

ON GLAISHER'S PARTITION THEOREM

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Euler's Partition Theorem

The grandfather of all partition identities is Euler's celebrated partition theorem:

Euler's Theorem (1748)

For any non-negative integer n ,

$$A(n) = B(n)$$

where $A(n)$ is the number of partitions of n into distinct parts and $B(n)$ is the number of partitions of n into odd parts.

This theorem contains all the elements that would suggest generalizations, and, over the centuries, generalizations have been found in profusion. The Rogers-Ramanujan identities and Schur's 1926 theorem kicked off the twentieth century's contributions. Henry Alder's* survey contains some of the results in the late 20th century. However, much little has ever been added to Euler's theorem itself.

*H. L. Alder, Partition identities—from Euler to the present, *Amer. Math. Monthly*, **76** (1969), 733–746.

Andrews, Kumar and Yee's work

In a recent work, Andrews, Kumar, and Yee provided two more elegant companion partition functions to $A(n)$ and $B(n)$. In particular, they have obtained the following partition identity.

Theorem AKY (FCNT, 2026)

For any positive integer n ,

$$A(n) = B(n) = C(n+1) = \frac{1}{2}D(n+1),$$

where $C(n)$ is the number of partitions of n with largest part even and parts not exceeding half of the largest part are distinct and $D(n)$ is the number of partitions of n into non-negative parts wherein the smallest part appear exactly twice and no other parts are repeated.

Andrews, Kumar, and Yee provided q -series proofs of $B(n) = C(n+1)$ and $C(n) = \frac{1}{2}D(n)$. They also provided bijective proofs of $A(n-1) = \frac{1}{2}D(n)$ and $B(n) = C(n+1)$.

An Example

For example, if $n = 6$, the four partition sets are the following.

$A(6)$	$B(6)$	$C(7)$	$D(7)$
6	$5 + 1$	$6 + 1$	$0 + 0 + 7$
$5 + 1$	$3 + 3$	$4 + 3$	$0 + 0 + 6 + 1$
$4 + 2$	$3 + 1 + 1 + 1$	$4 + 2 + 1$	$0 + 0 + 5 + 2$
$3 + 2 + 1$	$1 + 1 + 1 + 1 + 1 + 1$	$2 + 2 + 2 + 1$	$0 + 0 + 4 + 3$
			$0 + 0 + 4 + 2 + 1$
			$1 + 1 + 5$
			$1 + 1 + 2 + 3$
			$2 + 2 + 3$

Glaisher's Theorem

Let $m \geq 2$ be an integer. In 1883, J. W. L. Glaisher obtained the following partition theorem which is a natural extension of Euler's theorem.

Theorem G (1883)

For any non-negative integer n ,

$$A_m(n) = B_m(n),$$

where $A_m(n)$ is the number of partitions of n into parts repeating less than m times and $B_m(n)$ to be the number of partitions of n into parts not divisible by m .

For any integer $0 \leq j \leq m - 1$, define $B_m^{(j)}(n)$ to be the number of partitions of n into parts not divisible by m and largest part congruent to j modulo m . Clearly,

$$B_m(n) = \sum_{j=1}^{m-1} B_m^{(j)}(n).$$

Lin and Zhang's work

Recently, Lin and Zhang considered the set $B'_m(n)$ of all partitions of n into parts not divisible by m and largest part congruent to -1 modulo m . Thus, it is clear that

$$|B'_m(n)| = B_m^{(m-1)}(n).$$

They also considered the following generalization of Andrews, Kumar, and Yee's $C(n)$ function to the case of m -regular partitions (partitions whose parts are not divisible by m).

Define $C_m(n)$ to be the number of partitions of n in which the largest part is divisible by m (say mj) and all parts less than or equal to j repeat less than m times. Consider $C_m(0) := 1$.

Theorem LZ (2025)

For any positive integer n ,

$$|B'_m(n)| = C_m(n + 1).$$

Lin and Zhang provided both q -series and bijective proofs of Theorem LZ.

Notations and Conventions

For any complex variable A and non-negative integer N , we define the conventional q -Pochhammer symbol as

$$(A)_N = (A; q)_N := \begin{cases} 1 & \text{when } N = 0, \\ \prod_{i=0}^{N-1} (1 - Aq^i) & \text{otherwise, and} \end{cases}$$

$$(A)_\infty = (A; q)_\infty := \lim_{N \rightarrow \infty} (A)_N \text{ for } |q| < 1.$$

For non-negative integers a, b , we define the q -binomial coefficient as

$$\begin{bmatrix} a+b \\ b \end{bmatrix}_q := \begin{cases} \frac{(q)_{a+b}}{(q)_a (q)_b} & \text{for } a, b \geq 0, \\ 0 & \text{otherwise.} \end{cases}$$

Let $\zeta_n = e^{\frac{2\pi i}{n}}$ denote the primitive n^{th} root of unity.

Main Results

Define $D_m(n)$ to be the number of partitions of n into non-negative parts where the smallest part occurs exactly m times and all other parts repeat less than m times.

Theorem AD1 (2025)

For any non-negative integer n ,

$$C_m(n) = \frac{D_m(n) + E_m(n)}{m},$$

where $E_m(n)$ is such that the generating function of $E_m(n)$ is

$$\varepsilon_m(q) := \sum_{n=0}^{\infty} E_m(n)q^n = \sum_{n=0}^{\infty} q^{mn}(q^{n+1}; q)_{\infty} \sum_{j=1}^{m-1} (\zeta_m^j q^{n+1}; q)_{\infty}.$$

Proof of Theorem AD1

$$\begin{aligned}\sum_{n=0}^{\infty} C_m(n)q^n &= \sum_{n=0}^{\infty} \frac{\prod_{j=1}^n (1 + q^j + \dots + q^{(m-1)j})q^{mn}}{(q^{n+1}; q)_{(m-1)n}} \\ &= \sum_{n=0}^{\infty} \frac{(q^m; q^m)_n q^{mn}}{(q; q)_{mn}} \\ &= (q^m; q^m)_{\infty} \sum_{n=0}^{\infty} \frac{q^{mn}}{(q; q)_{mn} (q^{mn+m}; q^m)_{\infty}}.\end{aligned}\tag{1}$$

Using Euler's identity

$$\frac{1}{(t; q)_{\infty}} = \sum_{n=0}^{\infty} \frac{t^n}{(q; q)_n},\tag{2}$$

(replacing q by q^m and letting $t = q^{mn+m}$) in (1), we get

$$\sum_{n=0}^{\infty} C_m(n)q^n = (q^m; q^m)_{\infty} \sum_{n=0}^{\infty} \frac{q^{mn}}{(q; q)_{mn}} \sum_{j=0}^{\infty} \frac{q^{mnj+mj}}{(q^m; q^m)_j}$$

$$\begin{aligned}
&= (q^m; q^m)_\infty \sum_{n,j=0}^{\infty} \frac{q^{mn+mnj+mj}}{(q; q)_{mn} (q^m; q^m)_j} \\
&= (q^m; q^m)_\infty \sum_{n,j=0}^{\infty} \frac{1}{m} \left(\sum_{k=0}^{m-1} \zeta_m^{kn} \right) \frac{q^{n+nj+mj}}{(q; q)_n (q^m; q^m)_j} \\
&= \frac{(q^m; q^m)_\infty}{m} \sum_{j=0}^{\infty} \frac{q^{mj}}{(q^m; q^m)_j} \sum_{k=0}^{m-1} \sum_{n=0}^{\infty} \frac{(\zeta_m^k q^{j+1})^n}{(q; q)_n}. \tag{3}
\end{aligned}$$

Using (2) with $t = \zeta_m^k q^{j+1}$ in (3), we get

$$\begin{aligned}
\sum_{n=0}^{\infty} C_m(n) q^n &= \frac{(q^m; q^m)_\infty}{m} \sum_{j=0}^{\infty} \frac{q^{mj}}{(q^m; q^m)_j} \sum_{k=0}^{m-1} \frac{1}{(\zeta_m^k q^{j+1}; q)_\infty} \\
&= \frac{(q^m; q^m)_\infty}{m} \sum_{j=0}^{\infty} \frac{q^{mj}}{(q^m; q^m)_j (q^{j+1}; q)_\infty} \\
&\quad + \frac{(q^m; q^m)_\infty}{m} \sum_{j=0}^{\infty} \frac{q^{mj}}{(q^m; q^m)_j} \sum_{k=1}^{m-1} \frac{1}{(\zeta_m^k q^{j+1}; q)_\infty}
\end{aligned}$$

$$\begin{aligned}
&= \frac{1}{m} \sum_{j=0}^{\infty} \frac{q^{mj}(q^{m(j+1)}; q^m)_{\infty}}{(q^{j+1}; q)_{\infty}} \\
&\quad + \frac{(q^m; q^m)_{\infty}}{m} \sum_{j=0}^{\infty} \frac{q^{mj}(q^{j+1}; q)_{\infty}}{(q^m; q^m)_j (q^{m(j+1)}; q^m)_{\infty}} \sum_{k=1}^{m-1} (\zeta_m^k q^{j+1}; q)_{\infty} \\
&= \frac{1}{m} \sum_{j=0}^{\infty} q^{mj} \cdot \prod_{i=m+1}^{\infty} (1 + q^i + \dots + q^{(m-1)i}) \\
&\quad + \frac{1}{m} \sum_{j=0}^{\infty} q^{mj} (q^{j+1}; q)_{\infty} \sum_{k=1}^{m-1} (\zeta_m^k q^{j+1}; q)_{\infty} \\
\sum_{n=0}^{\infty} C_m(n) q^n &= \frac{1}{m} \sum_{n=0}^{\infty} D_m(n) q^n + \frac{1}{m} \sum_{n=0}^{\infty} E_m(n) q^n. \tag{4}
\end{aligned}$$

Comparing coefficients of q^n on both sides of (4), we have

$$C_m(n) = \frac{D_m(n) + E_m(n)}{m}.$$

□

We then have the following full extension of Theorem AKY.

Theorem AD2 (2025)

For any positive integer n ,

$$A_m(n) = B_m(n) = \sum_{k=1}^{m-2} B_m^{(k)}(n) + C_m(n+1) = \sum_{k=1}^{m-2} B_m^{(k)}(n) + \frac{D_m(n+1) + E_m(n+1)}{m}.$$

Theorem AD3 (2025)

$$\varepsilon_m(q) = P_m(q) + \sum_{k=0}^{\infty} (-1)^k q^{\binom{k+1}{2}} \sum_{j=0}^{m-1} \chi_m(k-j) \left(\left[\begin{matrix} m-1 \\ j \end{matrix} \right]_q - 1 \right),$$

where the polynomial $P_m(q)$ is defined as

$$P_m(q) := - \sum_{j=0}^{m-1} \left[\begin{matrix} m-1 \\ j \end{matrix} \right]_q \sum_{k=0}^{j-1} (-1)^k \chi_m(k-j) q^{\binom{k+1}{2}}$$

and $\chi_m(n)$ is defined as

$$\chi_m(n) := \sum_{j=1}^{m-1} \zeta_m^{jn} = \begin{cases} m-1 & \text{if } m \mid n, \\ -1 & \text{if } m \nmid n. \end{cases}$$

Proof of Theorem AD3

Define $\chi_m(n)$ as follows

$$\chi_m(n) := \sum_{j=1}^{m-1} \zeta_m^{jn} = \begin{cases} m-1 & \text{if } m \mid n, \\ -1 & \text{if } m \nmid n. \end{cases}$$

Using Euler's identity

$$(-t; q)_\infty = \sum_{k=0}^{\infty} \frac{t^k q^{\frac{k(k-1)}{2}}}{(q; q)_k}, \quad (5)$$

with $t = -\zeta_m^j q^n$ in the definition of $\varepsilon_m(q)$, we get

$$\varepsilon_m(q) = (q; q)_\infty \sum_{n=0}^{\infty} \frac{q^{mn}}{(q)_n} \sum_{k=0}^{\infty} \frac{(-1)^k \chi_m(k) q^{\binom{k+1}{2} + nk}}{(q; q)_k}. \quad (6)$$

Using Euler's identity (2) with $t = q^{k+m}$ after interchanging summations in (6), we have

$$\begin{aligned} \varepsilon_m(q) &= (q; q)_\infty \sum_{k=0}^{\infty} \frac{(-1)^k \chi_m(k) q^{\binom{k+1}{2}}}{(q; q)_k (q^{k+m}; q)_\infty} \\ &= \sum_{k=0}^{\infty} (-1)^k \chi_m(k) q^{\binom{k+1}{2}} (q^{k+1}; q)_{m-1}. \end{aligned} \quad (7)$$

Using the following finite analogue of q -binomial theorem

$$(z; q)_N = \sum_{j=0}^N \begin{bmatrix} N \\ j \end{bmatrix}_q (-1)^j z^j q^{\frac{j(j-1)}{2}}, \quad (8)$$

with the substitutions $N = m - 1$ and $z = q^{k+1}$ in (8), from (7) we get

$$\begin{aligned} \varepsilon_m(q) &= \sum_{k=0}^{\infty} (-1)^k \chi_m(k) q^{\binom{k+1}{2}} \sum_{j=0}^{m-1} (-1)^j \begin{bmatrix} m-1 \\ j \end{bmatrix}_q q^{\binom{j+1}{2} + kj} \\ &= \sum_{k=0}^{\infty} (-1)^k \chi_m(k) \sum_{j=0}^{m-1} (-1)^j \begin{bmatrix} m-1 \\ j \end{bmatrix}_q q^{\binom{j+k+1}{2}} \\ &= \sum_{j=0}^{m-1} (-1)^j \begin{bmatrix} m-1 \\ j \end{bmatrix}_q \sum_{k=0}^{\infty} (-1)^k \chi_m(k) q^{\binom{j+k+1}{2}} \\ &= \sum_{j=0}^{m-1} \begin{bmatrix} m-1 \\ j \end{bmatrix}_q \sum_{k=j}^{\infty} (-1)^k \chi_m(k-j) q^{\binom{k+1}{2}} \\ &= - \sum_{j=0}^{m-1} \begin{bmatrix} m-1 \\ j \end{bmatrix}_q \sum_{k=0}^{j-1} (-1)^k \chi_m(k-j) q^{\binom{k+1}{2}} \\ &\quad + \sum_{j=0}^{m-1} \begin{bmatrix} m-1 \\ j \end{bmatrix}_q \sum_{k=0}^{\infty} (-1)^k \chi_m(k-j) q^{\binom{k+1}{2}} \end{aligned}$$

$$\begin{aligned}
&= - \sum_{j=0}^{m-1} \left[\begin{matrix} m-1 \\ j \end{matrix} \right]_q \sum_{k=0}^{j-1} (-1)^k \chi_m(k-j) q^{\binom{k+1}{2}} \\
&\quad + \sum_{k=0}^{\infty} (-1)^k q^{\binom{k+1}{2}} \sum_{j=0}^{m-1} \chi_m(k-j) \left[\begin{matrix} m-1 \\ j \end{matrix} \right]_q \\
&= - \sum_{j=0}^{m-1} \left[\begin{matrix} m-1 \\ j \end{matrix} \right]_q \sum_{k=0}^{j-1} (-1)^k \chi_m(k-j) q^{\binom{k+1}{2}} \\
&\quad + \sum_{k=0}^{\infty} (-1)^k q^{\binom{k+1}{2}} \sum_{j=0}^{m-1} \chi_m(k-j) \left(\left[\begin{matrix} m-1 \\ j \end{matrix} \right]_q - 1 \right),
\end{aligned}$$

where we have used the fact that $\sum_{j=0}^{m-1} \chi_m(k-j) = 0$ in the second sum on the last line above. Hence, we have

$$\epsilon_m(q) = P_m(q) + \sum_{k=0}^{\infty} (-1)^k q^{\binom{k+1}{2}} \sum_{j=0}^{m-1} \chi_m(k-j) \left(\left[\begin{matrix} m-1 \\ j \end{matrix} \right]_q - 1 \right),$$

where $P_m(q) := - \sum_{j=0}^{m-1} \left[\begin{matrix} m-1 \\ j \end{matrix} \right]_q \sum_{k=0}^{j-1} (-1)^k \chi_m(k-j) q^{\binom{k+1}{2}}$.

The case $m = 3$

The complexity of the generating functions involved seemed to mitigate against an analog of the $C(n) = \frac{1}{2}D(n)$ portion of Theorem AKY; however, surprisingly, a similar result exists for $m = 3$.

Corollary AD4 (2025)

Let n be any positive integer that is not equal to a triangular number plus 1, then

$$C_3(n) = \frac{1}{3}D_3(n).$$

An Example

For example, if $n = 8$, we have

$C_3(8)$	$D_3(8)$
$6 + 2$	$0 + 0 + 0 + 8$
$6 + 1 + 1$	$0 + 0 + 0 + 7 + 1$
$3 + 3 + 2$	$0 + 0 + 0 + 6 + 2$
$3 + 3 + 1 + 1$	$0 + 0 + 0 + 6 + 1 + 1$
$3 + 2 + 2 + 1$	$0 + 0 + 0 + 5 + 3$
	$0 + 0 + 0 + 5 + 2 + 1$
	$0 + 0 + 0 + 4 + 4$
	$0 + 0 + 0 + 4 + 3 + 1$
	$0 + 0 + 0 + 4 + 2 + 2$
	$0 + 0 + 0 + 4 + 2 + 1 + 1$
	$0 + 0 + 0 + 3 + 3 + 2$
	$0 + 0 + 0 + 3 + 3 + 1 + 1$
	$0 + 0 + 0 + 3 + 2 + 2 + 1$
	$1 + 1 + 1 + 5$
	$1 + 1 + 1 + 2 + 3$

What happens in general?

We obtain an example of an *almost partition identity* (a partition identity that is true asymptotically 100% of the time and fails infinitely often).

Corollary AD5 (2025)

The identity

$$C_m(n) = \frac{1}{m} D_m(n) \quad (9)$$

is true for almost all n . Moreover, if $N(x)$ is the number of times that (9) is true for $n < x$, then

$$\lim_{x \rightarrow \infty} \frac{N(x)}{x} = 1.$$

Proof of Corollary AD5

The polynomial

$$\sum_{j=0}^{m-1} \chi_m(k-j) \left(\left[\begin{matrix} m-1 \\ j \end{matrix} \right]_q - 1 \right)$$

in $\varepsilon_m(q)$ has at most $2^{m-1} - m$ terms. Now, there are at most $\sqrt{2x} + 1$ triangular numbers less than x . Hence, the maximum possible number of non-zero coefficients in $\varepsilon_m(q)$ among the first x terms is at most $(2^{m-1} - m) \cdot (\sqrt{2x} + 1)$ leaving the number of coefficients that vanish as less than

$$x - (2^{m-1} - m) \cdot (\sqrt{2x} + 1)$$

which is asymptotic to x as $x \rightarrow \infty$. This shows that $E_m(n)$ is almost always zero.



Concluding remarks

We also obtain the following finite q -series identity.

Theorem AD6 (2025)

For any positive integer N , we have

$$1 + \sum_{n=1}^N \sum_{j=1}^{m-1} \frac{q^{mn-j}}{\left(\prod_{r=1}^{m-j} (q^r; q^m)_n \right) \cdot \left(\prod_{r=m-j+1}^{m-1} (q^r; q^m)_{n-1} \right)} = \frac{(q^m; q^m)_N}{(q; q)_{mN}}. \quad (10)$$

Letting $N \rightarrow \infty$ in (2) gives us the following q -series identity.

Corollary AD7 (2025)

$$1 + \sum_{n=1}^{\infty} \sum_{j=1}^{m-1} \frac{q^{mn-j}}{\left(\prod_{r=1}^{m-j} (q^r; q^m)_n \right) \cdot \left(\prod_{r=m-j+1}^{m-1} (q^r; q^m)_{n-1} \right)} = \frac{(q^m; q^m)_{\infty}}{(q; q)_{\infty}}. \quad (11)$$

- The $m = 2$ case of (11) above was recently proved by Andrews and Bachraoui in 2025. Clearly, the series on the left-hand side of (11) is the generating function of $\sum_{j=1}^{m-1} B_m^{(j)}(n)$ and the product on the right-hand side of (11) is the generating function of $B_m(n)$. The same observation (restricting all parts to be at most mN) proves (10).
- It would be very interesting to provide bijective proofs of $A_3(n-1) = \frac{1}{3}D_3(n)$ and $C_3(n) = \frac{1}{3}D_3(n)$ for $n \neq T_k + 1$ where $T_k = \frac{k(k+1)}{2}$.
- In 2019, Andrews and Ballantine proved an example of an *almost partition identity*, namely that the number of parts in all self-conjugate partitions of n is almost always equal to the number of partitions of n in which no odd part is repeated and there is exactly one even part (possibly repeated).

Somewhat in the shadow of partition identities exists related results, which we call almost partition identities. The first of these is Legendre's interpretation of Euler's Pentagonal Number Theorem (2):

$$p(n \mid \text{even \# of parts, all distinct}) = p(n \mid \text{odd \# of parts, all distinct}), \quad [2]$$

except when $n = j(3j \pm 1)/2$, in which case the two sides differ by ± 1 . Thus, [2] is an almost partition identity (i.e., [2] is asymptotically valid 100% of the time), failing for $n \in [1, N]$ on the order of \sqrt{N} times.

Part of the reason that almost partition identities attract little interest is that, until now, they have been immediate corollaries of a q series or theta series identity and are of the form

$$p(n \mid \text{even number of parts subject to } X) = p(n \mid \text{odd number of parts subject to } X). \quad [3]$$

References



G. E. Andrews, *The Theory of Partitions*, Cambridge University Press (1998).



G. E. Andrews, M. E. Bachraoui, On the generating functions for partitions with repeated smallest part, *J. Math. Anal. Appl.* **549** (2025), no. 1, Paper No. 129537, 16 pp.



G. E. Andrews, C. Ballantine, Almost partition identities, *Proc. Natl. Acad. Sci. USA* **116** (12) (2019), 5428–5436.



G. E. Andrews, R. Kumar, A. J. Yee, On Euler's Partition Theorem, *Frontiers in Combinatorics and Number Theory*, **1** (2026), pp. 26–30.



J. W. L. Glaisher, A theorem in partitions, *Messenger of Math.* **12** (1883), 158–170.



H. Lin, W. J. T. Zhang, Generalizations of Euler's Theorem to k -regular partitions, preprint, arXiv:2511.14594.

Thank You!

(This work is based on arXiv:2512.12346 [math.CO])