A BIJECTIVE PROOF OF AN IDENTITY OF BERKOVICH AND UNCU

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Background

A *partition* π of a positive integer n is a non-increasing finite sequence of non-negative integers $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_k$ such that

$$n = \lambda_1 + \lambda_2 + \ldots + \lambda_k.$$

The integers $\lambda_1, \lambda_2, \ldots, \lambda_k$ are called the *parts* of the partition π. We call $p(n)$ to be the number of partitions of n.

For example, $p(5) = 7$ where the seven partitions of 5 are 5, $(4, 1)$, $(3, 2)$, $(3, 1, 1)$, $(2, 2, 1)$, $(2, 1, 1, 1)$, and $(1, 1, 1, 1, 1)$.

We denote the number of parts of π by $\#(\pi)$ and the largest part of π by $l(\pi)$. λ_{2i-1} (resp. λ_{2i}) are called odd-indexed (resp. even-indexed) parts of π.

We call a partition into distinct parts a *strict* partition and denote it by π_d. Let $\mathcal{D}$ be the set of all strict partitions and $\mathcal{T}$ be the set of all triangular numbers, i.e., $\mathcal{T} := \left\{ \frac{n(n+1)}{2} : n \in \mathbb{Z} \right\}$.

Representation of strict partitions

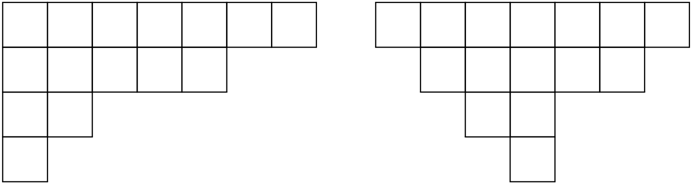

Figure: Young's diagram and shifted Young's diagram representing the strict partition $\pi_d = (7, 5, 2, 1)$

A partition statistic : BG-rank

BG-rank of a partition (Berkovich-Garvan, 2006)

For any partition π, the BG-rank, denoted by $BG(\pi)$, is defined as

$$BG(\pi) := i - j$$

where i is the number of odd-indexed odd parts and j is the number of even-indexed odd parts of π.

For example, for the partition $\pi = (7, 5, 2, 1)$, $BG(\pi) = i - j = 1 - 2 = -1$.

Notations and Conventions

For non-negative integers L, m, and n, we define the conventional q-Pochhammer symbol as

$$(a)_L = (a;q)_L := \prod_{k=0}^{L-1}(1-aq^k);$$
$$(a)_\infty = (a;q)_\infty := \lim_{L\to\infty}(a)_L \text{ where } |q|<1.$$

For $m, n \in \mathbb{Z}$, we define the q-binomial (Gaussian) coefficient as

$$\begin{bmatrix} n \\ m \end{bmatrix}_q := \begin{cases} \dfrac{(q)_n}{(q)_m (q)_{n-m}} & \text{for } 0 \le m \le n, \\ 0 & \text{otherwise.} \end{cases}$$

For $m, n \in \mathbb{Z}^+ \cup \{0\}$, $\begin{bmatrix} m+n \\ m \end{bmatrix}_q$ is the generating function for partitions π having $\#(\pi) \le m$ and $l(\pi) \le n$ and vice-versa.

For any sequence of integers $\Delta = \{d_1, \ldots, d_l\}$, we denote $l(\Delta) = l$, $|\Delta| = \sum_{i=1}^{l} d_i$, and $|\Delta|_{\text{alt}} = \sum_{i=1}^{l} (-1)^i d_i$.

Motivation

For any non-negative integer N and any integer k, let $B_N(k,q)$ denote the generating function of the number of strict partitions π_d having $l(\pi_d) \leq N$ and $\mathrm{BG}(\pi_d) = k$.

Theorem (Berkovich-Uncu, 2016)

For $\nu \in \{0,1\}$,

$$B_{2N+\nu}(k,q) = q^{2k^2-k} \begin{bmatrix} 2N+\nu \\ N+k \end{bmatrix}_{q^2}. \tag{1}$$

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Corollary

Letting $N \to \infty$ in (1), we have

$$B(k,q) = \frac{q^{2k^2-k}}{(q^2;q^2)_\infty} \tag{2}$$

where $B(k,q)$ denotes the generating function of the number of strict partitions π_d having $\mathrm{BG}(\pi_d) = k$.

State of the art and the main theorem

- In 2010, Vandervelde provided a bijective proof of (2) for $k = 0$ considering diagonal lengths of Young's diagrams of strict partitions.
- Building upon Vandervelde's bijection, in 2020, Fu and Tang provided a bijective proof of (2) for all integers k using certain unimodal sequences whose alternating sum equals zero.
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Theorem (D-Mukhopadhyay, 2023)

Let $\nu \in \{0, 1\}$, N be a non-negative integer, and k be any integer. Then, for any positive integer n, the number of strict partitions π_d of n having $\mathrm{BG}(\pi_d) = k$ and $l(\pi_d) \leq 2N + \nu$ is the same as the number of partitions π of $\frac{n-2k^2+k}{2}$ where $l(\pi) \leq N + \nu - k$, $\#(\pi) \leq N + k$ if $k \leq 0$ and $l(\pi) \leq N + k$, $\#(\pi) \leq N + \nu - k$ if $k > 0$.

(a, b)-sequences

(a, b)-sequences (Fu-Tang, 2020)

For any non-negative integer a and any integer $1 \leq b \leq l$, we call a sequence of l positive integers $\{d_1, \ldots, d_l\}$ an (a, b)-*sequence of length* l if

- $d_i = a + i$ for $1 \leq i \leq b$,
- d_i forms a non-increasing sequence of positive integers for $i \geq b$, and
- $\sum_{i=1}^{l} (-1)^i d_i = 0$.

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Example

$\{5, 6, 7, 8, 3, 3, 2, 2, 2, 1, 1\}$ is a $(4, 4)$-sequence of length 11.

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We denote the collection of all such sequences by $\mathcal{S}_{a,b}$ and define $\mathcal{S} := (\bigcup_{\substack{a \geq 0 \\ b \geq 1}} \mathcal{S}_{a,b}) \cup \{\varepsilon\}$ where ε is the empty sequence. If $\Delta \in \mathcal{S}_{a,b}$, we denote $a(\Delta) = a$ and $b(\Delta) = b$.

Proof of the main theorem : the beginnings

For any strict partition $\pi_d = (\lambda_1, \lambda_2, \ldots, \lambda_r)$, consider the shifted Young diagram of π_d and construct the sequence of column lengths (read from left to right) of its shifted Young diagram. These column lengths then form a unimodal sequence $c(\pi_d) = \{c_1, c_2, \ldots, c_{\lambda_1}\}$.

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Proposition (Fu-Tang, 2020)

There exists a unique integer $0 \leq m \leq r$ such that $\sum_{i=1}^{m} (-1)^i c_i = \sum_{i=1}^{\lambda_1} (-1)^i c_i$, i.e., $|\Delta|_{\text{alt}} = 0$ where $\Delta := \{c_{m+1}, c_{m+2}, \ldots, c_{\lambda_1}\}$.

Fu and Tang's map ι

Lemma (Fu-Tang, 2020)

There is an injection $\iota : \mathcal{D} \to \mathcal{T} \times \mathcal{S}$. Suppose $\iota(\pi_d) = (t, \Delta)$, then $|\pi_d| = t + |\Delta|$. Moreover, $(t, \Delta) \in \iota(\mathcal{D})$ if and only if

1. $a(\Delta) = m$, or
2. $a(\Delta) \leq m - 1$ and $b(\Delta) = 1$, or
3. $\Delta = \varepsilon$,

where $t = 1 + 2 + \ldots + m = \binom{m+1}{2}$ for some $m \geq 0$.

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To see that ι is an injection, simply note that for any $(t, \Delta) \in \mathcal{T} \times \mathcal{S}$ that satisfies either (1) (the case $m < r$) or (2) (the case $m = r < \lambda_1$) or (3) (the case $m = r = \lambda_1$ for which π_d is the staircase partition $(m, m-1, \ldots, 2, 1)$), we can uniquely recover its preimage by appending columns of length $1, 2, \ldots, m$ to the left of the columns of length given by the elements of Δ and obtaining a valid shifted Young diagram.

An example illustrating ι

Example

Consider $\pi_d = (8, 7, 4, 2)$. So, $c(\pi_d) = \{1, 2, 3, 4, 4, 3, 2, 2\}$. Hence, $m = 2$, the triangular part $t = 3$, and $\Delta = \{3, 4, 4, 3, 2, 2\} \in \mathcal{S}_{2,2}$.

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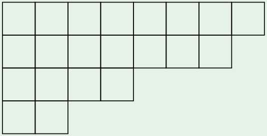

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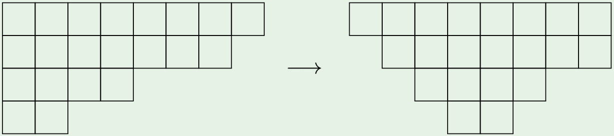

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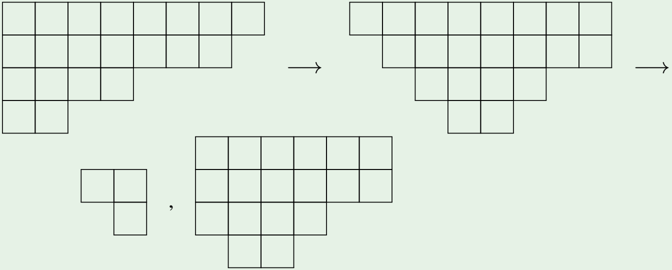

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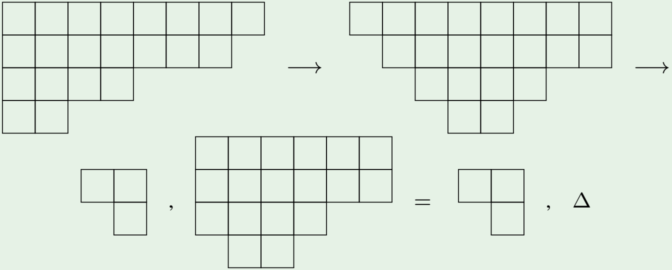

Fu and Tang's map ϕ_a

In 1993, Chu defined the d-Durfee rectangle for the Young's diagram of a partition to be an $i \times (i+d)$ rectangle (having i rows and $i+d$ columns) which is obtained by choosing the largest possible i such that the $i \times (i+d)$ rectangle is contained in the Young's diagram for a fixed integer d.

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For integers $a \geq 0$ and $b \geq 1$, consider the map $\phi_a : \mathcal{S}_{a,b} \to \mathcal{P}_{a,b}$ where $\mathcal{P}_{a,b}$ is the set of all integer partitions $\lambda = (\lambda_1, \lambda_2, \ldots)$ whose a-Durfee rectangle has size $\lceil \frac{b}{2} \rceil \times (\lceil \frac{b}{2} \rceil + a)$ and $\lambda_{\frac{b}{2}} > a + b/2$ if b is even or $\lambda_{\frac{b+1}{2}} = a + (b+1)/2$ if b is odd.

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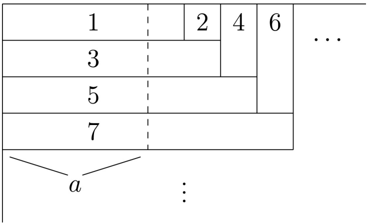

Figure: Block diagram configuration for $\phi_a(\mathcal{S}_{a,b})$ with labeled blocks

In the block diagram configuration, we call the ith labeled block B_i. B_i has size $1 \times \left(a + \frac{i+1}{2}\right)$ (resp. $\frac{i}{2} \times 1$) if i is odd (resp. even). We denote the area of B_i, i.e., the number of cells, by n_i. So, $n_1 = a+1$, $n_2 = 1$, $n_3 = a+2$, $n_4 = 2$, and so on. For any sequence $\Delta = \{d_1, d_2, \ldots, d_l\} \in \mathcal{S}_{a,b}$, we obtain $\phi_a(\Delta)$ by performing the following operations:

- Fill up B_1 in the block diagram with $d_1 = a+1$ cells which is equivalent to labeling the $a+1$ cells in B_1 with '1'.
- Use d_i cells first to *double cover* the already existing cells in B_{i-1} for $2 \leq i \leq l$ and then use the remaining cells to fill B_i. This is equivalent to using d_i cells to re-label the already existing n_{i-1} cells in B_{i-1} by '2' first for $2 \leq i \leq l$ and then labeling the remaining $d_i - n_{i-1}$ cells by '1' to fill B_i.
- Filling of B_i's (labeling by '1' and re-labeling by '2') are done from left to right if i is odd and from top to bottom if i is even.
- After having used up all the d_i's where $1 \leq i \leq l$, the *doubly covered* cells (cells which are labeled by '2') form the Young diagram of a partition (say) $\lambda = \phi_a(\Delta)$.

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The notion of double covering of the cells in the block diagram configuration is equivalent to coloring the cells by yellow and then re-coloring the cells by green so that in the end, all the cells are colored in green. This is exactly the reason why the base in the q-binomial coefficient in (1) is q^2 instead of just q as we are counting the cells twice.

An example illustrating ϕ_a

Example

Suppose $\Delta = \{3, 4, 4, 3, 2, 2\} \in \mathcal{S}_{2,2}$. We get $\lambda = \phi_2(\Delta) = (4, 3, 2) \in \mathcal{P}_{2,2}$ and $|\lambda| = |\Delta|/2 = 9$.

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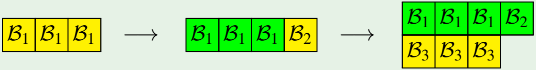

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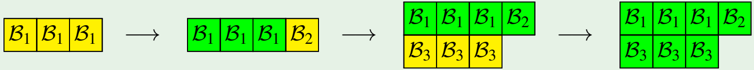

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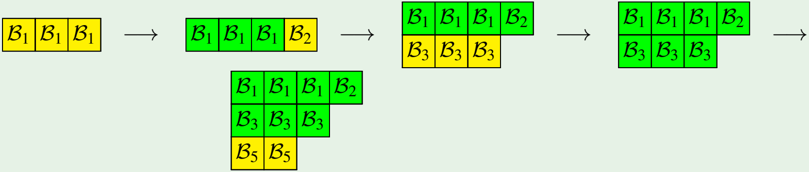

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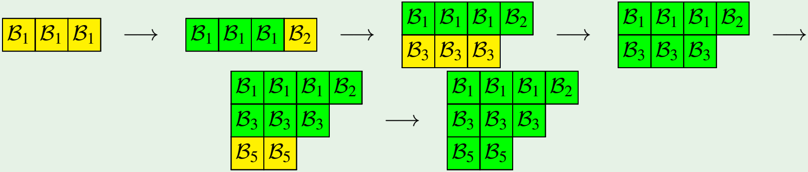

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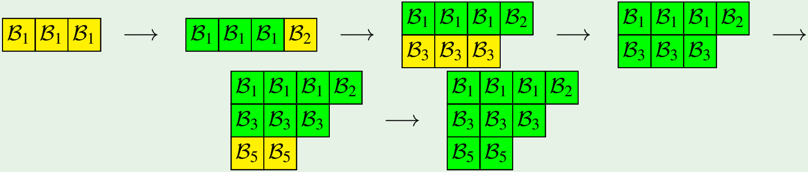

All *singly covered* (equivalent to being labeled by '1' or counted once) cells are colored yellow and all *doubly covered* (equivalent to being labeled by '2' or counted twice) cells are colored green. All the cells labeled $\mathcal{B}_i$ form a sub-region of the ith block B_i and b_i is the number of doubly covered cells (colored green) labeled $\mathcal{B}_i$ for $i \in \{1,2,3,4,5\}$. Here, $b_1 = 3$, $b_2 = 1$, $b_3 = 3$, $b_4 = 0$, and $b_5 = 2$ form the partition $\lambda = (4,3,2)$.

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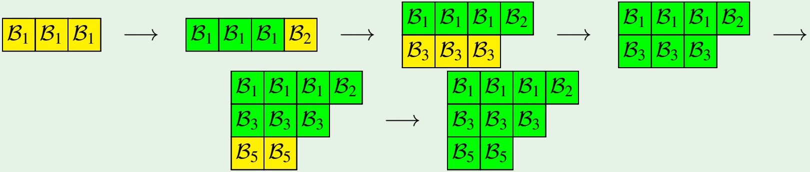

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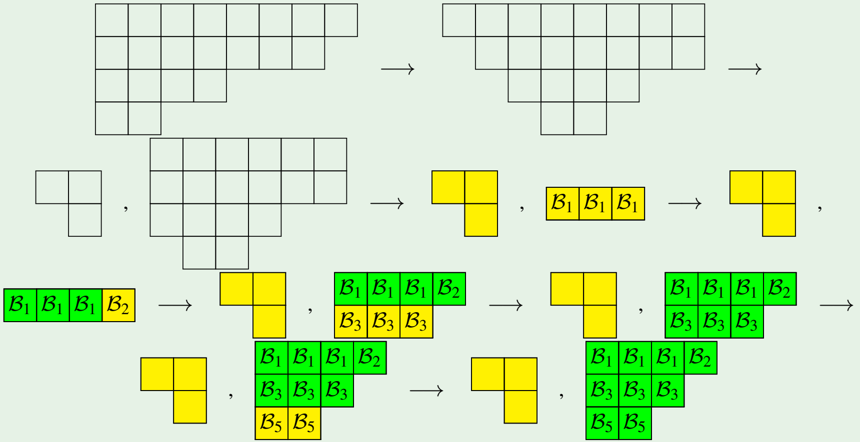
Theorem (Fu-Tang, 2020)

For a fixed $a \geq 0$ and any $b \geq 1$, ϕ_a is a bijection from $\mathcal{S}_{a,b}$ to $\mathcal{P}_{a,b}$, such that $|\Delta| = 2|\phi_a(\Delta)|$, for any $\Delta \in \mathcal{S}_{a,b}$.

So now?

Example



So, starting from $\pi_d = (8,7,4,2)$, we obtain (t, π) where $t = 3$ is the triangular part and π is the *doubly covered* partition $(4,3,2)$. Observe that $|\pi_d| = 21 = 3 + 2 \cdot 9 = t + 2|\pi|$.

Two important lemmas

Lemma 1

For the $\Delta \in \mathcal{S}_{a,b}$ obtained from the shifted Young diagram of π_d,

$$a = a(\Delta) = \begin{cases} -2k & \text{if } k \leq 0, \\ 2k-1 & \text{if } k > 0. \end{cases}$$

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Proof. For π_d having BG-rank k, we have $k = -\left|\{1,2,\ldots,a\}\right|_{\text{alt}} = 1-2+3-\ldots+(-1)^{a+1}a$. Now, we consider two cases concerning the parity of a:

- Case I: a is even
  Let $a = 2t$ for some $t \geq 0$. Then,

$$\begin{aligned} k &= 1-2+3-\ldots-2t \\ &= (1+3+\ldots+(2t-1)) - 2(1+2+\ldots+t) \\ &= t^2 - 2\cdot\frac{t(t+1)}{2} \\ &= t^2 - t^2 - t \\ &= -t \\ &= -\frac{a}{2}. \end{aligned}$$

- Case II: a is odd
  Let $a = 2t - 1$ for some $t \geq 1$. Then,

$$\begin{aligned} k &= 1 - 2 + 3 - \ldots + (2t-1) \\ &= (1 + 3 + \ldots + (2t-1)) - 2(1 + 2 + \ldots + (t-1)) \\ &= t^2 - 2 \cdot \frac{t(t-1)}{2} \\ &= t^2 - t^2 + t \\ &= t \\ &= \frac{a+1}{2}. \end{aligned}$$

Hence, $a = -2k$ if $k \leq 0$ and $a = 2k - 1$ if $k > 0$.

- Case II: a is odd
  Let $a = 2t - 1$ for some $t \geq 1$. Then,

$$\begin{aligned} k &= 1 - 2 + 3 - \ldots + (2t-1) \\ &= (1 + 3 + \ldots + (2t-1)) - 2(1 + 2 + \ldots + (t-1)) \\ &= t^2 - 2 \cdot \frac{t(t-1)}{2} \\ &= t^2 - t^2 + t \\ &= t \\ &= \frac{a+1}{2}. \end{aligned}$$

Hence, $a = -2k$ if $k \leq 0$ and $a = 2k - 1$ if $k > 0$.

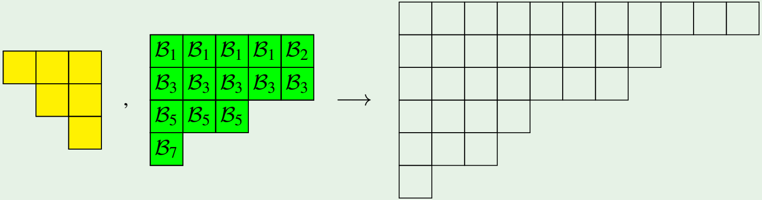
Lemma 2

The index of the last block present in the block diagram representation of the Young diagram of π is at most $l(\pi_d) - a - 1$.

Proof. In the shifted Young diagram of π_d, the length of the unimodal sequence whose alternating sum is zero is equal to $l(\pi_d) - a$. So, the number of blocks that can be *doubly covered* by the elements of this sequence is at most $l(\pi_d) - a - 1$.

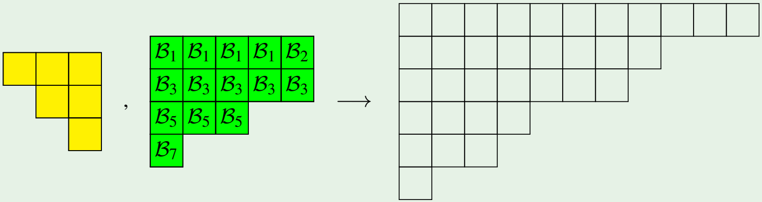
Is the process reversible?

Example



Is the process reversible?

Example



We are given the triangular part $t = 6$ and the (*doubly covered*) partition $\pi = (5, 5, 3, 1)$. So, the solutions to $2k^2 - k = 6$ are $k = 2$ and $k = -\frac{3}{2}$. Since $k \in \mathbb{Z}$, $k = 2$. Now, $a = 2 \cdot 2 - 1 = 3$ since $k = 2 > 0$ which implies $b_1 = a + 1 = 4$, $b_2 = 1$, $b_3 = 5$, $b_4 = 0$, $b_5 = 3$, $b_6 = 0$, and $b_7 = 1$. Now, we obtain $d_1 = b_1 = 4, d_2 = b_1 + b_2 = 5, d_3 = b_2 + b_3 = 6, d_4 = b_3 + b_4 = 5, d_5 = b_4 + b_5 = 3, d_6 = b_5 + b_6 = 3$, $d_7 = b_6 + b_7 = 1$, and $d_8 = b_7 + b_8 = 1$ since $b_8 = 0$. Thus, we obtain the sequence $\{4, 5, 6, 5, 3, 3, 1, 1\}$ which we write column-wise and if we append columns of length $\{1, 2, 3\}$ to the left of the column of length 4, we retrieve back the shifted Young diagram of the partition $\pi_d = (11, 8, 7, 4, 3, 1)$.

How to obtain the bounds?

We now obtain bounds on the largest part and the number of parts of π. We consider two cases according to the sign of the BG-rank k of π_d. In both the cases, we also show that the we can retrieve back the bound on the largest part of π_d.

- Case I: $k \leq 0$

  If $k \leq 0$, then from Lemma 1, we have $a = -2k$.

  If I is the index of the last present block in the block diagram representation of the Young diagram of π, using Lemma 2, we have

$$\begin{aligned} I &\leq l(\pi_d) - a - 1 \\ &\leq 2N + \nu - a - 1 \\ &= 2N + \nu + 2k - 1 \\ &= 2(N+k) + \nu - 1 \\ &= \begin{cases} 2(N+k) - 1 & \text{if } \nu = 0, \\ 2(N+k) & \text{if } \nu = 1. \end{cases} \end{aligned}$$

  Therefore, $\#(\pi) \leq N + k$.

  If E is the number of even-indexed blocks present in the block diagram representation of the Young diagram of π, then

$$E \leq \sum_{\substack{i=2 \\ 2|i}}^{l(\pi_d)-a-1} 1.$$

Again from the block diagram representation of the Young diagram of π, we have

$$
\begin{aligned}
l(\pi) &= a + 1 + E \\
&\leq a + 1 + \sum_{\substack{i=2 \\ 2|i}}^{l(\pi_d)-a-1} 1 \\
&\leq a + 1 + \sum_{\substack{i=2 \\ 2|i}}^{2N+\nu-a-1} 1 \\
&= -2k + 1 + \sum_{\substack{i=2 \\ 2|i}}^{2N+\nu+2k-1} 1 \\
&= \begin{cases} -2k + 1 + \sum\limits_{\substack{i=2 \\ 2|i}}^{2N+2k-1} 1 & \text{if } \nu = 0, \\ -2k + 1 + \sum\limits_{\substack{i=2 \\ 2|i}}^{2N+2k} 1 & \text{if } \nu = 1 \end{cases} \\
&= \begin{cases} -2k + 1 + N + k - 1 & \text{if } \nu = 0, \\ -2k + 1 + N + k & \text{if } \nu = 1 \end{cases} \\
&= \begin{cases} N - k & \text{if } \nu = 0, \\ N - k + 1 & \text{if } \nu = 1. \end{cases}
\end{aligned}
$$

Hence, $l(\pi) \leq N + \nu - k$.

For the reverse direction, since $k \leq 0$, we know that $a = -2k$ and we also know that $l(\pi) \leq N + \nu - k$ and $\#(\pi) \leq N + k$. Clearly, $l(\pi_d) = a + l(\Delta) = -2k + l(\Delta) = -2k + I + 1$ where I is the index of the last present block in the block diagram representation of the Young diagram of π. Now, we consider two sub-cases regarding the parity of I:

- **Sub-Case IA: I is odd**

  Since $\#(\pi) \leq N + k$,

$$
\begin{aligned}
I &\leq 2(N + k) - 1 \\
&= 2N + 2k - 1 \\
&\leq 2N + \nu + 2k - 1 \qquad (3)
\end{aligned}
$$

  where (3) follows from the fact that $\nu \in \{0, 1\}$.

  Therefore, from (3), it follows that $l(\pi_d) = -2k + I + 1 \leq 2N + \nu$.

- **Sub-Case IB: I is even**

  Since $l(\pi) \leq N + \nu - k$,

$$
\begin{aligned}
I &\leq 2((N + \nu - k) - (a + 1)) \\
&= 2N + 2\nu - 2k - 2a - 2 \\
&= 2N + 2\nu + 2k - 2 \\
&= 2N + \nu + 2k - 1 + \nu - 1 \\
&\leq 2N + \nu + 2k - 1 + \nu - 1 + 1 - \nu \qquad (4) \\
&= 2N + \nu + 2k - 1 \qquad (5)
\end{aligned}
$$

  where (4) follows from the fact that $1 - \nu \in \{0, 1\}$.

  Therefore, from (5), it follows that $l(\pi_d) = -2k + I + 1 \leq 2N + \nu$.

- Case II: $k > 0$

If $k > 0$, then from Lemma 1, we have $a = 2k - 1$, i.e, a is odd.

Let I be the index of the last present block in the block diagram representation of the Young diagram of π. From Lemma 2, we know that

$$\begin{aligned} I &\leq l(\pi_d) - a - 1 \\ &\leq 2N + \nu - a - 1 \\ &= 2N + \nu - 2k \\ &= 2(N + \nu - k) - \nu \\ &= \begin{cases} 2(N + \nu - k) & \text{if } \nu = 0, \\ 2(N + \nu - k) - 1 & \text{if } \nu = 1. \end{cases} \end{aligned}$$

Therefore, $\#(\pi) \leq N + \nu - k$.

If E is the number of even-indexed blocks present in the block diagram representation of the Young diagram of π,

$$E \leq \sum_{\substack{i=2 \\ 2|i}}^{l(\pi_d)-a-1} 1.$$

Again, from the block diagram representation of the Young diagram of π, we have

$$
\begin{aligned}
l(\pi) &= a + 1 + E \\
&\leq a + 1 + \sum_{\substack{i=2 \\ 2|i}}^{l(\pi_d)-a-1} 1 \\
&\leq a + 1 + \sum_{\substack{i=2 \\ 2|i}}^{2N+\nu-a-1} 1 \\
&= 2k + \sum_{\substack{i=2 \\ 2|i}}^{2N+\nu-2k} 1 \\
&= \begin{cases} 2k + \sum\limits_{\substack{i=2 \\ 2|i}}^{2N-2k} 1 & \text{if } \nu = 0, \\ 2k + \sum\limits_{\substack{i=2 \\ 2|i}}^{2N-2k+1} 1 & \text{if } \nu = 1 \end{cases} \\
&= \begin{cases} 2k + N - k & \text{if } \nu = 0, \\ 2k + N - k & \text{if } \nu = 1. \end{cases}
\end{aligned}
$$

Hence, $l(\pi) \leq N + k$.

For the reverse direction, since $k > 0$, we know that $a = 2k - 1$, $l(\pi) \leq N + k$, and $\#(\pi) \leq N + \nu - k$. Clearly, $l(\pi_d) = 2k - 1 + l(\Delta) = 2k - 1 + I + 1 = 2k + I$ where I is the index of the last present block in the block diagram representation of the Young diagram of π. Now, we consider two sub-cases regarding the parity of I:

- Sub-Case IIA: I is odd

  Since $\#(\pi) \leq N + \nu - k$,

$$\begin{aligned} I &\leq 2(N + \nu - k) - 1 \\ &= 2N + 2\nu - 2k - 1 \\ &= 2N + \nu - 2k + \nu - 1 \\ &\leq 2N + \nu - 2k + \nu - 1 + 1 - \nu \quad (6) \\ &= 2N + \nu - 2k \quad (7) \end{aligned}$$

  where (6) follows from the fact that $1 - \nu \in \{0, 1\}$.

  Therefore, from (7), it follows that $l(\pi_d) = 2k + I \leq 2N + \nu$.

- Sub-Case IIB: I is even

  Since $l(\pi) \leq N + k$,

$$\begin{aligned} I &\leq 2((N + k) - (a + 1)) \\ &= 2N - 2k \\ &= 2N + \nu - 2k - \nu \\ &\leq 2N + \nu - 2k - \nu + \nu \quad (8) \\ &= 2N + \nu - 2k \quad (9) \end{aligned}$$

  where (8) follows from the fact that $\nu \in \{0, 1\}$.

  Therefore, from (9), it follows that $l(\pi_d) = 2k + I \leq 2N + \nu$.

Concluding remarks and a future question

- In their study of the Ariki-Koike algebras and Kleshchev multipartitions, Li, Seo, Stanton and Yee proved (1) using the 2-abacus of staircase 2-cores of strict partitions having bounded largest part.
- Berkovich and Dhar recently proved (1) using Littlewood decomposition of strict partitions with bounded largest part into 2-cores and 2-quotients.
- If $\tilde{B}_N(k,q)$ denotes the generating function for the number of partitions into parts less than or equal to N with BG-rank equal to k, then in 2016, Berkovich and Uncu showed that

$$\tilde{B}_{2N+\nu}(k,q) = \frac{q^{2k^2-k}}{(q^2;q^2)_{N+k}(q^2;q^2)_{N+\nu-k}}. \tag{10}$$

Find a bijective proof of (10) without resorting to 2-cores and 2-quotients.

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Thank You!

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