinct parts.

Euler's partition identity—Example

- Of the eleven partitions of 6, four of them have only odd parts:

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- and four of them have distinct parts:

$$6 \qquad 5+1 \qquad 4+2 \qquad 3+2+1.$$

Frequency Notation for Partitions

Any partition $\lambda_1 + \lambda_2 + \lambda_3 + \cdots + \lambda_r$ may be written in the form

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$$f_1 \cdot 1 + f_2 \cdot 2 + f_3 \cdot 3 + f_4 \cdot 4 + \dots,$$

or more briefly, as

$$(f_1, f_2, f_3, f_4, \dots),$$

where f_i represents the number of appearances of the positive integer i in the partition.

Frequency Notation for Partitions

For example, the partition

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Frequency Notation for Partitions

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$$\begin{aligned}&6+6+6+6+4+4+3+2+2+2+2+1+1\\ &= 2\cdot 1+4\cdot 2+1\cdot 3+2\cdot 4+0\cdot 5+4\cdot 6+0\cdot 7+0\cdot 8+0\cdot 9+\dots\end{aligned}$$

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$$6+6+6+6+4+4+3+2+2+2+2+1+1$$
$$= 2\cdot 1 + 4\cdot 2 + 1\cdot 3 + 2\cdot 4 + 0\cdot 5 + 4\cdot 6 + 0\cdot 7 + 0\cdot 8 + 0\cdot 9 + \dots$$

may be represented by the frequency sequence

$$(2,4,1,2,0,4,0,0,0,0,0,0,\dots).$$

Thus each sequence $\{f_i\}_{i=1}^{\infty}$, where each f_i is a nonnegative integer and only finitely many of the f_i are nonzero, represents a partition of the integer $\sum_{i=1}^{\infty} i\ f_i$.

Glaisher's proof of Euler's identity

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- Replace each f_i with its binary expansion

$$\cdots + a_{i3} \cdot 8 + a_{i2} \cdot 4 + a_{i1} \cdot 2 + a_{i0} \cdot 1.$$

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So,

$$f_1 \cdot 1 + f_3 \cdot 3 + f_5 \cdot 5 + f_7 \cdot 7 + \cdots$$

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$$
\begin{aligned}
& f_1 \cdot 1 + f_3 \cdot 3 + f_5 \cdot 5 + f_7 \cdot 7 + \cdots \\
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& +(\cdots + a_{3,3} \cdot 8 + a_{3,2} \cdot 4 + a_{3,1} \cdot 2 + a_{3,0} \cdot 1) \cdot 3 \\
& +(\cdots + a_{5,3} \cdot 8 + a_{5,2} \cdot 4 + a_{5,1} \cdot 2 + a_{5,0} \cdot 1) \cdot 5 \\
& \vdots
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& \vdots \\
= \; & a_{1,0} + 2a_{1,1} + 3a_{3,0} + 4a_{1,2} + 5a_{5,0} + 6a_{3,1} + 7a_{7,0} + \ldots
\end{aligned}$$

Replace "binary expansion" with "base m expansion" and we obtain the theorem:

Glaisher's partition identity

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The number of partitions of n into nonmultiples of m equals the number of partitions of n where no part appears more than $m-1$ times.

A partition ideal C is a set of partitions such that for each $\lambda \in C$, if one or more parts is removed from λ, the resulting partition is also in C.

- Let $R_1(n)$ denote the number of partitions of n into parts congruent to $\pm 1 \pmod 5$.

The First Rogers-Ramanujan Identity

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- Let $R_2(n)$ denote the number of partitions $\lambda_1 + \cdots + \lambda_r$ of n such that $\lambda_i - \lambda_{i+1} \geq 2$.

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- Then $R_1(n) = R_2(n)$ for all n.

- The partitions enumerated by $R_1(n)$ are those for which $f_i = 0$ whenever $i \not\equiv \pm 1 \pmod{5}$.

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- The partitions enumerated by $R_2(n)$ are those for which $f_i + f_{i+1} \leq 1$. **(order 2)**

Minimal Bounding Sequence

Let the sequence $(d_1^C, d_2^C, d_3^C, \ldots)$ be defined by

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each d_i is a nonnegative integer or $+\infty$.

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- $$\{d_j^{\mathcal{O}}\}_{j=1}^{\infty} = (\infty, 0, \infty, 0, \infty, 0, \dots)$$
- Let $\mathcal{D}$ denote the set of all partitions with distinct parts.

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- Let $\mathcal{D}$ denote the set of all partitions with distinct parts.
- $$\{d_j^{\mathcal{D}}\}_{j=1}^{\infty} = (1, 1, 1, 1, 1, 1, \dots).$$

- Let $p(C, n)$ denote the number of partitions of an integer n in the partition ideal C.

Equivalent Partition Ideals

- Let $p(C, n)$ denote the number of partitions of an integer n in the partition ideal C.
- We say that two partition ideals C_1 and C_2 are equivalent, and write $C_1 \sim C_2$, if $p(C_1, n) = p(C_2, n)$ for all integers n.

Euler's Partition Identity

$$\mathcal{O} \sim \mathcal{D}.$$

The Multiset Associated with a Partition Ideal of Order 1

- Define the multiset associated with C, $M(C)$, as follows:

$$M(C) := \{ j(d_j^C + 1) \,\Big|\, j \in \mathbb{Z}_+ \text{ and } d_j^C < \infty \}.$$

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$$M(C) := \{ j(d_j^C + 1) \, \big| \, j \in \mathbb{Z}_+ \text{ and } d_j^C < \infty \}.$$

- Andrews proved $C_1 \sim C_2$ if and only if $M(C_1) = M(C_2)$.

The Multiset Associated with a Partition Ideal of Order 1

$$M(\mathcal{O}) = M(\mathcal{D}) = \{2, 4, 6, 8, 10, 12, \dots\}.$$

Example: Schur's theorem (easy part)

- Let S_1 denote the set of partitions into parts $\equiv \pm 1 \pmod 6$; $\{d^{S_1}\}_{j=1}^{\infty} = (\infty, 0, 0, 0, \infty, 0, \infty, 0, 0, 0, \infty, 0, \dots)$.

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- Let S_2 denote the set of partitions into distinct nonmultiples of 3; $\{d_j^{S_2}\}_{j=1}^{\infty} = (1, 1, 0, 1, 1, 0, \dots)$

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- Let S_3 denote the set of partitions into odd parts where no part appears more than twice; $\{d^{S_3}\}_{j=1}^{\infty} = (2, 0, 2, 0, 2, 0, \dots)$. (Alladi)

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Then $S_1 \sim S_2 \sim S_3$ because

$$M(S_1) = M(S_2) = M(S_3) = \{2, 4, 6, 8, 10, \dots\} \cup \{3, 9, 15, 21, \dots\}.$$

The "meaning" of $M(C)$

$$\sum_{n=0}^{\infty} p(C,n)q^n = \left(\prod_{\substack{j \in M(C) \\ \text{with mult.}}} (1-q^j) \right) \left(\prod_{k=1}^{\infty} \frac{1}{1-q^k} \right).$$

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$$1 + \sum_{n=1}^{\infty} p(C,n)q^n = \prod_{i=1}^{\infty} \frac{1}{(1-q^i)^{a_i}};$$

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Notice $a_i = 1 - m_i$, where $m_i = m_i(C)$ = the number of times i appears in $M(C)$.

Partitions of Infinity

MacMahon defined a *partition of infinity* to be a formal expression of the form

$$(g_1 - 1)\cdot 1 + (g_2 - 1)\cdot g_1 + (g_3 - 1)\cdot (g_1 g_2) + (g_4 - 1)\cdot (g_1 g_2 g_3) + \dots$$

where

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or for some fixed k,

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or for some fixed k,

- $g_1, g_2, g_3, \dots, g_{k-1} > 1$,
- $g_k = \infty$, and
- $g_{k+1} = g_{k+2} = g_{k+3} = \cdots = 1$.

Note that a partition of infinity may be thought of as a minimal bounding sequence for a partition ideal of order one with

$$\begin{aligned} d_1 &= g_1 - 1 \\ d_{g_1} &= g_2 - 1 \\ d_{g_1 g_2} &= g_3 - 1 \\ d_{g_1 g_2 g_3} &= g_4 - 1 \\ &\vdots \end{aligned}$$

and

$$d_i = 0 \text{ if } i \notin \{1, g_1, g_1 g_2, g_1 g_2 g_3, \dots\}.$$

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...in the sense that the base m expansions of positive integers are in in bijection with partitions into $1, m, m^2, m^3, \dots$ where no part appears more than $m-1$ times.

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- and any equivalent partition ideal of order 1 C',
- there exists a collection of partitions of infinity which gives rise to a “Glaisher-type bijection” from C to C'.
- Further, there is an explicit algorithm for finding the required partitions of infinity.

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  - and all other positive integers may appear without restriction.
- $C \sim C'$.

An Inelegant Partition Identity

- $$d_j^C = \begin{cases} 0 & \text{if } j \in \{2, 9, 10, 12, 18, 20\} \\ \infty & \text{otherwise.} \end{cases}$$

$$\{d_j^{C'}\}_{j=1}^{\infty} = (1, 4, 2, 2, \infty, \infty, \infty, \infty, 1, 1, \infty, \infty, \infty, \infty, \dots).$$

- $$d_j^C = \begin{cases} 0 & \text{if } j \in \{2,9,10,12,18,20\} \\ \infty & \text{otherwise.} \end{cases}$$

  $$\{d_j^{C'}\}_{j=1}^{\infty} = (1,4,2,2,\infty,\infty,\infty,\infty,1,1,\infty,\infty,\infty,\infty,\dots).$$

- Any partition of n in C can be written in the form

  $$n = f_1 \cdot 1 + \sum_{i=3}^{8} f_i \cdot i + f_{11} \cdot 11 + \sum_{i=13}^{17} f_i \cdot i$$
  $$+ f_{19} \cdot 19 + \sum_{i=21}^{\infty} f_i \cdot i,$$

  where each f_i is a nonnegative integer.

An Inelegant Partition Identity

- Expand f_1 by the partition of infinity defined by $g_{1,1} = 2$, $g_{1,2} = 5$, $g_{1,3} = 2$, $g_{1,4} = \infty$, $g_{1,k} = 1$ if $k > 4$.

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- Expand f_3 by the partition of infinity defined by $g_{3,1} = 3$, $g_{3,2} = 2$, $g_{3,3} = \infty$, $g_{3,k} = 1$ if $k > 3$.

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- Expand f_3 by the partition of infinity defined by $g_{3,1} = 3$, $g_{3,2} = 2$, $g_{3,3} = \infty$, $g_{3,k} = 1$ if $k > 3$.
- Expand f_4 by the partition of infinity defined by $g_{4,1} = 3$, $g_{4,2} = \infty$, $g_{4,k} = 1$ if $k > 2$.

$$
\begin{aligned}
n \quad = \quad & \big(a_{1,0}(1) + a_{1,1}(2) + a_{1,2}(2\cdot 5) + a_{1,3}(2\cdot 5\cdot 2)\big)1 \\
&+ \big(a_{2,0}(1) + a_{2,1}(3) + a_{2,2}(3\cdot 2)\big)3 \\
&+ \big(a_{4,0}(1) + a_{4,1}(3)\big)4 \\
&+ \big(a_{5,0}(1)\big)5 \\
&+ \big(a_{6,0}(1)\big)6 \\
&+ \big(a_{7,0}(1)\big)7 \\
&+ \big(a_{8,0}(1)\big)8 \\
&+ \big(a_{11,0}(1)\big)11 \\
&\vdots
\end{aligned}
$$

where $0 \le a_{j,k} \le g_{j,k+1} - 1 = d_{jg_{j,1}g_{j,2}\cdots g_{j,k}}$.

Apply the distributive property to obtain

$$
\begin{aligned}
n \quad = \quad & a_{1,0}(1) + a_{1,1}(2) + a_{1,2}(10) + a_{1,4}(20) \\
&+ a_{2,0}(3) + a_{2,1}(9) + a_{2,2}(18) \\
&+ a_{4,0}(4) + a_{4,1}(12) \\
&+ a_{5,0}(5) \\
&+ a_{6,0}(6) \\
&\vdots
\end{aligned}
$$

where, in particular,

$$
a_{1,0} \le 1, \quad a_{1,1} \le 4, \quad a_{1,2} \le 2, \quad a_{2,0} \le 2,
$$

$$
a_{2,1} \le 1, \quad a_{4,0} \le 2.
$$

Open problems

Given a multiset M, find all partition ideals C of order 1 that have M as their associated multiset.

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Partition ideals C of order greater than 1 also have an associated multiset $M(C)$, but they can't be found directly since they don't have a minimal bounding sequence $\{d_j\}$. Can we find another way to calculate $M(C)$?

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Possible Hint: Recall that for

$$1 + \sum_{n=1}^{\infty} p(C,n)q^n = \prod_{i=1}^{\infty} \frac{1}{(1-q^i)^{a_i}},$$

we have recursive and direct formulas to write $p(C,n)$ and a_i in terms of each other.

For partition ideals C of order 2 and higher, there is no minimal bounding sequence, but is there an analogous mathematical object that can be used to link it with the $M(C)$?

THANK YOU!