NEW BORWEIN-TYPE CONJECTURES

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On a Conjecture of Peter Borwein

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We consider several conjectures by Peter Borwein concerning products of the form $\prod_{n=1}^{pN}(1-q^n)/\prod_{n=1}^{N}(1-q^{pn})$, $p = 3, 5$.

1. Introduction

Peter Borwein (1990) has recently uncovered several charming and intriguing mysteries of great simplicity. The most easily stated is the

FIRST BORWEIN CONJECTURE (Polynomial). For the polynomials $A_n(q)$, $B_n(q)$ and $C_n(q)$ defined by

$$\prod_{j=1}^{n}(1-q^{3j-2})(1-q^{3j-1}) = A_n(q^3) - qB_n(q^3) - q^2C_n(q^3), \tag{1.1}$$

each has non-negative coefficients.

The natural setting for this problem is the theory of partitions.

FIRST BORWEIN CONJECTURE (Partitions). Let $B_e(n, N)$ (respectively $B_0(n, N)$) denote the number of partitions of N into an even (respectively odd) number of distinct non-multiples of 3 each $< 3n$. Then

$$B_e(n, N) - B_0(n, N) \begin{cases} \geq 0 & \text{if } 3|N \\ \leq 0 & \text{otherwise.} \end{cases} \tag{1.2}$$

This is the sort of intriguing simply stated problem that devotees of the theory of partitions love. It should be stated at the beginning that we are unable to prove these conjectures. We proceed by listing the remaining ones.

SECOND BORWEIN CONJECTURE. For the polynomials $\alpha_n(q)$, $\beta_n(q)$ and $\gamma_n(q)$ defined by

$$\prod_{j=1}^{n}(1-q^{3j-2})^2(1-q^{3j-1})^2 = \alpha_n(q^3) - q\beta_n(q^3) - q^2\gamma_n(q^3), \tag{1.3}$$

each has non-negative coefficients.

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Introduction: Borwein's 1990 Conjectures (Andrews 1995)

Conjecture 1 (*First Borwein Conjecture*)

For the polynomials $A_n(q)$, $B_n(q)$, and $C_n(q)$ defined by

$$\prod_{j=1}^{n}(1-q^{3j-2})(1-q^{3j-1}) = A_n(q^3) - qB_n(q^3) - q^2 C_n(q^3),$$

each has non-negative coefficients.

Conjecture 2 (*Second Borwein Conjecture*)

For the polynomials $\alpha_n(q)$, $\beta_n(q)$, and $\gamma_n(q)$ defined by

$$\prod_{j=1}^{n}(1-q^{3j-2})^2(1-q^{3j-1})^2 = \alpha_n(q^3) - q\beta_n(q^3) - q^2\gamma_n(q^3),$$

each has non-negative coefficients.

Conjecture 3 (*Third Borwein Conjecture*)

For the polynomials $\nu_n(q)$, $\phi_n(q)$, $\chi_n(q)$, $\psi_n(q)$, and $\omega_n(q)$ defined by

$$\prod_{j=1}^{n}(1-q^{5j-4})(1-q^{5j-3})(1-q^{5j-2})(1-q^{5j-1})$$
$$= \nu_n(q^5) - q\phi_n(q^5) - q^2\chi_n(q^5) - q^3\psi(q^5) - q^4\omega(q^5),$$

each has non-negative coefficients.

Although Andrews (1995) did not prove any of the above conjectures, he derived explicit sum representations of the polynomials $A_n(q)$, $B_n(q)$, and $C_n(q)$ respectively. The *First Borwein Conjecture* was recently proved analytically by Chen Wang (2022) using the saddle point method. Apart from other tools, his proof relied on a theorem of Andrews where the following three recursive relations were given by Andrews:

$$\begin{aligned}
A_n(q) &= (1+q^{2n-1})A_{n-1}(q) + q^n B_{n-1}(q) + q^n C_{n-1}(q),\\
B_n(q) &= (1+q^{2n-1})B_{n-1}(q) + q^{n-1} A_{n-1}(q) - q^n C_{n-1}(q),\\
C_n(q) &= (1+q^{2n-1})C_{n-1}(q) + q^{n-1} A_{n-1}(q) - q^{n-1} B_{n-1}(q).
\end{aligned}$$

Open Question

Find recursive relations for polynomials in Conjecture 2 and Conjecture 3.

Recent Progress : Work of Wang and Krattenthaler

Recently, Chen Wang and Christian Krattenthaler (2022) proved Conjecture 1 and Conjecture 2. Unlike Wang (2022) who considered Andrews' sum representations for his proof, Wang and Krattenthaler applied the saddle point method directly to the "*First Borwein polynomial*" $\prod_{j=1}^{n}(1-q^{3j-2})(1-q^{3j-1})$ and its higher powers. They also gave the following cubic conjecture which was missed by both Borwein and Andrews:

Conjecture 4 (*Cubic Borwein Conjecture*)

For the polynomials $\kappa_n(q)$, $\delta_n(q)$, and $\theta_n(q)$ defined by

$$\prod_{j=1}^{n}(1-q^{3j-2})^3(1-q^{3j-1})^3 = \kappa_n(q^3) - q\delta_n(q^3) - q^2\theta_n(q^3),$$

each has non-negative coefficients.

They proved "two thirds" of Conjecture 4 above by showing that all coefficients of $\kappa_n(q)$ are non-negative, "one half" of the coefficients of $\delta_n(q)$ are non-negative, and "one half" of the coefficients of $\theta_n(q)$ are non-negative. They also provided ideas along similar lines for (at least) "three fifths" of a proof of Conjecture 3 when the modulus 3 is replaced by 5.

AN ASYMPTOTIC APPROACH TO BORWEIN-TYPE SIGN PATTERN THEOREMS

CHEN WANG AND CHRISTIAN KRATTENTHALER

ABSTRACT. The celebrated (First) Borwein Conjecture predicts that for all positive integers n the sign pattern of the coefficients of the "Borwein polynomial"

$$(1-q)(1-q^2)(1-q^4)(1-q^5)\cdots(1-q^{3n-2})(1-q^{3n-1})$$

is $+--+--\cdots$. It was proved by the first author in [*Adv. Math.* **394** (2022), Paper No. 108028]. In the present paper, we extract the essentials from the former paper and enhance them to a conceptual approach for the proof of "Borwein-like" sign pattern statements. In particular, we provide a new proof of the original (First) Borwein Conjecture, a proof of the Second Borwein Conjecture (predicting that the sign pattern of the square of the "Borwein polynomial" is also $+--+--\cdots$), and a partial proof of a "cubic" Borwein Conjecture due to the first author (predicting the same sign pattern for the cube of the "Borwein polynomial"). Many further applications are discussed.

1. INTRODUCTION

It was in 1993 at a workshop at Cornell University, when what became known as *the Borwein Conjecture* was born. (One of the authors was an intrigued witness of this event.) George Andrews delivered a two-part lecture on *"AXIOM and the Borwein Conjecture"*, in which he — first of all — stated three conjectures that had been communicated to him by Peter Borwein (the first of which became known as "the Borwein Conjecture"), and then reported the lines of attack that he had tried, all of which had failed to give a proof, stressing (quoting from [1], which contains Andrews' findings in printed form) that *"this is the sort of intriguing simply stated problem that devotees of the theory of partitions love."* Indeed, the statement of the first conjecture, dubbed the "First Borwein Conjecture" in [1], is the following.

Conjecture 1.1 (P. BORWEIN). *For all positive integers n, the sign pattern of the coefficients in the expansion of the polynomial $P_n(q)$ defined by*

$$P_n(q) := (1-q)(1-q^2)(1-q^4)(1-q^5)\cdots(1-q^{3n-2})(1-q^{3n-1}) \qquad (1.1)$$

is $+--+--+--\cdots$, with a coefficient 0 being considered as both $+$ and $-$.

The *Second Borwein Conjecture* from [1] predicts the same sign behaviour of the coefficients for the square of the "Borwein polynomial".

Conjecture 1.2 (P. BORWEIN). *For all positive integers n, the sign pattern of the coefficients in the expansion of the polynomial $P_n^2(q)$, where $P_n(q)$ is defined by (1.1), is $+--+--+--\cdots$, with the same convention concerning zero coefficients.*

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Notation & Convention

For non-negative integers n, consider the standard notation for q-Pochammer symbols,

$$(a)_n = (a;q)_n := \prod_{k=0}^{n-1}(1-aq^k), \text{ for } n \geq 1,$$
$$(a)_0 = (a;q)_0 := 1.$$

For positive integers n and k, consider polynomials of the form

$$P_{n,k}(q) := \frac{(q;q)_{kn}}{(q^k;q^k)_n} = \prod_{j=1}^{k-1}(q^j;q^k)_n.$$

Motivation

For $k = 7$, Wang and Krattenthaler (2022) stated the following interesting modulus 7 "Borwein-type" Conjecture:

Conjecture 5 (*Modulus 7 Borwein Conjecture*)

For positive integers n, consider the expansion of the polynomial

$$P_{n,7}(q) := \frac{(q;q)_{7n}}{(q^7;q^7)_n} = \sum_{m=0}^{21n^2} d_m(n) q^m.$$

Then

$$d_{7m}(n) \geq 0 \quad \text{and} \quad d_{7m+1}(n), d_{7m+3}(n), d_{7m+4}(n), d_{7m+6}(n) \leq 0, \quad \text{for all } m \text{ and } n,$$

while

$$d_{7m+5}(n) \begin{cases} \geq 0, & \text{for } m \leq 3\alpha(n)n^2, \\ \leq 0, & \text{for } m > 3\alpha(n)n^2, \end{cases}$$

where $\alpha(n)$ seems to stabilise around 0.302.

Conjecture 6 (Berkovich-D (2024))

Let n be a positive integer and $i \in \{4, 5, 6, 7, 8\}$. Consider the expansion of the polynomial $P^{\,i}_{n,3}(q)$ defined by

$$P^{\,i}_{n,3}(q) := \frac{(q;q)^{i}_{3n}}{(q^3;q^3)^{i}_{n}} = \sum_{m=0}^{3in^2} c^{(i)}_{m}(n)q^m.$$

Then

$$c^{(i)}_{3m}(n) \geq 0, \quad \text{for all } m \text{ and } n,$$

while

$$c^{(i)}_{3m+2}(n) \begin{cases} \geq 0, & \text{for } \ m \leq \alpha^{(i)}(n)n^2, \\ \leq 0, & \text{for } \ m > \alpha^{(i)}(n)n^2, \end{cases}$$

where $\alpha^{(i)}(n)$ tends to a limiting value as n tends to ∞ according to the following table:

Table for Conjecture 6

value of i	$\lim_{n\to\infty} \alpha^{(i)}(n)$
4	$0.750\cdots$
5	$1.270\cdots$
6	$1.778\cdots$
7	$2.282\cdots$
8	$2.784\cdots$

We verified Conjecture 6 using q-series package in Maple. For instance, we got

$$\alpha^{(4)}(65) \cdot 65^2 = 3170.$$

Since the polynomial $P_{n,3}^{i}(q)$ is palindromic for any positive integer i, we can also predict the signs of the coefficients $c_{3m+1}^{(i)}(n)$ in Conjecture 6. This is true because of the following observation.

Remark 1

Consider the polynomial $P_{n,3}^{i}(q)$ as defined in Conjecture 6. If we write

$$P_{n,3}^{i}(q) = X_{n,i}(q^3) + qY_{n,i}(q^3) + q^2 Z_{n,i}(q^3),$$

then it is easy to show that the following duality relations hold

$$X_{n,i}(q) = q^{in^2} X_{n,i}(1/q)$$

and

$$Z_{n,i}(q) = q^{in^2-1} Y_{n,i}(1/q).$$

Conjecture 7 (Berkovich-D (2024))

Let n be a positive integer and $i \in \{2,3\}$. Consider the expansion of the polynomial $P^{i}_{n,5}(q)$ defined by

$$P^{i}_{n,5}(q) := \frac{(q;q)^{i}_{5n}}{(q^5;q^5)^{i}_{n}} = \sum_{m=0}^{10in^2} d^{(i)}_{m}(n) q^m.$$

Then

$$d^{(i)}_{5m}(n) \geq 0, \quad \text{for all } m \text{ and } n,$$

while

$$d^{(i)}_{5m+3}(n) \begin{cases} \geq 0, & \text{for } m \leq \tilde{\alpha}^{(i)}(n) n^2, \\ \leq 0, & \text{for } m > \tilde{\alpha}^{(i)}(n) n^2, \end{cases}$$

and

$$d^{(i)}_{5m+4}(n) \begin{cases} \geq 0, & \text{for } m \leq \tilde{\beta}^{(i)}(n) n^2, \\ \leq 0, & \text{for } m > \tilde{\beta}^{(i)}(n) n^2, \end{cases}$$

where $\tilde{\alpha}^{(i)}(n)$ and $\tilde{\beta}^{(i)}(n)$ tend to limiting values as n tends to ∞ according to the following table:

Table for Conjecture 7

value of i	$\lim_{n\to\infty} \tilde{\alpha}^{(i)}(n)$	$\lim_{n\to\infty} \tilde{\beta}^{(i)}(n)$
2	$1.133\cdots$	$1.011\cdots$
3	$2.132\cdots$	$2.001\cdots$

We verified Conjecture 7 using q-series package in Maple. For instance, we got

$$\tilde{\beta}^{(2)}(65) \cdot 65^2 = 4274.$$

Since the polynomial $P^{i}_{n,5}(q)$ is palindromic for any positive integer i, we can also predict the signs of the coefficients $d^{(i)}_{5m+1}(n)$ and $d^{(i)}_{5m+2}(n)$ in Conjecture 8. This is true because of the following observation.

Remark 2

Consider the polynomial $P^{i}_{n,5}(q)$ as defined in Conjecture 7. If we write

$$P^{i}_{n,5}(q) = A_{n,i}(q^5) + qB_{n,i}(q^5) + q^2 C_{n,i}(q^5) + q^3 D_{n,i}(q^5) + q^4 E_{n,i}(q^5),$$

then it is easy to show that the following relations hold

$$A_{n,i}(q) = q^{2in^2} A_{n,i}(1/q),$$

$$D_{n,i}(q) = q^{2in^2-1} C_{n,i}(1/q),$$

and

$$E_{n,i}(q) = q^{2in^2-1} B_{n,i}(1/q).$$

Conclusion and Future Work

- It would be very interesting to investigate whether the asymptotic approach of Wang and Krattenthaler (2022) applies to Conjecture 6 and Conjecture 7.
- The sign-pattern of coefficients of $P^{i}_{n,3}(q)$ is much more complicated when $i \geq 9$. For instance, we have

$$
\begin{aligned}
X_{11,9}(q) = {} & 1 - 3q + 8q^3 - 9q^4 + 17q^6 - 27q^7 + 46q^9 - 57q^{10} + 260q^{12} \\
& + 1899q^{13} + \cdots + 1899q^{1076} + 260q^{1077} - 57q^{1079} + 46q^{1080} \\
& - 27q^{1082} + 17q^{1083} - 9q^{1085} + 8q^{1086} - 3q^{1088} + q^{1089}.
\end{aligned}
$$

The sign-pattern of $X_{11,9}(q)$ above is definitely worth exploring further.

- Similarly, the sign-pattern of coefficients of $P^{i}_{n,5}(q)$ is complicated when $i \geq 4$. For instance, we have

$$\begin{aligned}B_{2,4}(q) = &-4 - 26q - 108q^2 - 326q^3 - 748q^4 - 1536q^5 - 2644q^6 - 4352q^7 \\ &- 6200q^8 - 8418q^9 - 10544q^{10} - 11369q^{11} - 12772q^{12} - 11134q^{13} \\ &- 10072q^{14} - 7762q^{15} - 3696q^{16} - 2348q^{17} + 1420q^{18} + 2281q^{19} \\ &+ 2924q^{20} + 3422q^{21} + 2300q^{22} + 1983q^{23} + 1044q^{24} + 522q^{25} \\ &+ 168q^{26} - 15q^{27} - 48q^{28} - 42q^{29} - 20q^{30} - 5q^{31}.\end{aligned}$$

  The sign pattern of $B_{2,4}(q)$ above is again worth exploring more.

- It is a challenge to find partition theoretic explanation of why $X_{n,i}(q)$ for $1 \leq i \leq 8$ has non-negative coefficients for all n. Such interpretation, if any, would lead to a great insight into positivity problems in the theory of partitions and q-series.

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Thank You!

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