On partitions with bounded largest part and fixed integral GBG-rank modulo primes

Aritram Dhar

(Joint work with Alexander Berkovich)

(preprint available at https://arxiv.org/abs/2312.15117)

UF UNIVERSITY of FLORIDA

Online Seminar in Partition Theory, q-Series and Related Topics (MTU)

April 18, 2024

Thanks to...

Prof. Alexander Berkovich

Preliminary Background on partitions

A partition is a non-increasing finite sequence $\pi = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ of positive integers $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_k \ (\geq 1)$ where λ_i's are called the parts of π. The size of π, denoted by $|\pi|$, is defined as

$$|\pi| = \sum_{i=1}^{k} \lambda_i.$$

We say that π is a partition of n if $|\pi| = n$. For example, the seven partitions of 5 are $5, 4+1, 3+2, 3+1+1, 2+2+1, 2+1+1+1$, and $1+1+1+1+1$.

We may also write a partition π in terms of its frequency of parts as

$$\pi = (1^{f_1}, 2^{f_2}, 3^{f_3}, \ldots, k^{f_k}),$$

where $f_i (\geq 0)$ is the frequency of the part 'i'. For example, the seven partitions of 5 are $(5^1), (1^1, 4^1), (2^1, 3^1), (1^2, 3^1), (1^1, 2^2), (1^3, 2^1)$, and (1^5). Let $\mathcal{P}$ denote the set of all partitions.

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t-residue diagrams of partitions

The Young diagram of π is a way of representing π graphically wherein the parts of π are depicted as rows of cells. Given the Young diagram of π, we label a cell in the ith row and jth column by the least non-negative integer $\equiv j - i \pmod t$. The resulting diagram is called a t-residue diagram of π.

0	1	2	0	1	2	0	1	2	0
2	0	1	2	0	1	2			
1	2	0	1						
0	1	2							

Figure 1: 3-residue diagram of the partition $\pi = (10, 7, 4, 3)$

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r-vector and n-vector associated to a partition

r-vector

For every partition π and positive integer t, we define

$$\vec{r} = \vec{r}(\pi, t) = (r_0(\pi, t), r_1(\pi, t), \ldots, r_{t-1}(\pi, t))$$

where, for $0 \leq i \leq t-1$,

$$r_i(\pi, t) = r_i$$

is the number of cells labelled i in the t-residue diagram of π.

n-vector

For every partition π and positive integer t, we define

$$\vec{n} = \vec{n}(\pi, t) = (n_0, n_1, \ldots, n_{t-1}),$$

where, for $0 \leq i \leq t-2$,

$$n_i = r_i - r_{i+1}$$

and

$$n_{t-1} = r_{t-1} - r_0.$$

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Extended t-residue diagrams of partitions

We can also label cells in the infinite 0th column and the infinite 0th row in the same fashion and call the resulting diagram the extended t-residue diagram of π.

⋮											
⋯ 0	1	2	0	1	2	0	1	2	0	1	2 ⋯
2	0	1	2	0	1	2	0	1	2	0	
1	2	0	1	2	0	1	2				
0	1	2	0	1							
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1											
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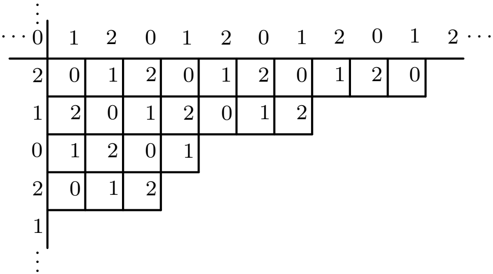

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Two partition statistics : GBG-rank & BG-rank

GBG-rank of a partition (Berkovich-Garvan, *Adv. in Appl. Math.*, 2008)

For any partition π, the GBG-rank of π mod t, denoted by $\mathrm{GBG}^{(t)}(\pi)$, is defined as

$$\mathrm{GBG}^{(t)}(\pi) := \sum_{j=0}^{t-1} r_j(\pi, t)\omega_t^j,$$

where $\omega_t := e^{\frac{2\pi\iota}{t}}$ is a tth root of unity and $\iota = \sqrt{-1}$.

For example, in Figure 1, for the partition $\pi = (10, 7, 4, 3)$, $\mathrm{GBG}^{(3)}(\pi) = r_0 + r_1\omega_3 + r_2\omega_3^2 = 8 + 8\omega_3 + 8\omega_3^2 = 0$.

BG-rank of a partition (Berkovich-Garvan, *Trans. Amer. Math. Soc.*, 2006)

The special case $t = 2$ is called the BG-rank of a partition π defined as

$$\mathrm{GBG}^{(2)}(\pi) = \mathrm{BG}(\pi) := r_0(\pi) - r_1(\pi) = i - j,$$

where i is the number of odd-indexed odd parts and j is the number of even-indexed odd parts.

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Notations and Conventions

For variables a, q and non-negative integers L, m, and n, we define the conventional q-Pochammer symbol as

$$(a)_L = (a;q)_L := \prod_{k=0}^{L-1} (1 - aq^k);$$
$$(a)_\infty = (a;q)_\infty := \lim_{L\to\infty} (a)_L \text{ for } |q| < 1.$$

For $m, n \in \mathbb{Z}$, we define the q-binomial (Gaussian) coefficient as

$$\begin{bmatrix} n \\ m \end{bmatrix}_q := \begin{cases} \dfrac{(q)_n}{(q)_m (q)_{n-m}} & \text{for } 0 \le m \le n, \\ 0 & \text{otherwise.} \end{cases}$$

For $m, n \ge 0$, $\dfrac{1}{(q)_m}$ is the generating function for partitions into at most m parts and $\begin{bmatrix} m+n \\ m \end{bmatrix}_q$ is the generating function for partitions into at most m parts each of size at most n, or vice versa.

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Motivation

For any non-negative integer N and any integer k, let $B_N(k,q)$ denote the generating function for the number of partitions into parts less than or equal to N with BG-rank equal to k. Then,

Theorem (Berkovich-Uncu, *J. Number Theory*, 2016)

For $\nu \in \{0,1\}$,

$$B_{2N+\nu}(k,q) = \frac{q^{2k^2-k}}{(q^2;q^2)_{N+\nu-k}(q^2;q^2)_{N+k}}.$$

For any non-negative integer N and any integer k, let $\tilde{B}_N(k,q)$ denote the generating function of the number of distinct part partitions into parts less than or equal to N with BG-rank equal to k. Then,

Theorem (Berkovich-Uncu, *J. Number Theory*, 2016)

For $\nu \in \{0,1\}$,

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Rim cell, Rim hook & t-cores

Rim cell

If some cell of π shares a vertex or edge with the rim of the Young diagram of π, we call this cell a rim cell of π.

Rim hook

A connected collection of rim cells of π is called a rim hook if (Young diagram of π)$\setminus$(rim hook) represents a legitimate partition.

Example

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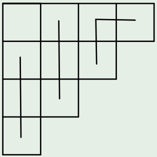

t-core

A partition, denoted by $\pi_{t\text{-core}}$, is called a t-core if its Young diagram has no rim hooks of length t.

Example

All staircase partitions (of triangular numbers) are 2-cores, for instance,

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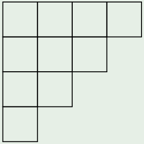

Regions

A region r in the extended t-residue diagram of π is the set of all cells (i, j) satisfying $t(r-1) \leq j - i < tr$. A cell of π is called exposed if it is at the end of a row in the extended t-residue diagram of π and not exposed otherwise.

Figure 3: Extended 3-residue diagram of $\pi = (10, 7, 4, 3)$ with regions

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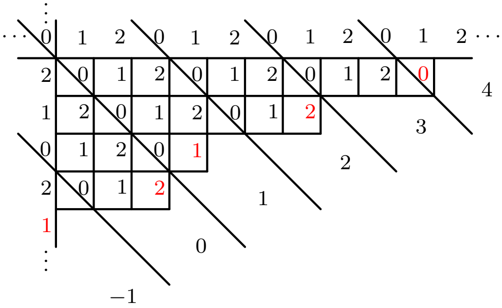

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Binary Words

One can construct t infinite binary words $W_0, W_1, \ldots, W_{t-1}$ of two letters N, E as follows: The rth letter of W_i is E if there is an exposed cell labelled i in the region r, otherwise the rth letter of W_i is N.

For example, the three bi-infinite words W_0, W_1, W_2 for the partition $(10, 7, 4, 3)$ in Figure 1 are as follows:

Region :	$\cdots\cdots$	-3	-2	-1	0	1	2	3	4	5	$\cdots\cdots$
W_0 :	$\cdots\cdots$	E	E	E	N	N	N	N	E	N	$\cdots\cdots$
W_1 :	$\cdots\cdots$	E	E	E	N	E	N	N	N	N	$\cdots\cdots$
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Littlewood decomposition: t-cores & t-quotients

Let $\mathcal{P}_{t\text{-core}}$ denote the set of all t-cores. There is a well-known bijection

$$\phi_1 : \mathcal{P} \longrightarrow \mathcal{P}_{t\text{-core}} \times \underbrace{\mathcal{P} \times \mathcal{P} \times \mathcal{P} \times \cdots \times \mathcal{P}}_{t \text{ times}}$$

due to D. E. Littlewood (1951)

$$\phi_1(\pi) = (\pi_{t\text{-core}}, (\hat{\pi}_0, \hat{\pi}_1, \ldots, \hat{\pi}_{t-1}))$$

such that

$$|\pi| = |\pi_{t\text{-core}}| + t \sum_{i=0}^{t-1} |\hat{\pi}_i|.$$

The vector partition $(\hat{\pi}_0, \hat{\pi}_1, \ldots, \hat{\pi}_{t-1})$ is called the t-quotient of π and is denoted by $\pi_{t\text{-quotient}}$.

Theorem (Garvan-Kim-Stanton, *Invent. Math.*, 1990)

$$|\pi_{t\text{-core}}| = \frac{t}{2}\vec{n} \cdot \vec{n} + \vec{b}_t \cdot \vec{n},$$

where $\vec{b}_t := (0, 1, 2, \ldots, t-1)$ and $\vec{n} = \vec{n}(\pi, t)$.

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Definition

For $0 \leq i \leq t-1$, define $\chi_i(\pi, t)$ to be the largest region in the extended t-residue diagram of π where the cell labeled i is exposed.

Lemma (Berkovich-Garvan, *Adv. in Appl. Math.*, 2008)

$$\chi_i(\pi, t) = v_i + n_i,$$

where v_i is the number of parts in the ith component of the t-quotient of π and n_i is the ith component of $\vec{n}(\pi, t)$.

For example, for the partition $\pi = (10, 7, 4, 3)$ in Figure 1, from its three bi-infinite words W_0, W_1, and W_2, we have $\chi_0(\pi, 3) = 4$, $\chi_1(\pi, 3) = 1$, and $\chi_2(\pi, 3) = 2$. Note that $\vec{n}(\pi, 3) = (0, 0, 0)$. Therefore, from Lemma, it follows that the 1st component of $\pi_{3\text{-quotient}}$ has 4 parts, the 2nd component of $\pi_{3\text{-quotient}}$ has 1 part, and the 3rd component of $\pi_{3\text{-quotient}}$ has 2 parts.

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Main Results - Part I

Let $G_{N,t}(k,q)$ denote the generating function for the number of partitions into parts less than or equal to N with GBG-rank mod t equal to k, i.e.,

$$G_{N,t}(k,q) := \sum_{\substack{\pi \in \mathcal{P} \\ \mathsf{GBG}^{(t)}(\pi)=k \\ l(\pi)\le N}} q^{|\pi|}.$$

Theorem 1 (Berkovich-D, 2024)

For any prime t, a non-negative integer N, and any integer k, we have

$$G_{tN+\nu,t}(k,q) = \frac{q^{tk^2-(t-1)k}}{\prod_{i=0}^{t-1} (\tilde{q};\tilde{q})_{N+\lceil \frac{\nu-i}{t} \rceil - k\delta_{i,0} + k\delta_{i,t-1}}},$$

where $\tilde{q} := q^t$ and $\nu \in \{0,1,2,\ldots,t-1\}$.

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$$G_{tN+\nu,t}(k,q) = \frac{q^{tk^2-(t-1)k}}{\prod_{i=0}^{t-1} (\tilde{q};\tilde{q})_{N+\lceil \frac{\nu-i}{t} \rceil - k\delta_{i,0} + k\delta_{i,t-1}}},$$

where $\tilde{q} := q^t$ and $\nu \in \{0, 1, 2, \ldots, t-1\}$.

Let $\tilde{G}_{N,t}(k,q)$ denote the generating function for the number of partitions into parts repeating no more than $t-1$ times and less than or equal to N with GBG-rank mod t equal to k, i.e.,

$$\tilde{G}_{N,t}(k,q) := \sum_{\substack{\pi=(1^{f_1},2^{f_2},\ldots,N^{f_N})\in\mathcal{P} \\ f_i\le t-1, 1\le i\le N \\ \mathsf{GBG}^{(t)}(\pi)=k}} q^{|\pi|}.$$

Theorem 2 (Berkovich-D, 2024)

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Proof of Theorem 1

Let $\pi = (\lambda_1, \lambda_2, \lambda_3, \ldots) \in \mathcal{P}$ be a partition with $\lambda_1 \leq tN + \nu$ where $0 \leq \nu \leq t-1$ and consider the Littlewood decomposition of π

$$\phi_1(\pi) = (\pi_{t\text{-core}}, \pi_{t\text{-quotient}})$$

where

$$\pi_{t\text{-quotient}} = (\hat{\pi}_0, \hat{\pi}_1, \ldots, \hat{\pi}_{t-1}).$$

Now, let t be any prime and $\mathrm{GBG}^{(t)}(\pi) = k$ be an integer. Then $\mathrm{GBG}^{(t)}(\pi) = k$ is an integer if and only if $r_0(\pi) = k + r$ and $r_i(\pi) = r$ for any $r \in \mathbb{Z}^+ \cup \{0\}$ and $1 \leq i \leq t-1$. Therefore,

$$\vec{n}(\pi, t) = (k, 0, 0, \ldots, 0, 0, -k),$$

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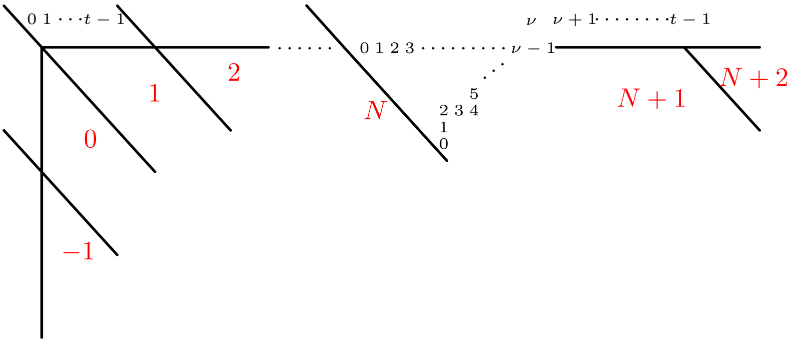

Figure 4: Generic extended t-residue diagram of a partition $\pi = (\lambda_1, \lambda_2, \lambda_3, \dots) \in \mathcal{P}$ with $\lambda_1 = tN + \nu$ where $0 \le \nu \le t-1$ (the regions are labeled in red).

Case I: $\nu > 0$

The cell labeled i may be exposed in the region marked $N+1$ for $0 \le i \le \nu - 1$ and is not exposed in the region marked $N+1$ for $\nu \le i \le t-1$. Thus, it follows that

$$N + 1 \geq \chi_i(\pi, t) \;, \quad 0 \leq i \leq \nu - 1,$$

and

$$N \geq \chi_i(\pi, t) \;, \quad \nu \leq i \leq t - 1.$$

Lemma then implies that

$$N + 1 \geq v_i + k\delta_{i,0} \;, \quad 0 \leq i \leq \nu - 1,$$

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□

Proof of Theorem 2

Let

$$\pi = (1^{tq_1+f_1}, 2^{tq_2+f_2}, 3^{tq_3+f_3} \ldots, N^{tq_N+f_N}),$$

where $0 \leq f_j \leq t-1$ and $q_j \geq 0$ for $1 \leq j \leq N$.

Observe that π is in one-one correspondence with the pair of partitions (π_1, π_2) where

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Now, note that

$$\sum_{\pi_1} q^{|\pi_1|} = \frac{1}{(q^t;q^t)_N},$$

and so,

$$\begin{aligned}\sum_{\pi} q^{|\pi|} &= \sum_{\pi_1} q^{|\pi_1|} \cdot \sum_{\pi_2} q^{|\pi_2|} \\ &= \frac{1}{(q^t;q^t)_N} \sum_{\pi_2} q^{|\pi_2|},\end{aligned}$$

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Hence,

$$\sum_{\pi_2} q^{|\pi_2|} = (q^t;q^t)_N \sum_{\pi} q^{|\pi|},$$

where π_2 is a partition whose parts repeat no more than $t-1$ times.

Also, observe that $\mathsf{GBG}^{(t)}(\pi) = \mathsf{GBG}^{(t)}(\pi_2) = k$ since removal of parts which repeat t times in succession keeps the GBG-rank mod t value of π invariant. Therefore, using the above observation and replacing N by $tN+\nu$ for $0 \le \nu \le t-1$ along with Theorem 1 gives us the desired generating function. □

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Self-conjugate partitions

Definition

The conjugate of a partition π, denoted by π', is associated to the Young diagram obtained by reflecting the diagram for π across the main diagonal. We say that π is *self-conjugate* if $\pi = \pi'$. Let $\mathcal{SCP}$ denote the set of all self-conjugate partitions.

Generating function

$(-q;q^2)_L$ is the generating function for partitions into district odd parts having largest part at most $2L-1$. Since self-conjugate partitions are in bijection with distinct odd part partitions, $(-q;q^2)_L$ is also the generating function for self-conjugate partitions with number of parts at most L.

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Littlewood decomposition on self-conjugate partitions

Under conjugation,

$$\vec{n}(\pi) = (n_0, n_1, \dots, n_{t-2}, n_{t-1})$$

becomes

$$\vec{\tilde{n}} = \vec{n}(\pi') = (-n_{t-1}, -n_{t-2}, \dots, -n_1, -n_0).$$

Let $\pi \in \mathcal{SCP}$. Then,

$$\phi_1(\pi) = (\pi_{t\text{-core}}, \pi_{t\text{-quotient}})$$

where

$$\pi_{t\text{-core}} = \pi'_{t\text{-core}}$$

and

$$\pi_{t\text{-quotient}} = (\hat{\pi}_0, \hat{\pi}_1, \hat{\pi}_2, \dots, \hat{\pi}_{t-1})$$

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Main Results - Part II

Let $GSC_{N,t}(k,q)$ denote the generating function for the number of self-conjugate partitions into parts less than or equal to N with GBG-rank mod t equal to k, i.e.,

$$GSC_{N,t}(k,q) := \sum_{\substack{\pi \in \mathcal{SCP} \\ \mathsf{GBG}^{(t)}(\pi) = k \\ l(\pi) \leq N}} q^{|\pi|}.$$

Theorem 3 (Berkovich-D, 2024)

For any non-negative integer N and any integer k, we have

$$GSC_{2N+\nu,2}(k,q) = q^{2k^2-k} \begin{bmatrix} 2N+\nu \\ N+k \end{bmatrix}_{q^4},$$

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where $\nu \in \{0,1,2,\ldots,t-1\}$.

Proof of Theorem 3

Let $\pi \in \mathcal{SCP}$ be a self-conjugate partition having $l(\pi) \leq 2N+1$ and consider the Littlewood decomposition of π

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$$N+1 \geq k+v_0,$$

which implies that

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$$q^{2(|\hat{\pi}_0|+|\hat{\pi}_1|)} = q^{4|\hat{\pi}_0|}.$$

Thus, we get the term

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in the required generating function. Analogously, one can prove the required generating function for $\nu = 0$. □

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$$\begin{bmatrix} 2N + \nu \\ N + k \end{bmatrix}_{q^4}$$

in the required generating function. Analogously, one can prove the required generating function for $\nu = 0$. □

Proof of Theorem 4

Here, again for any $\pi \in \mathcal{SCP}$, we have $\vec{n}(\pi) = (k, 0, 0, \ldots, 0, 0, -k)$ and $|\pi_{t\text{-core}}| = \frac{t}{2} \cdot 2k^2 + (t-1) \cdot (-k) = tk^2 - (t-1)k$. Consider the pair of partitions $(\hat{\pi}_0, \hat{\pi}_{t-1})$ from $\pi_{t\text{-quotient}}$ and observe that $\hat{\pi}_0 = \hat{\pi}'_{t-1}$. Therefore, this pair contributes to the generating function as

$$q^{t(|\hat{\pi}_0|+|\hat{\pi}_{t-1}|)} = q^{2t|\hat{\pi}_0|}.$$

Now, we have $\#(\hat{\pi}_0) \leq N - k + \lceil \frac{\nu}{t} \rceil$ and $\#(\hat{\pi}_{t-1}) \leq N + k$ which is the same as $l(\hat{\pi}_0) \leq N + k$. So, we get the term

$$\begin{bmatrix} 2N + \lceil \frac{\nu}{t} \rceil \\ N + k \end{bmatrix}_{q^{2t}}$$

in the required generating function. Next, for $1 \leq i \leq (t-3)/2$, we consider the pair of partitions $(\hat{\pi}_i, \hat{\pi}_{t-1-i})$ from $\pi_{t\text{-quotient}}$ and observe that $\hat{\pi}_i = \hat{\pi}'_{t-1-i}$. Therefore, this pair contributes to the generating function as

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in the required generating function. Finally, observe that $\hat{\pi}_{\frac{t-1}{2}} = \hat{\pi}'_{\frac{t-1}{2}}$ and similarly, we have $\#(\hat{\pi}_{\frac{t-1}{2}}) \leq N + \lceil \frac{\nu - \frac{t-1}{2}}{t} \rceil$. Thus, we get the term

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What about complex GBG-rank mod t values?

Theorem 5 (Berkovich-D, 2024)

For any odd prime t, a non-negative integer N, and any integer k, we have

$$G_{tN+\nu,t}(k\omega_t^j, q) = \frac{q^{tk^2+k}}{\prod_{i=0}^{t-1} (\tilde{q};\tilde{q})_{N+k\delta_{i,j-1}-k\delta_{i,j}+\lceil \frac{\nu-i}{t} \rceil}},$$

where $1 \le j \le t-1$, $\tilde{q} := q^t$, and $\nu \in \{0, 1, 2, \ldots, t-1\}$.

(Here $r_i(\pi) = r + k\delta_{i,j}$ which implies $n_i(\pi) = -k\delta_{i,j-1} + k\delta_{i,j}$ for any $r \in \mathbb{Z}^+ \cup \{0\}$, $0 \le i \le t-1$, and $1 \le j \le t-1$.)

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Possible future work

- It would be interesting to investigate whether results analogous to Theorem 1, Theorem 4, and Theorem 5 exist for t being a composite integer ≥ 4.
- Dhar and Mukhopadhyay (2023) provided a direct combinatorial proof of Theorem 2 for $t = 2$ (Berkovich and Uncu's 2016 identity for distinct part partitions) based on Fu & Tang's 2020 combinatorial proof for the limiting case $N \to \infty$ where they used certain unimodal sequences whose alternating sum equals zero. In the same spirit, it would be natural to ask whether a direct combinatorial proof of Theorem 2 for any odd prime t exists or not.

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Thank you for your attention!