# The Rogers-Ramanujan dissection of a theta function


Atul Dixit

Indian Institute of Technology Gandhinagar


(Joint work with Gaurav Kumar)

Seminar in Partition Theory and $q$-series

January 30, 2025

- For $a \in \mathbb{C}\backslash\{0\}$, and $b \in \mathbb{C}$,

$$\sum_{m=0}^{\infty} \frac{a^{-2m} q^{m^2}}{(bq)_m} \sum_{n=0}^{\infty} \frac{a^n b^n q^{n^2/4}}{(q)_n} + \sum_{m=0}^{\infty} \frac{a^{-2m-1} q^{m^2+m}}{(bq)_m} \sum_{n=0}^{\infty} \frac{a^n b^n q^{(n+1)^2/4}}{(q)_n}$$
$$= \frac{1}{(bq)_\infty} \sum_{n=-\infty}^{\infty} a^n q^{n^2/4} - (1-b) \sum_{n=1}^{\infty} a^n q^{n^2/4} \sum_{\ell=0}^{n-1} \frac{b^\ell}{(q)_\ell}. \qquad (1)$$

## Page 27 of Ramanujan's Lost Notebook

- For $a \in \mathbb{C}\backslash\{0\}$, and $b \in \mathbb{C}$,

$$\sum_{m=0}^{\infty} \frac{a^{-2m} q^{m^2}}{(bq)_m} \sum_{n=0}^{\infty} \frac{a^n b^n q^{n^2/4}}{(q)_n} + \sum_{m=0}^{\infty} \frac{a^{-2m-1} q^{m^2+m}}{(bq)_m} \sum_{n=0}^{\infty} \frac{a^n b^n q^{(n+1)^2/4}}{(q)_n}$$
$$= \frac{1}{(bq)_\infty} \sum_{n=-\infty}^{\infty} a^n q^{n^2/4} - (1-b) \sum_{n=1}^{\infty} a^n q^{n^2/4} \sum_{\ell=0}^{n-1} \frac{b^\ell}{(q)_\ell}. \qquad (1)$$

- **(Andrews, *Partitions: Yesterday and Today*)**
  Let $a = b = 1$, replace $q$ by $q^4$, multiply both sides of the resulting identity by $(-q^2; q^2)_\infty$, and use Rogers' identities

- For $a \in \mathbb{C}\backslash\{0\}$, and $b \in \mathbb{C}$,

$$\sum_{m=0}^{\infty}\frac{a^{-2m}q^{m^2}}{(bq)_m}\sum_{n=0}^{\infty}\frac{a^nb^nq^{n^2/4}}{(q)_n}+\sum_{m=0}^{\infty}\frac{a^{-2m-1}q^{m^2+m}}{(bq)_m}\sum_{n=0}^{\infty}\frac{a^nb^nq^{(n+1)^2/4}}{(q)_n}$$
$$=\frac{1}{(bq)_\infty}\sum_{n=-\infty}^{\infty}a^nq^{n^2/4}-(1-b)\sum_{n=1}^{\infty}a^nq^{n^2/4}\sum_{\ell=0}^{n-1}\frac{b^\ell}{(q)_\ell}. \tag{1}$$

- **(Andrews, *Partitions: Yesterday and Today*)** Let $a=b=1$, replace $q$ by $q^4$, multiply both sides of the resulting identity by $(-q^2;q^2)_\infty$, and use Rogers' identities

$$G(q)=(-q^2;q^2)_\infty\sum_{n=0}^{\infty}\frac{q^{n^2}}{(q^4;q^4)_n},\quad H(q)=(-q^2;q^2)_\infty\sum_{n=0}^{\infty}\frac{q^{n^2+2n}}{(q^4;q^4)_n},$$

where $G(q):=\sum_{n=0}^{\infty}\frac{q^{n^2}}{(q)_n}$ and $H(q):=\sum_{n=0}^{\infty}\frac{q^{n^2+n}}{(q)_n}$.

# Why is the identity interesting?

- This leads to the modular relation

$$G(q)G(q^4) + qH(q)H(q^4) = \frac{\varphi(q)}{(q^2;q^2)_\infty} = (-q;q^2)_\infty^2, \tag{2}$$

where $\varphi(q) := \sum_{n=-\infty}^{\infty} q^{n^2}$ is the Jacobi theta function.

# Why is the identity interesting?

- This leads to the modular relation

$$G(q)G(q^4) + qH(q)H(q^4) = \frac{\varphi(q)}{(q^2;q^2)_\infty} = (-q;q^2)_\infty^2, \tag{2}$$

where $\varphi(q) := \sum_{n=-\infty}^{\infty} q^{n^2}$ is the Jacobi theta function.

- Regarding (2), Andrews says (in his prefatorial introduction to the Lost Notebook):

*This sort of identity has always appeared to me to lie totally within the realm of modular functions and to be completely resistant to q-series generalization. One of the greatest shocks I got from the Lost Notebook was the following assertion...*

# Another special case of the generalized modular relation

- Letting $a = b = -1$ in (1), replacing $q$ by $q^4$, multiplying the resulting two sides by $(-q^2; q^2)_\infty$, and then using Rogers' identities yields

[1] G.E. Andrews, *Partitions: Yesterday and Today*, New Zealand Math. Soc., Wellington, 1979.

# Another special case of the generalized modular relation

- Letting $a = b = -1$ in (1), replacing $q$ by $q^4$, multiplying the resulting two sides by $(-q^2;q^2)_\infty$, and then using Rogers' identities yields

$$G(q)f_0(q^4) - qH(q)f_1(q^4) \\ = (-q^2;q^4)_\infty \sum_{n=-\infty}^{\infty} (-1)^n q^{n^2} + 2(-q^2;q^2)_\infty \sum_{n,\ell \geq 0} \frac{(-1)^n q^{(n+\ell+1)^2}}{(q^4;q^4)_\ell}, \tag{3}$$

where $f_0(q) := \sum_{n=0}^{\infty} \frac{q^{n^2}}{(-q;q)_n}$ and $f_1(q) := \sum_{n=0}^{\infty} \frac{q^{n^2+n}}{(-q;q)_n}$ are two fifth order mock theta functions of Ramanujan.

[1] G.E. Andrews, *Partitions: Yesterday and Today*, New Zealand Math. Soc., Wellington, 1979.

# Another special case of the generalized modular relation

- Letting $a=b=-1$ in (1), replacing $q$ by $q^4$, multiplying the resulting two sides by $(-q^2;q^2)_\infty$, and then using Rogers' identities yields

$$G(q)f_0(q^4)-qH(q)f_1(q^4) \tag{3}$$
$$=(-q^2;q^4)_\infty\sum_{n=-\infty}^{\infty}(-1)^nq^{n^2}+2(-q^2;q^2)_\infty\sum_{n,\ell\geq 0}\frac{(-1)^nq^{(n+\ell+1)^2}}{(q^4;q^4)_\ell},$$

where $f_0(q):=\sum_{n=0}^{\infty}\frac{q^{n^2}}{(-q;q)_n}$ and $f_1(q):=\sum_{n=0}^{\infty}\frac{q^{n^2+n}}{(-q;q)_n}$ are two fifth order mock theta functions of Ramanujan.

- Andrews[1] says, '*Nothing like* (3) *appears in any of the literature on mock theta functions.*

[1] G.E. Andrews, *Partitions: Yesterday and Today*, New Zealand Math. Soc., Wellington, 1979.

# Ramanujan's generalized modular relation

Recall that for $a \in \mathbb{C}\backslash\{0\}$, and $b \in \mathbb{C}$,

$$\sum_{m=0}^{\infty} \frac{a^{-2m} q^{m^2}}{(bq)_m} \sum_{n=0}^{\infty} \frac{a^n b^n q^{n^2/4}}{(q)_n} + \sum_{m=0}^{\infty} \frac{a^{-2m-1} q^{m^2+m}}{(bq)_m} \sum_{n=0}^{\infty} \frac{a^n b^n q^{(n+1)^2/4}}{(q)_n}$$
$$= \frac{1}{(bq)_\infty} \sum_{n=-\infty}^{\infty} a^n q^{n^2/4} - (1-b) \sum_{n=1}^{\infty} a^n q^{n^2/4} \sum_{\ell=0}^{n-1} \frac{b^\ell}{(q)_\ell}.$$

[2] A. Dixit and G. Kumar, *The Rogers-Ramanujan dissection of a theta function*, submitted for publication.

# Ramanujan's generalized modular relation

Recall that for $a \in \mathbb{C}\backslash\{0\}$, and $b \in \mathbb{C}$,

$$\sum_{m=0}^{\infty} \frac{a^{-2m} q^{m^2}}{(bq)_m} \sum_{n=0}^{\infty} \frac{a^n b^n q^{n^2/4}}{(q)_n} + \sum_{m=0}^{\infty} \frac{a^{-2m-1} q^{m^2+m}}{(bq)_m} \sum_{n=0}^{\infty} \frac{a^n b^n q^{(n+1)^2/4}}{(q)_n}$$
$$= \frac{1}{(bq)_\infty} \sum_{n=-\infty}^{\infty} a^n q^{n^2/4} - (1-b) \sum_{n=1}^{\infty} a^n q^{n^2/4} \sum_{\ell=0}^{n-1} \frac{b^\ell}{(q)_\ell}.$$

- Andrews proved this identity by showing that the coefficients of $a^N, -\infty < N < \infty$, on both sides are identical.

[2] A. Dixit and G. Kumar, *The Rogers-Ramanujan dissection of a theta function*, submitted for publication.

# Ramanujan's generalized modular relation

Recall that for $a \in \mathbb{C}\backslash\{0\}$, and $b \in \mathbb{C}$,

$$\sum_{m=0}^{\infty} \frac{a^{-2m}q^{m^2}}{(bq)_m} \sum_{n=0}^{\infty} \frac{a^n b^n q^{n^2/4}}{(q)_n} + \sum_{m=0}^{\infty} \frac{a^{-2m-1}q^{m^2+m}}{(bq)_m} \sum_{n=0}^{\infty} \frac{a^n b^n q^{(n+1)^2/4}}{(q)_n}$$
$$= \frac{1}{(bq)_\infty} \sum_{n=-\infty}^{\infty} a^n q^{n^2/4} - (1-b) \sum_{n=1}^{\infty} a^n q^{n^2/4} \sum_{\ell=0}^{n-1} \frac{b^\ell}{(q)_\ell}.$$

- Andrews proved this identity by showing that the coefficients of $a^N, -\infty < N < \infty$, on both sides are identical.
- However, this requires knowing the identity in advance.

[2] A. Dixit and G. Kumar, *The Rogers-Ramanujan dissection of a theta function*, submitted for publication.

# Ramanujan's generalized modular relation

Recall that for $a \in \mathbb{C}\backslash\{0\}$, and $b \in \mathbb{C}$,

$$\sum_{m=0}^{\infty} \frac{a^{-2m}q^{m^2}}{(bq)_m} \sum_{n=0}^{\infty} \frac{a^n b^n q^{n^2/4}}{(q)_n} + \sum_{m=0}^{\infty} \frac{a^{-2m-1}q^{m^2+m}}{(bq)_m} \sum_{n=0}^{\infty} \frac{a^n b^n q^{(n+1)^2/4}}{(q)_n}$$
$$= \frac{1}{(bq)_\infty} \sum_{n=-\infty}^{\infty} a^n q^{n^2/4} - (1-b) \sum_{n=1}^{\infty} a^n q^{n^2/4} \sum_{\ell=0}^{n-1} \frac{b^\ell}{(q)_\ell}.$$

- Andrews proved this identity by showing that the coefficients of $a^N, -\infty < N < \infty$, on both sides are identical.
- However, this requires knowing the identity in advance.
- One of the goals of our work[2] was to see if a natural proof of (1) could be obtained.

[2] A. Dixit and G. Kumar, *The Rogers-Ramanujan dissection of a theta function*, submitted for publication.

# A question on page 26 of the Lost Notebook

- Ramanujan wrote

$$\sum_{n=0}^{\infty} \frac{a^n q^{n^2/(2s)}}{(q)_n} \sum_{n=0}^{\infty} \frac{a^{-ns} q^{n^2 s/2}}{(q)_n} \qquad \text{as } q \to 1?? \tag{4}$$

# A question on page 26 of the Lost Notebook

- Ramanujan wrote

$$\sum_{n=0}^{\infty} \frac{a^n q^{n^2/(2s)}}{(q)_n} \sum_{n=0}^{\infty} \frac{a^{-ns} q^{n^2 s/2}}{(q)_n} \qquad \text{as } q \to 1?? \tag{4}$$

- When $s = 2$, the above product is the special case $b = 1$ of

$$\sum_{m=0}^{\infty} \frac{a^{-2m} q^{m^2}}{(bq)_m} \sum_{n=0}^{\infty} \frac{a^n b^n q^{n^2/4}}{(q)_n}.$$

# A question on page 26 of the Lost Notebook

- Ramanujan wrote

$$\sum_{n=0}^{\infty} \frac{a^n q^{n^2/(2s)}}{(q)_n} \sum_{n=0}^{\infty} \frac{a^{-ns} q^{n^2 s/2}}{(q)_n} \qquad \text{as } q \to 1?? \tag{4}$$

- When $s = 2$, the above product is the special case $b = 1$ of

$$\sum_{m=0}^{\infty} \frac{a^{-2m} q^{m^2}}{(bq)_m} \sum_{n=0}^{\infty} \frac{a^n b^n q^{n^2/4}}{(q)_n}.$$

- Andrews and Berndt remark, *'Ramanujan provides no indication either why this is of interest for arbitrary s or what the asymptotics should be.'*

# Second goal

- Among other things, we answer Ramanujan's question:

**Theorem (D. - Kumar (2024))**

*Let $a > 0$ and $s \in \mathbb{N}$. Let $z_1$ denote the real positive root of $az^{1/s} + z - 1 = 0$. As $q \to 1^-$,*

$$\sum_{n=0}^{\infty} \frac{a^n q^{n^2/(2s)}}{(q)_n} \sum_{n=0}^{\infty} \frac{a^{-ns} q^{n^2 s/2}}{(q)_n} \sim \frac{\sqrt{s}}{1+(s-1)z_1} \exp\left(\frac{-1}{\log(q)}\left(\frac{\pi^2}{6} + \frac{s}{2}\log^2(a)\right)\right).$$

# Second goal

- Among other things, we answer Ramanujan's question:

**Theorem (D. - Kumar (2024))**

*Let $a > 0$ and $s \in \mathbb{N}$. Let $z_1$ denote the real positive root of $az^{1/s} + z - 1 = 0$. As $q \to 1^-$,*

$$\sum_{n=0}^{\infty} \frac{a^n q^{n^2/(2s)}}{(q)_n} \sum_{n=0}^{\infty} \frac{a^{-ns} q^{n^2 s/2}}{(q)_n} \sim \frac{\sqrt{s}}{1+(s-1)z_1} \exp\left(\frac{-1}{\log(q)}\left(\frac{\pi^2}{6} + \frac{s}{2}\log^2(a)\right)\right).$$

- But the very fact that Ramanujan considers

$$\sum_{n=0}^{\infty} \frac{a^n q^{n^2/(2s)}}{(q)_n} \sum_{n=0}^{\infty} \frac{a^{-ns} q^{n^2 s/2}}{(q)_n}$$

also raises an important question - *is there a generalization of* (1) *which reduces to it when $s = 2$?*

# Main theorem

**Theorem (D. - Kumar (2024))**

*Let $s \in \mathbb{N}$. For $a \in \mathbb{C}$, $a \neq 0$, and $b \in \mathbb{C}$,*

$$\sum_{k=0}^{s-1}\left\{\sum_{m=0}^{\infty}\frac{a^{-sm-k}q^{(sm+k)^2/(2s)}}{(bq)_m}\sum_{n=0}^{\infty}\frac{a^nb^nq^{n(n+2js-2k)/(2s)}}{(q)_n}\right\}$$
$$=\frac{1}{(bq)_\infty}\sum_{n=-\infty}^{\infty}a^nq^{n^2/(2s)}-(1-b)\sum_{n=1}^{\infty}a^nq^{n^2/(2s)}\sum_{\ell=0}^{n-1}\frac{b^\ell}{(q)_\ell}, \tag{5}$$

*where*

$$j=\begin{cases}0, & \textit{if } k=0,\\ 1, & \textit{otherwise.}\end{cases}$$

# Comparison with other formulas for theta functions

- There are several formulas in the literature which express a product of theta function with a $q$-series or another theta function as a finite sum of theta functions or their products or powers.

# Comparison with other formulas for theta functions

- There are several formulas in the literature which express a product of theta function with a $q$-series or another theta function as a finite sum of theta functions or their products or powers.
- For example, Schröter's formula, Ramanujan's circular summation of theta functions.

# Comparison with other formulas for theta functions

- There are several formulas in the literature which express a product of theta function with a $q$-series or another theta function as a finite sum of theta functions or their products or powers.
- For example, Schröter's formula, Ramanujan's circular summation of theta functions.
- The series in our result are generalized Rogers-Ramanujan functions of the form

$$\sum_{n=0}^{\infty} \frac{a^n q^{(cn^2+dn)/s}}{(bq)_n},$$

where $a, b \in \mathbb{C}$, $c, d \in \mathbb{R}$, and $s \in \mathbb{N}$.

**Theorem (D. - Kumar (2024))**

*For* $s \in \mathbb{N}$,

$$\sum_{k=0}^{s-1}\left\{\sum_{m=0}^{\infty}\frac{a^{-sm-k}q^{(sm+k)^2/(2s)}}{(bq)_m}\sum_{n=sm+k+1}^{\infty}\frac{a^nb^nq^{n(n+2js-2k)/(2s)}}{(q)_n}\right\}$$

$$=\frac{1}{(bq)_\infty}\sum_{n=1}^{\infty}a^nq^{n^2/(2s)}-(1-b)\sum_{n=1}^{\infty}a^nq^{n^2/(2s)}\sum_{\ell=0}^{n-1}\frac{b^\ell}{(q)_\ell}.$$

# Representations for partial theta functions

**Theorem (D. - Kumar (2024))**

*For* $s \in \mathbb{N}$,

$$\sum_{k=0}^{s-1}\left\{\sum_{m=0}^{\infty}\frac{a^{-sm-k}q^{(sm+k)^2/(2s)}}{(bq)_m}\sum_{n=0}^{sm+k}\frac{a^nb^nq^{n(n+2js-2k)/(2s)}}{(q)_n}\right\}$$
$$=\frac{1}{(bq)_\infty}\sum_{n=-\infty}^{0}a^nq^{n^2/(2s)}.$$

# The Rogers-Ramanujan dissection of a theta function

$$\sum_{k=0}^{s-1}\left\{\sum_{m=0}^{\infty}\frac{a^{-sm-k}q^{(sm+k)^2/(2s)}}{(bq)_m}\sum_{n=0}^{\infty}\frac{a^nb^nq^{n(n+2js-2k)/(2s)}}{(q)_n}\right\}$$
$$=\frac{1}{(bq)_\infty}\sum_{n=-\infty}^{\infty}a^nq^{n^2/(2s)}-(1-b)\sum_{n=1}^{\infty}a^nq^{n^2/(2s)}\sum_{\ell=0}^{n-1}\frac{b^\ell}{(q)_\ell},$$

where $j=0$ if $k=0$, and 1 otherwise.

# The Rogers-Ramanujan dissection of a theta function

$$\sum_{k=0}^{s-1}\left\{\sum_{m=0}^{\infty}\frac{a^{-sm-k}q^{(sm+k)^2/(2s)}}{(bq)_m}\sum_{n=0}^{\infty}\frac{a^nb^nq^{n(n+2js-2k)/(2s)}}{(q)_n}\right\}$$
$$=\frac{1}{(bq)_\infty}\sum_{n=-\infty}^{\infty}a^nq^{n^2/(2s)}-(1-b)\sum_{n=1}^{\infty}a^nq^{n^2/(2s)}\sum_{\ell=0}^{n-1}\frac{b^\ell}{(q)_\ell},$$

where $j=0$ if $k=0$, and 1 otherwise.

- The special case $b=1$ of this identity is what we call *the Rogers-Ramanujan dissection of the theta function* $\sum_{n=-\infty}^{\infty}a^nq^{n^2/(2s)}$.

# The Rogers-Ramanujan dissection of a theta function

$$\sum_{k=0}^{s-1}\left\{\sum_{m=0}^{\infty}\frac{a^{-sm-k}q^{(sm+k)^2/(2s)}}{(bq)_m}\sum_{n=0}^{\infty}\frac{a^nb^nq^{n(n+2js-2k)/(2s)}}{(q)_n}\right\}$$
$$=\frac{1}{(bq)_\infty}\sum_{n=-\infty}^{\infty}a^nq^{n^2/(2s)}-(1-b)\sum_{n=1}^{\infty}a^nq^{n^2/(2s)}\sum_{\ell=0}^{n-1}\frac{b^\ell}{(q)_\ell},$$

where $j=0$ if $k=0$, and 1 otherwise.

- The special case $b=1$ of this identity is what we call *the Rogers-Ramanujan dissection of the theta function* $\sum_{n=-\infty}^{\infty}a^nq^{n^2/(2s)}$.
- Clearly, Ramanujan's result is the special case $s=2$ of the above.

# A generalization of the Jacobi triple product identity

- Letting $s = 1$ in the main result leads to

**Corollary**

*For $a \in \mathbb{C}$, $a \neq 0$, and $b \in \mathbb{C}$,*

$$\sum_{m=0}^{\infty} \frac{a^{-m} q^{m^2/2}}{(bq)_m} \sum_{n=0}^{\infty} \frac{a^n b^n q^{n^2/2}}{(q)_n}$$

$$= \frac{1}{(bq)_\infty} \sum_{n=-\infty}^{\infty} a^n q^{n^2/2} - (1-b) \sum_{n=1}^{\infty} a^n q^{n^2/2} \sum_{\ell=0}^{n-1} \frac{b^\ell}{(q)_\ell}.$$

# A generalization of the Jacobi triple product identity

- Letting $s = 1$ in the main result leads to

**Corollary**

*For $a \in \mathbb{C}$, $a \neq 0$, and $b \in \mathbb{C}$,*

$$\sum_{m=0}^{\infty} \frac{a^{-m} q^{m^2/2}}{(bq)_m} \sum_{n=0}^{\infty} \frac{a^n b^n q^{n^2/2}}{(q)_n}$$

$$= \frac{1}{(bq)_\infty} \sum_{n=-\infty}^{\infty} a^n q^{n^2/2} - (1-b) \sum_{n=1}^{\infty} a^n q^{n^2/2} \sum_{\ell=0}^{n-1} \frac{b^\ell}{(q)_\ell}.$$

- Letting $b = 1$ and replacing $q$ by $q^2$ leads to the Jacobi triple product identity.

# The special case $s = 3$

## Corollary

*For $a \in \mathbb{C}\backslash\{0\}$ and $b \in \mathbb{C}$,*

$$\begin{aligned}
&\sum_{m=0}^{\infty} \frac{a^{-3m} q^{3m^2/2}}{(bq)_m} \sum_{n=0}^{\infty} \frac{a^n b^n q^{n^2/6}}{(q)_n} \\
&+ q^{1/6} \sum_{m=0}^{\infty} \frac{a^{-3m-1} q^{3m^2/2+m}}{(bq)_m} \sum_{n=0}^{\infty} \frac{a^n b^n q^{n(n+4)/6}}{(q)_n} \\
&+ q^{2/3} \sum_{m=0}^{\infty} \frac{a^{-3m-2} q^{3m^2/2+2m}}{(bq)_m} \sum_{n=0}^{\infty} \frac{a^n b^n q^{n(n+2)/6}}{(q)_n} \\
&= \frac{1}{(bq)_\infty} \sum_{n=-\infty}^{\infty} a^n q^{n^2/6} - (1-b) \sum_{n=1}^{\infty} a^n q^{n^2/6} \sum_{\ell=0}^{n-1} \frac{b^\ell}{(q)_\ell}.
\end{aligned} \tag{6}$$

# The special case $s = 3$

## Corollary

*For $a \in \mathbb{C}\backslash\{0\}$ and $b \in \mathbb{C}$,*

$$\begin{aligned}
&\sum_{m=0}^{\infty} \frac{a^{-3m} q^{3m^2/2}}{(bq)_m} \sum_{n=0}^{\infty} \frac{a^n b^n q^{n^2/6}}{(q)_n} \\
&+ q^{1/6} \sum_{m=0}^{\infty} \frac{a^{-3m-1} q^{3m^2/2+m}}{(bq)_m} \sum_{n=0}^{\infty} \frac{a^n b^n q^{n(n+4)/6}}{(q)_n} \\
&+ q^{2/3} \sum_{m=0}^{\infty} \frac{a^{-3m-2} q^{3m^2/2+2m}}{(bq)_m} \sum_{n=0}^{\infty} \frac{a^n b^n q^{n(n+2)/6}}{(q)_n} \\
&= \frac{1}{(bq)_\infty} \sum_{n=-\infty}^{\infty} a^n q^{n^2/6} - (1-b) \sum_{n=1}^{\infty} a^n q^{n^2/6} \sum_{\ell=0}^{n-1} \frac{b^\ell}{(q)_\ell}.
\end{aligned} \tag{6}$$

If we let $a = b = 1$, do we get an analogue of

$$G(q)G(q^4) + qH(q)H(q^4) = \frac{\varphi(q)}{(q^2;q^2)_\infty} = (-q;q^2)_\infty^2? \tag{7}$$

# Unlikelihood of such relations for $s > 2$

- The key ingredients to get (7) from (1) were Rogers' identities

$$G(q) = (-q^2; q^2)_\infty \sum_{n=0}^{\infty} \frac{q^{n^2}}{(q^4; q^4)_n}, \quad H(q) = (-q^2; q^2)_\infty \sum_{n=0}^{\infty} \frac{q^{n^2+2n}}{(q^4; q^4)_n}.$$

# Unlikelihood of such relations for $s > 2$

- The key ingredients to get (7) from (1) were Rogers' identities

$$G(q) = (-q^2;q^2)_\infty \sum_{n=0}^{\infty} \frac{q^{n^2}}{(q^4;q^4)_n}, \quad H(q) = (-q^2;q^2)_\infty \sum_{n=0}^{\infty} \frac{q^{n^2+2n}}{(q^4;q^4)_n}.$$

- It is unlikely that analogues of these identities exist for $s > 2$.

# Unlikelihood of such relations for $s > 2$

- The key ingredients to get (7) from (1) were Rogers' identities

$$G(q) = (-q^2; q^2)_\infty \sum_{n=0}^{\infty} \frac{q^{n^2}}{(q^4; q^4)_n}, \quad H(q) = (-q^2; q^2)_\infty \sum_{n=0}^{\infty} \frac{q^{n^2+2n}}{(q^4; q^4)_n}.$$

- It is unlikely that analogues of these identities exist for $s > 2$.
- Bressoud's has not only generalized Rogers' first identity to have

$$\sum_{n=0}^{\infty} \frac{q^{n^2} a^n}{(q)_n} = (-aq^2; q^2)_\infty \sum_{n=0}^{\infty} \frac{q^{n^2} a^n}{(q^2; q^2)_n (-aq^2; q^2)_n}$$

but has, in turn, also generalized this identity for any $s \in \mathbb{N}$ so that

$$\sum_{m=0}^{\infty} \frac{q^{m+\frac{sm(m-1)}{2}} a^m}{(q)_m} = (-aq^s; q^s)_\infty \sum_{n_1, \cdots, n_{s-1} \geq 0} \frac{a^N q^{\frac{sN(N-1)}{2} + n_1 + 2n_2 + \cdots + (s-1)n_{s-1}}}{(q^s; q^s)_{n_1} \cdots (q^s; q^s)_{n_{s-1}} (-aq^s; q^s)_N} \tag{8}$$

where $N = n_1 + n_2 + \cdots + n_{s-1}$.

# Some more special cases of the main result

## Corollary

*Let* $\varphi(q) := \sum_{n=-\infty}^{\infty} q^{n^2}$. *Then,*

$$\sum_{m=0}^{\infty} \frac{(-1)^m q^{m^2}}{(-bq^2;q^2)_m} = \frac{\varphi(-q)}{(-bq)_\infty} + (1+b) \sum_{m=1}^{\infty} \frac{(-1)^{m-1} q^{m^2}}{(-bq;q^2)_m}.$$

*In particular,*

$$\phi(q) + 2\psi(q) = (-q;q^2)_\infty^3 (q^2;q^2)_\infty,$$

*where,* $\phi(q)$ *and* $\psi(q)$ *are two of Ramanujan's third order mock theta functions defined by*

$$\phi(q) := \sum_{m=0}^{\infty} \frac{q^{m^2}}{(-q^2;q^2)_m} \quad \text{and} \quad \psi(q) := \sum_{m=1}^{\infty} \frac{q^{m^2}}{(q;q^2)_m}.$$

# Annihilation of the theta function

- Let $a=-1$ in (1) and replace $q$ by $q^{1/(2s)}$ in $\sum_{n=-\infty}^{\infty}(-1)^n q^{n^2-n}=0.$

# Annihilation of the theta function

- Let $a=-1$ in (1) and replace $q$ by $q^{1/(2s)}$ in $\sum_{n=-\infty}^{\infty}(-1)^n q^{n^2-n}=0$.
- This annihilates the theta function and we get

$$\sum_{k=0}^{s-1}\left\{\sum_{m=0}^{\infty}\frac{(-1)^{sm-k}q^{(sm+k)(sm+k+1)/(2s)}}{(bq)_m}\sum_{n=0}^{\infty}\frac{(-b)^n q^{(n(n+2js-2k)-n)/(2s)}}{(q)_n}\right\}$$
$$=-(1-b)\sum_{n=1}^{\infty}(-1)^n q^{n(n-1)/(2s)}\sum_{\ell=0}^{n-1}\frac{b^{\ell}}{(q)_{\ell}}. \tag{9}$$

# Annihilation of the theta function

- Let $a = -1$ in (1) and replace $q$ by $q^{1/(2s)}$ in $\sum_{n=-\infty}^{\infty}(-1)^n q^{n^2-n} = 0$.

- This annihilates the theta function and we get

$$\sum_{k=0}^{s-1}\left\{\sum_{m=0}^{\infty}\frac{(-1)^{sm-k}q^{(sm+k)(sm+k+1)/(2s)}}{(bq)_m}\sum_{n=0}^{\infty}\frac{(-b)^n q^{(n(n+2js-2k)-n)/(2s)}}{(q)_n}\right\}$$
$$= -(1-b)\sum_{n=1}^{\infty}(-1)^n q^{n(n-1)/(2s)}\sum_{\ell=0}^{n-1}\frac{b^\ell}{(q)_\ell}. \tag{9}$$

- Now let $s = 2, b = -1$ and replace $q$ by $q^2$ in (9) so as to obtain

# A $q$-series of Ramanujan

## Corollary

$$\sum_{m=0}^{\infty}\frac{q^{m(2m+1)}}{(-q^2;q^2)_m}\sum_{n=0}^{\infty}\frac{q^{n(n-1)/2}}{(q^2;q^2)_n}-\sum_{m=0}^{\infty}\frac{q^{2m^2+3m+1}}{(-q^2;q^2)_m}\sum_{n=0}^{\infty}\frac{q^{n(n+1)/2}}{(q^2;q^2)_n}$$
$$=-2\sum_{n=1}^{\infty}(-1)^nq^{n(n-1)/2}\sum_{\ell=0}^{n-1}\frac{(-1)^{\ell}}{(q^2;q^2)_{\ell}}. \qquad (10)$$

[3] G. E. Andrews, *Ramanujan's "Lost" Notebook* IV. *Stacks and alternating parity in partitions*, Adv. Math. **53** (1984), 55–74.

# A $q$-series of Ramanujan

## Corollary

$$\sum_{m=0}^{\infty}\frac{q^{m(2m+1)}}{(-q^2;q^2)_m}\sum_{n=0}^{\infty}\frac{q^{n(n-1)/2}}{(q^2;q^2)_n}-\sum_{m=0}^{\infty}\frac{q^{2m^2+3m+1}}{(-q^2;q^2)_m}\sum_{n=0}^{\infty}\frac{q^{n(n+1)/2}}{(q^2;q^2)_n}$$
$$=-2\sum_{n=1}^{\infty}(-1)^n q^{n(n-1)/2}\sum_{\ell=0}^{n-1}\frac{(-1)^{\ell}}{(q^2;q^2)_{\ell}}. \qquad (10)$$

- Andrews[3] studied the function $\sum_{n=0}^{\infty} q^{\frac{n(n+1)}{2}}/(q^2;q^2)_n$ in conjunction with four identities from the Lost Notebook out of which two are quite difficult to prove.

[3] G. E. Andrews, *Ramanujan's "Lost" Notebook* IV. *Stacks and alternating parity in partitions*, Adv. Math. **53** (1984), 55–74.

# A $q$-series of Ramanujan

## Corollary

$$\sum_{m=0}^{\infty}\frac{q^{m(2m+1)}}{(-q^2;q^2)_m}\sum_{n=0}^{\infty}\frac{q^{n(n-1)/2}}{(q^2;q^2)_n}-\sum_{m=0}^{\infty}\frac{q^{2m^2+3m+1}}{(-q^2;q^2)_m}\sum_{n=0}^{\infty}\frac{q^{n(n+1)/2}}{(q^2;q^2)_n}$$

$$=-2\sum_{n=1}^{\infty}(-1)^n q^{n(n-1)/2}\sum_{\ell=0}^{n-1}\frac{(-1)^{\ell}}{(q^2;q^2)_{\ell}}. \qquad (10)$$

- Andrews[3] studied the function $\sum_{n=0}^{\infty} q^{\frac{n(n+1)}{2}}/(q^2;q^2)_n$ in conjunction with four identities from the Lost Notebook out of which two are quite difficult to prove.
- Two of the series are special cases of the series $\sum_{n=0}^{\infty} z^n q^{n^2/2}/(q^2;q^2)_n$ which is a $q$-analogue of Airy function and which arises as a hypergeometric solution of a $q$-Painlevé equation.

[3] G. E. Andrews, *Ramanujan's "Lost" Notebook* IV. *Stacks and alternating parity in partitions*, Adv. Math. **53** (1984), 55–74.

# A congruence implied by the above result

**Corollary**

$$\sum_{m=0}^{\infty}\frac{q^{m(2m+1)}}{(-q^2;q^2)_m}\left\{1+\sum_{n=1}^{\infty}\frac{q^{n(n-1)/2}}{(q^2;q^2)_n}-\frac{1}{(-q;-q)_\infty}\sum_{m=0}^{\infty}\frac{q^{2m^2+3m+1}}{(-q^2;q^2)_m}\right\}\equiv 0 \pmod 2. \tag{11}$$

# A congruence implied by the above result

## Corollary

$$\sum_{m=0}^{\infty}\frac{q^{m(2m+1)}}{(-q^2;q^2)_m}\left\{1+\sum_{n=1}^{\infty}\frac{q^{n(n-1)/2}}{(q^2;q^2)_n}-\frac{1}{(-q;-q)_\infty}\sum_{m=0}^{\infty}\frac{q^{2m^2+3m+1}}{(-q^2;q^2)_m}\right\}\equiv 0 \pmod 2. \tag{11}$$

- While the coefficients of the sum over $n$ form the sequence A179080 in OEIS, those of the other two (over $m$) are not even included there.

# A congruence implied by the above result

**Corollary**

$$\sum_{m=0}^{\infty}\frac{q^{m(2m+1)}}{(-q^2;q^2)_m}\left\{1+\sum_{n=1}^{\infty}\frac{q^{n(n-1)/2}}{(q^2;q^2)_n}-\frac{1}{(-q;-q)_\infty}\sum_{m=0}^{\infty}\frac{q^{2m^2+3m+1}}{(-q^2;q^2)_m}\right\}\equiv 0 \pmod 2. \tag{11}$$

- While the coefficients of the sum over $n$ form the sequence A179080 in OEIS, those of the other two (over $m$) are not even included there.
- All three sums seem to not have been studied before, but they, indeed, deserve a serious study.

# Further annihilation

We have

**Corollary**

$$\sum_{k=0}^{s-1}\left\{\sum_{m=0}^{\infty}\frac{(-1)^{sm-k}q^{(sm+k)(sm+k+1)/(2s)}}{(q)_m}\sum_{n=0}^{\infty}\frac{(-1)^n q^{n(n+2js-2k-1)/(2s)}}{(q)_n}\right\}=0. \tag{12}$$

# Further annihilation

We have

**Corollary**

$$\sum_{k=0}^{s-1}\left\{\sum_{m=0}^{\infty}\frac{(-1)^{sm-k}q^{(sm+k)(sm+k+1)/(2s)}}{(q)_m}\sum_{n=0}^{\infty}\frac{(-1)^{n}q^{n(n+2js-2k-1)/(2s)}}{(q)_n}\right\}=0. \tag{12}$$

- Letting $s=1$, we obtain the trivial identity

$$\sum_{m=0}^{\infty}\frac{(-1)^{m}q^{m(m+1)/2}}{(q)_m}\sum_{n=0}^{\infty}\frac{(-1)^{n}q^{n(n-1)/2}}{(q)_n}=0,$$

# Further annihilation

We have

**Corollary**

$$\sum_{k=0}^{s-1}\left\{\sum_{m=0}^{\infty}\frac{(-1)^{sm-k}q^{(sm+k)(sm+k+1)/(2s)}}{(q)_m}\sum_{n=0}^{\infty}\frac{(-1)^n q^{n(n+2js-2k-1)/(2s)}}{(q)_n}\right\}=0. \tag{12}$$

- Letting $s=1$, we obtain the trivial identity

$$\sum_{m=0}^{\infty}\frac{(-1)^m q^{m(m+1)/2}}{(q)_m}\sum_{n=0}^{\infty}\frac{(-1)^n q^{n(n-1)/2}}{(q)_n}=0,$$

- Letting $s=2$ in (12) and then replacing $q$ by $q^2$, we obtain

$$\sum_{m=0}^{\infty}\frac{q^{m(2m+1)}}{(q^2;q^2)_m}\sum_{n=0}^{\infty}\frac{(-1)^n q^{n(n-1)/2}}{(q^2;q^2)_n}=\sum_{m=0}^{\infty}\frac{q^{(m+1)(2m+1)}}{(q^2;q^2)_m}\sum_{n=0}^{\infty}\frac{(-1)^n q^{n(n+1)/2}}{(q^2;q^2)_n}.$$

# Further new results

It also follows easily by applying McIntosh's identity twice (here $\mu \in \mathbb{N}$):

$$\sum_{n=0}^{\infty} \frac{q^{(2n+\mu)(2n+\mu+1)/2}}{(q^2;q^2)_n} = (-q)_\infty \sum_{n=0}^{\infty} \frac{(-1)^n q^{n(n+1)/2-\mu n}}{(q^2;q^2)_n}.$$

# Further new results

It also follows easily by applying McIntosh's identity twice (here $\mu \in \mathbb{N}$):

$$\sum_{n=0}^{\infty} \frac{q^{(2n+\mu)(2n+\mu+1)/2}}{(q^2;q^2)_n} = (-q)_\infty \sum_{n=0}^{\infty} \frac{(-1)^n q^{n(n+1)/2-\mu n}}{(q^2;q^2)_n}.$$

The special cases for $s > 2$ are, to the best of our knowledge, new. For example, when $s = 3$,

# Further new results

It also follows easily by applying McIntosh's identity twice (here $\mu \in \mathbb{N}$):

$$\sum_{n=0}^{\infty} \frac{q^{(2n+\mu)(2n+\mu+1)/2}}{(q^2;q^2)_n} = (-q)_\infty \sum_{n=0}^{\infty} \frac{(-1)^n q^{n(n+1)/2-\mu n}}{(q^2;q^2)_n}.$$

The special cases for $s > 2$ are, to the best of our knowledge, new. For example, when $s = 3$,

$$\begin{aligned}\sum_{m=0}^{\infty} \frac{(-1)^m q^{m(3m+1)/2}}{(q)_m} \sum_{n=0}^{\infty} \frac{(-1)^n q^{n(n-1)/6}}{(q)_n} \\ - \sum_{m=0}^{\infty} \frac{(-1)^m q^{(3m+1)(3m+2)/6}}{(q)_m} \sum_{n=0}^{\infty} \frac{(-1)^n q^{n(n+3)/6}}{(q)_n} \\ + \sum_{m=0}^{\infty} \frac{(-1)^m q^{(m+1)(3m+2)/2}}{(q)_m} \sum_{n=0}^{\infty} \frac{(-1)^n q^{n(n+1)/6}}{(q)_n} = 0.\end{aligned}$$

# Ingredients in the proof of the main identity

## Theorem (D. - Kumar (2024))

*Let $s \in \mathbb{N}$. For any complex numbers $a, b$ and $q$ such that $|q| < 1$,*

$$\sum_{n=0}^{\infty} \frac{a^n b^n q^{n^2}}{(aq)_n (bq)_n} = \sum_{k=0}^{s-1} \sum_{n=0}^{\infty} \frac{a^{sn+k} b^{n+j} q^{(sn+k)(n+j)}}{(aq)_n (bq)_{sn+k}} = 1 + b \sum_{n=1}^{\infty} \frac{a^n q^n}{(bq)_n}.$$

# Ingredients in the proof of the main identity

**Theorem (D. - Kumar (2024))**

*Let $s \in \mathbb{N}$. For any complex numbers $a, b$ and $q$ such that $|q| < 1$,*

$$\sum_{n=0}^{\infty} \frac{a^n b^n q^{n^2}}{(aq)_n (bq)_n} = \sum_{k=0}^{s-1} \sum_{n=0}^{\infty} \frac{a^{sn+k} b^{n+j} q^{(sn+k)(n+j)}}{(aq)_n (bq)_{sn+k}} = 1 + b \sum_{n=1}^{\infty} \frac{a^n q^n}{(bq)_n}.$$

This gives, in particular,

$$\sum_{k=0}^{s-1} \sum_{m=0}^{\infty} \frac{b^{sm+k} q^{(sm+k+n)(m+j)}}{(bq)_m (q^{n+1})_{sm+k}} = 1 + \sum_{m=1}^{\infty} \frac{b^m q^{m+n}}{(q^{n+1})_m},$$

$$\sum_{k=0}^{s-1} \sum_{n=0}^{\infty} \frac{b^{sn+k} q^{(sn+k)(m+n+j)}}{(bq^{m+1})_n (q)_{sn+k}} = \frac{1}{(bq^{m+1})_\infty},$$

both of which are required in our proofs.

# Another identity of Ramanujan

- On page 26 of the Lost Notebook, we find

$$\sum_{n=-\infty}^{\infty} a^n q^{n^2/4} \sum_{n=0}^{\infty} \frac{(-a)^n q^{n^2/4}}{(q)_n} - \sum_{n=-\infty}^{\infty} (-a)^n q^{n^2/4} \sum_{n=0}^{\infty} \frac{a^n q^{n^2/4}}{(q)_n}$$
$$= 2q^{1/4}(q)_\infty \sum_{m=0}^{\infty} \frac{a^{-2m-1} q^{m^2+m}}{(bq)_m}. \quad (13)$$

[4] G. N. Watson, *The mock theta functions* (2), Proc. London Math. Soc. **42** (1937), 274–304.

# Another identity of Ramanujan

- On page 26 of the Lost Notebook, we find

$$\sum_{n=-\infty}^{\infty} a^n q^{n^2/4} \sum_{n=0}^{\infty} \frac{(-a)^n q^{n^2/4}}{(q)_n} - \sum_{n=-\infty}^{\infty} (-a)^n q^{n^2/4} \sum_{n=0}^{\infty} \frac{a^n q^{n^2/4}}{(q)_n}$$
$$= 2q^{1/4}(q)_\infty \sum_{m=0}^{\infty} \frac{a^{-2m-1} q^{m^2+m}}{(bq)_m}. \tag{13}$$

- It was proved by Andrews in *Partitions: Yesterday and Today*.

[4] G. N. Watson, *The mock theta functions* (2), Proc. London Math. Soc. **42** (1937), 274–304.

# Another identity of Ramanujan

- On page 26 of the Lost Notebook, we find

$$\sum_{n=-\infty}^{\infty} a^n q^{n^2/4} \sum_{n=0}^{\infty} \frac{(-a)^n q^{n^2/4}}{(q)_n} - \sum_{n=-\infty}^{\infty} (-a)^n q^{n^2/4} \sum_{n=0}^{\infty} \frac{a^n q^{n^2/4}}{(q)_n}$$
$$= 2q^{1/4}(q)_\infty \sum_{m=0}^{\infty} \frac{a^{-2m-1} q^{m^2+m}}{(bq)_m}. \quad (13)$$

- It was proved by Andrews in *Partitions: Yesterday and Today*.
- He also showed that Watson's identities[4]

$$G(-q)\varphi(q) - G(q)\varphi(-q) = 2qH(q^4)\psi(q^2),$$
$$H(-q)\varphi(q) + H(q)\varphi(-q) = 2G(q^4)\psi(q^2),$$

where $\psi(q) = (q^2;q^2)_\infty/(q;q^2)_\infty$, follow as special cases of (13).

[4] G. N. Watson, *The mock theta functions* (2), Proc. London Math. Soc. **42** (1937), 274–304.

# An identity from the list of forty

- Ramanujan's list of forty identities[5] involving $G(q)$ and $H(q)$ is famous.

[5] B. C. Berndt, G. Choi, Y.-S. Choi, H. Hahn, B. P. Yeap, A. J. Yee, H. Yesilyurt and J. Yee, *Ramanujan's forty identities for the Rogers-Ramanujan functions* Mem. Amer. Math. Soc. **188** (2007), no. 880, 96 pp.

# An identity from the list of forty

- Ramanujan's list of forty identities[5] involving $G(q)$ and $H(q)$ is famous.
- One of them is

$$G(q)H(-q) + G(-q)H(q) = 2(-q^2; q^2)_\infty^2. \tag{14}$$

[5] B. C. Berndt, G. Choi, Y.-S. Choi, H. Hahn, B. P. Yeap, A. J. Yee, H. Yesilyurt and J. Yee, *Ramanujan's forty identities for the Rogers-Ramanujan functions* Mem. Amer. Math. Soc. **188** (2007), no. 880, 96 pp.

# An identity from the list of forty

- Ramanujan's list of forty identities[5] involving $G(q)$ and $H(q)$ is famous.
- One of them is

$$G(q)H(-q) + G(-q)H(q) = 2(-q^2; q^2)_\infty^2. \tag{14}$$

- It follows easily from (13) and (15).

[5] B. C. Berndt, G. Choi, Y.-S. Choi, H. Hahn, B. P. Yeap, A. J. Yee, H. Yesilyurt and J. Yee, *Ramanujan's forty identities for the Rogers-Ramanujan functions* Mem. Amer. Math. Soc. **188** (2007), no. 880, 96 pp.

# Asymptotic analysis

Recall the generalized modular relation:

$$\sum_{m=0}^{\infty}\frac{a^{-2m}q^{m^2}}{(bq)_m}\sum_{n=0}^{\infty}\frac{a^nb^nq^{n^2/4}}{(q)_n}+\sum_{m=0}^{\infty}\frac{a^{-2m-1}q^{m^2+m}}{(bq)_m}\sum_{n=0}^{\infty}\frac{a^nb^nq^{(n+1)^2/4}}{(q)_n}$$
$$=\frac{1}{(bq)_\infty}\sum_{n=-\infty}^{\infty}a^nq^{n^2/4}-(1-b)\sum_{n=1}^{\infty}a^nq^{n^2/4}\sum_{\ell=0}^{n-1}\frac{b^\ell}{(q)_\ell}. \tag{15}$$

# Asymptotic analysis

Recall the generalized modular relation:

$$\sum_{m=0}^{\infty}\frac{a^{-2m}q^{m^2}}{(bq)_m}\sum_{n=0}^{\infty}\frac{a^nb^nq^{n^2/4}}{(q)_n}+\sum_{m=0}^{\infty}\frac{a^{-2m-1}q^{m^2+m}}{(bq)_m}\sum_{n=0}^{\infty}\frac{a^nb^nq^{(n+1)^2/4}}{(q)_n}$$
$$=\frac{1}{(bq)_\infty}\sum_{n=-\infty}^{\infty}a^nq^{n^2/4}-(1-b)\sum_{n=1}^{\infty}a^nq^{n^2/4}\sum_{\ell=0}^{n-1}\frac{b^\ell}{(q)_\ell}. \qquad (15)$$

- How in the world could someone conceive such a relation?

# Asymptotic analysis

Recall the generalized modular relation:

$$\sum_{m=0}^{\infty}\frac{a^{-2m}q^{m^2}}{(bq)_m}\sum_{n=0}^{\infty}\frac{a^nb^nq^{n^2/4}}{(q)_n}+\sum_{m=0}^{\infty}\frac{a^{-2m-1}q^{m^2+m}}{(bq)_m}\sum_{n=0}^{\infty}\frac{a^nb^nq^{(n+1)^2/4}}{(q)_n}$$
$$=\frac{1}{(bq)_\infty}\sum_{n=-\infty}^{\infty}a^nq^{n^2/4}-(1-b)\sum_{n=1}^{\infty}a^nq^{n^2/4}\sum_{\ell=0}^{n-1}\frac{b^\ell}{(q)_\ell}. \qquad (15)$$

- How in the world could someone conceive such a relation?
- Ramanujan was a master of asymptotic analysis, and we believe he may have first got an idea about the existence of (15) through such an analysis.

# Asymptotic analysis

- Before Ramanujan obtained his own proofs of the Rogers-Ramanujan identities

$$G(q)=\frac{1}{(q;q^5)_\infty(q^4;q^5)_\infty}, H(q)=\frac{1}{(q^2;q^5)_\infty(q^3;q^5)_\infty},$$

his belief in them stemmed from several pieces of evidence he found for their existence.

# Asymptotic analysis

- Before Ramanujan obtained his own proofs of the Rogers-Ramanujan identities

$$G(q)=\frac{1}{(q;q^5)_\infty(q^4;q^5)_\infty}, H(q)=\frac{1}{(q^2;q^5)_\infty(q^3;q^5)_\infty},$$

  his belief in them stemmed from several pieces of evidence he found for their existence.

- For example, one of them was that both sides of the first identity are asymptotically equal to $\exp\left(\frac{\pi^2}{15(1-q)}\right)$ as $q\to 1^-$.

# A general result on asymptotics

In his notebooks as well as on page 359 of the Lost Notebook, Ramanujan obtained an asymptotic formula for the series $\sum_{n=0}^{\infty} a^n q^{bn^2+cn}/(q)_n$.

# A general result on asymptotics

In his notebooks as well as on page 359 of the Lost Notebook, Ramanujan obtained an asymptotic formula for the series $\sum_{n=0}^{\infty} a^n q^{bn^2+cn}/(q)_n$.

## Theorem (Ramanujan)

*Let $a, b, c,$ and $q$ be real numbers such that $a > 0, b > 0$ and $|q| < 1$. Let $z$ denote the positive root of $az^{2b} + z - 1 = 0$. Then as $q \to 1^-$,*

$$\sum_{n=0}^{\infty} \frac{a^n q^{bn^2+cn}}{(q)_n} \sim \frac{z^c}{\sqrt{z + 2b(1-z)}} \exp\left(-\frac{1}{\log(q)} \left(\mathrm{Li}_2(az^{2b}) + b\log^2(z)\right)\right),$$

*where the $\mathrm{Li}_2(z)$ is the dilogarithm function defined for $|z| < 1$, by $\mathrm{Li}_2(z) := \sum_{n=1}^{\infty} z^n/n^2$, and for any $z \in \mathbb{C}$ by $\mathrm{Li}_2(z) := -\int_0^z \log(1-u)/u\, du$.*

# Asymptotic formula for product of two such series

**Theorem**

*Let $a > 0$ and $s \in \mathbb{N}$. Let $z_1$ denote the real positive root of $az^{1/s} + z - 1 = 0$. As $q \to 1^-$,*

$$\sum_{n=0}^{\infty} \frac{a^n q^{\frac{n^2}{2s}}}{(q)_n} \sum_{n=0}^{\infty} \frac{a^{-ns} q^{\frac{n^2 s}{2}}}{(q)_n} \sim \frac{\sqrt{s}}{1+(s-1)z_1} \exp\left(-\frac{1}{\log(q)}\left(\frac{\pi^2}{6} + \frac{s}{2}\log^2(a)\right)\right), \tag{16}$$

# Asymptotic formula for product of two such series

**Theorem**

*Let $a > 0$ and $s \in \mathbb{N}$. Let $z_1$ denote the real positive root of $az^{1/s} + z - 1 = 0$. As $q \to 1^-$,*

$$\sum_{n=0}^{\infty} \frac{a^n q^{\frac{n^2}{2s}}}{(q)_n} \sum_{n=0}^{\infty} \frac{a^{-ns} q^{\frac{n^2 s}{2}}}{(q)_n} \sim \frac{\sqrt{s}}{1+(s-1)z_1} \exp\left(-\frac{1}{\log(q)}\left(\frac{\pi^2}{6} + \frac{s}{2}\log^2(a)\right)\right), \tag{16}$$

Observe that the asymptotic formula does not involve a dilogarithm, and, instead, only elementary functions.

# Proof of the asymptotic formula

Letting $b = 1/(2s)$ and $c = 0$ in the above theorem, we see that

$$\sum_{n=0}^{\infty} \frac{a^n q^{n^2/(2s)}}{(q)_n} \sim \frac{1}{\sqrt{z_1 + (1 - z_1)/s}} \exp\left(-\frac{1}{\log(q)}\left(\mathrm{Li}_2(az_1^{1/s}) + \frac{1}{2s}\log^2(z_1)\right)\right), \tag{17}$$

where $z_1$ is the positive root of $az^{1/s} + z - 1 = 0$, and,
$z_1 + (1 - z_1)/s > 0$.

# Proof of the asymptotic formula

Letting $b = 1/(2s)$ and $c = 0$ in the above theorem, we see that

$$\sum_{n=0}^{\infty} \frac{a^n q^{n^2/(2s)}}{(q)_n} \sim \frac{1}{\sqrt{z_1 + (1 - z_1)/s}} \exp\left(-\frac{1}{\log(q)}\left(\mathrm{Li}_2(az_1^{1/s}) + \frac{1}{2s}\log^2(z_1)\right)\right), \tag{17}$$

where $z_1$ is the positive root of $az^{1/s} + z - 1 = 0$, and,
$z_1 + (1 - z_1)/s > 0$.
Similarly, letting $b = s/2$ and $c = 0$ in the theorem, and replacing $a$ by $a^{-s}$, we have

$$\sum_{n=0}^{\infty} \frac{a^{-ns} q^{n^2 s/2}}{(q)_n} \sim \frac{1}{\sqrt{z_2 + s(1 - z_2)}} \exp\left(-\frac{1}{\log(q)}\left(\mathrm{Li}_2(a^{-s} z_2^{s}) + \frac{s}{2}\log^2(z_2)\right)\right), \tag{18}$$

where $z_2$ is the positive root of

$$a^{-s} z^s + z - 1 = 0, \tag{19}$$

and $z_2 + s(1 - z_2) > 0$.

# Proof of the asymptotic formula

Letting $b = 1/(2s)$ and $c = 0$ in the above theorem, we see that

$$\sum_{n=0}^{\infty} \frac{a^n q^{n^2/(2s)}}{(q)_n} \sim \frac{1}{\sqrt{z_1 + (1-z_1)/s}} \exp\left(-\frac{1}{\log(q)}\left(\mathrm{Li}_2(az_1^{1/s}) + \frac{1}{2s}\log^2(z_1)\right)\right), \tag{17}$$

where $z_1$ is the positive root of $az^{1/s} + z - 1 = 0$, and,
$z_1 + (1 - z_1)/s > 0$.
Similarly, letting $b = s/2$ and $c = 0$ in the theorem, and replacing $a$ by $a^{-s}$, we have

$$\sum_{n=0}^{\infty} \frac{a^{-ns} q^{n^2 s/2}}{(q)_n} \sim \frac{1}{\sqrt{z_2 + s(1-z_2)}} \exp\left(-\frac{1}{\log(q)}\left(\mathrm{Li}_2(a^{-s}z_2^{s}) + \frac{s}{2}\log^2(z_2)\right)\right), \tag{18}$$

where $z_2$ is the positive root of

$$a^{-s}z^s + z - 1 = 0, \tag{19}$$

and $z_2 + s(1 - z_2) > 0$. Thus, $z_1 = a^{-s}z_2^s$ and $z_1 + z_2 = 1$.

# Proof of the asymptotic formula

$$\sum_{n=0}^{\infty}\frac{a^n q^{n^2/(2s)}}{(q)_n}\sum_{n=0}^{\infty}\frac{a^{-ns}q^{n^2s/2}}{(q)_n}\sim\frac{1}{\sqrt{(z_1+(1-z_1)/s)(z_2+s(1-z_2))}}$$
$$\times\exp\left(\tfrac{-1}{\log(q)}\left(\mathrm{Li}_2(az_1^{1/s})+\mathrm{Li}_2(a^{-s}z_2^s)+\tfrac{1}{2s}\log^2(z_1)+\tfrac{s}{2}\log^2(z_2)\right)\right).$$

# Proof of the asymptotic formula

$$\sum_{n=0}^{\infty} \frac{a^n q^{n^2/(2s)}}{(q)_n} \sum_{n=0}^{\infty} \frac{a^{-ns} q^{n^2 s/2}}{(q)_n} \sim \frac{1}{\sqrt{(z_1 + (1-z_1)/s)(z_2 + s(1-z_2))}}$$

$$\times \exp\left(\tfrac{-1}{\log(q)}\left(\mathrm{Li}_2(a z_1^{1/s}) + \mathrm{Li}_2(a^{-s} z_2^s) + \tfrac{1}{2s}\log^2(z_1) + \tfrac{s}{2}\log^2(z_2)\right)\right).$$

Using the functional equation

$$\mathrm{Li}_2(z) + \mathrm{Li}_2(1-z) = \frac{\pi^2}{6} - \log(z)\log(1-z),$$

and the facts $z_1 = a^{-s} z_2^s$ and $z_1 + z_2 = 1$, we obtain

# Proof of the asymptotic formula

$$\sum_{n=0}^{\infty}\frac{a^n q^{n^2/(2s)}}{(q)_n}\sum_{n=0}^{\infty}\frac{a^{-ns}q^{n^2s/2}}{(q)_n}\sim\frac{1}{\sqrt{(z_1+(1-z_1)/s)(z_2+s(1-z_2))}}$$
$$\times\exp\left(\tfrac{-1}{\log(q)}\left(\mathrm{Li}_2(az_1^{1/s})+\mathrm{Li}_2(a^{-s}z_2^s)+\tfrac{1}{2s}\log^2(z_1)+\tfrac{s}{2}\log^2(z_2)\right)\right).$$

Using the functional equation

$$\mathrm{Li}_2(z)+\mathrm{Li}_2(1-z)=\frac{\pi^2}{6}-\log(z)\log(1-z),$$

and the facts $z_1=a^{-s}z_2^s$ and $z_1+z_2=1$, we obtain

$$\mathrm{Li}_2(az_1^{1/s})+\mathrm{Li}_2(a^{-s}z_2^s)+\frac{1}{2s}\log^2(z_1)+\frac{s}{2}\log^2(z_2)$$
$$=\frac{\pi^2}{6}+\frac{s}{2}\log^2(a).$$

# Proof of the asymptotic formula

$$\sum_{n=0}^{\infty}\frac{a^n q^{n^2/(2s)}}{(q)_n}\sum_{n=0}^{\infty}\frac{a^{-ns}q^{n^2s/2}}{(q)_n}\sim\frac{1}{\sqrt{(z_1+(1-z_1)/s)(z_2+s(1-z_2))}}$$

$$\times\exp\left(\tfrac{-1}{\log(q)}\left(\mathrm{Li}_2(az_1^{1/s})+\mathrm{Li}_2(a^{-s}z_2^s)+\tfrac{1}{2s}\log^2(z_1)+\tfrac{s}{2}\log^2(z_2)\right)\right).$$

Using the functional equation

$$\mathrm{Li}_2(z)+\mathrm{Li}_2(1-z)=\frac{\pi^2}{6}-\log(z)\log(1-z),$$

and the facts $z_1=a^{-s}z_2^s$ and $z_1+z_2=1$, we obtain

$$\mathrm{Li}_2(az_1^{1/s})+\mathrm{Li}_2(a^{-s}z_2^s)+\frac{1}{2s}\log^2(z_1)+\frac{s}{2}\log^2(z_2)$$
$$=\frac{\pi^2}{6}+\frac{s}{2}\log^2(a).$$

Finally routine simplification leads us to the required asymptotic:

$$(z_1+(1-z_1)/s)(z_2+s(1-z_2))=(1+(s-1)z_1)^2/s.$$

# Guessing the theta function in Ramanujan's formula from the above asymptotic

Letting $s = 2$ in the above Theorem we see that as $q \to 1^-$,

$$\sum_{n=0}^{\infty} \frac{a^n q^{n^2/4}}{(q)_n} \sum_{m=0}^{\infty} \frac{a^{-2m} q^{m^2}}{(q)_m} \sim \frac{2\sqrt{2}}{4 + a^2 - a\sqrt{4 + a^2}} \exp\left\{ -\frac{1}{\log(q)} \left( \frac{\pi^2}{6} + \log^2(a) \right) \right\}.$$

# Guessing the theta function in Ramanujan's formula from the above asymptotic

Letting $s = 2$ in the above Theorem we see that as $q \to 1^-$,

$$\sum_{n=0}^{\infty} \frac{a^n q^{n^2/4}}{(q)_n} \sum_{m=0}^{\infty} \frac{a^{-2m} q^{m^2}}{(q)_m} \sim \frac{2\sqrt{2}}{4 + a^2 - a\sqrt{4 + a^2}} \exp\left\{-\frac{1}{\log(q)}\left(\frac{\pi^2}{6} + \log^2(a)\right)\right\}.$$

Similarly, it can be seen that

$$\sum_{m=0}^{\infty} \frac{a^{-2m-1} q^{m^2+m}}{(q)_m} \sum_{n=0}^{\infty} \frac{a^n q^{(n+1)^2/4}}{(q)_n} \sim \sqrt{2} q^{1/4} \frac{(2 + a^2 - a\sqrt{4 + a^2})}{(4 + a^2 - a\sqrt{4 + a^2})} \times \exp\left\{-\frac{1}{\log(q)}\left(\frac{\pi^2}{6} + \log^2(a)\right)\right\}.$$

# Guessing the theta function in Ramanujan's formula from the above asymptotic

Thus, as $q \to 1^-$,

$$\begin{aligned}
&\sum_{m=0}^{\infty} \frac{a^{-2m} q^{m^2}}{(q)_m} \sum_{n=0}^{\infty} \frac{a^n q^{n^2/4}}{(q)_n} + \sum_{m=0}^{\infty} \frac{a^{-2m-1} q^{m^2+m}}{(q)_m} \sum_{n=0}^{\infty} \frac{a^n q^{(n+1)^2/4}}{(q)_n} \\
&\sim \sqrt{2} \frac{\left(2 + q^{1/4}(2 + a^2 - a\sqrt{4+a^2})\right)}{(4 + a^2 - a\sqrt{4+a^2})} \exp\left\{ -\frac{1}{\log(q)} \left( \frac{\pi^2}{6} + \log^2(a) \right) \right\} \\
&\sim \sqrt{2} \exp\left\{ -\frac{1}{\log(q)} \left( \frac{\pi^2}{6} + \log^2(a) \right) \right\}. \qquad (20)
\end{aligned}$$

# Guessing the theta function in Ramanujan's formula from the above asymptotic

Thus, as $q \to 1^-$,

$$\sum_{m=0}^{\infty} \frac{a^{-2m} q^{m^2}}{(q)_m} \sum_{n=0}^{\infty} \frac{a^n q^{n^2/4}}{(q)_n} + \sum_{m=0}^{\infty} \frac{a^{-2m-1} q^{m^2+m}}{(q)_m} \sum_{n=0}^{\infty} \frac{a^n q^{(n+1)^2/4}}{(q)_n}$$
$$\sim \sqrt{2} \frac{\left(2 + q^{1/4}(2 + a^2 - a\sqrt{4+a^2})\right)}{(4 + a^2 - a\sqrt{4+a^2})} \exp\left\{-\frac{1}{\log(q)}\left(\frac{\pi^2}{6} + \log^2(a)\right)\right\}$$
$$\sim \sqrt{2} \exp\left\{-\frac{1}{\log(q)}\left(\frac{\pi^2}{6} + \log^2(a)\right)\right\}. \tag{20}$$

Now, as $q \to 1^-$,

$$\frac{1}{(q;q)_\infty} \sim \sqrt{\frac{-\log(q)}{2\pi}} \exp\left(-\frac{\pi^2}{6\log(q)}\right),$$

whereas

$$\sum_{n=-\infty}^{\infty} a^n q^{n^2/4} \sim 2\sqrt{\frac{\pi}{-\log(q)}} \exp\left(-\frac{\log^2(a)}{\log(q)}\right)$$

# The identity

- This may have led Ramanujan to

$$\sum_{m=0}^{\infty}\frac{a^{-2m}q^{m^2}}{(q)_m}\sum_{n=0}^{\infty}\frac{a^nq^{n^2/4}}{(q)_n}+\sum_{m=0}^{\infty}\frac{a^{-2m-1}q^{m^2+m}}{(q)_m}\sum_{n=0}^{\infty}\frac{a^nq^{(n+1)^2/4}}{(q)_n}$$

$$=\frac{1}{(q;q)_\infty}\sum_{n=-\infty}^{\infty}a^nq^{n^2/4}.$$

# The identity

- This may have led Ramanujan to

$$\sum_{m=0}^{\infty}\frac{a^{-2m}q^{m^2}}{(q)_m}\sum_{n=0}^{\infty}\frac{a^nq^{n^2/4}}{(q)_n}+\sum_{m=0}^{\infty}\frac{a^{-2m-1}q^{m^2+m}}{(q)_m}\sum_{n=0}^{\infty}\frac{a^nq^{(n+1)^2/4}}{(q)_n}$$
$$=\frac{1}{(q;q)_\infty}\sum_{n=-\infty}^{\infty}a^nq^{n^2/4}.$$

- But he goes further and obtains

$$\sum_{m=0}^{\infty}\frac{a^{-2m}q^{m^2}}{(bq)_m}\sum_{n=0}^{\infty}\frac{a^nb^nq^{n^2/4}}{(q)_n}+\sum_{m=0}^{\infty}\frac{a^{-2m-1}q^{m^2+m}}{(bq)_m}\sum_{n=0}^{\infty}\frac{a^nb^nq^{(n+1)^2/4}}{(q)_n}$$
$$=\frac{1}{(bq)_\infty}\sum_{n=-\infty}^{\infty}a^nq^{n^2/4}-(1-b)\sum_{n=1}^{\infty}a^nq^{n^2/4}\sum_{\ell=0}^{n-1}\frac{b^\ell}{(q)_\ell}.$$

# Concluding remarks

- After finding two algebraic relations between $G(q)$ and $H(q)$, Ramanujan[6] wrote '*Each of these formulae is the simplest of a large class*.

[6] S. Ramanujan, *Algebraic relations between certain infinite products*, Proc. London Math. Soc. (2) (1920), p. xviii.

# Concluding remarks

- After finding two algebraic relations between $G(q)$ and $H(q)$, Ramanujan[6] wrote '*Each of these formulae is the simplest of a large class*.
- It is often believed that the '*large class*' in Ramanujan's remark is the set of forty identities for $G(q)$ and $H(q)$.

[6] S. Ramanujan, *Algebraic relations between certain infinite products*, Proc. London Math. Soc. (2) (1920), p. xviii.

# Concluding remarks

- After finding two algebraic relations between $G(q)$ and $H(q)$, Ramanujan[6] wrote '*Each of these formulae is the simplest of a large class*.
- It is often believed that the '*large class*' in Ramanujan's remark is the set of forty identities for $G(q)$ and $H(q)$.
- In light of the fact that (1) and (13) lead to some of these identities as corollaries, could it be that Ramanujan is referring to such generalized modular relations when he says '*large class*'?

[6] S. Ramanujan, *Algebraic relations between certain infinite products*, Proc. London Math. Soc. (2) (1920), p. xviii.

# Concluding remarks

- Our main result shows that a theta function can be dissected as a sum of $s$ number of products of generalized Rogers-Ramanujan functions, where $s$ is *any* natural number.

# Concluding remarks

- Our main result shows that a theta function can be dissected as a sum of $s$ number of products of generalized Rogers-Ramanujan functions, where $s$ is *any* natural number.
- Its special case $s = 2$ corresponds to Ramanujan's result whose special cases are two modular relations for $G(q)$ and $H(q)$ (when $a = b = 1$), or a result connecting the fifth order mock theta functions with $G(q)$ and $H(q)$ (when $a = b = -1$.

# Concluding remarks

- Our main result shows that a theta function can be dissected as a sum of $s$ number of products of generalized Rogers-Ramanujan functions, where $s$ is *any* natural number.
- Its special case $s = 2$ corresponds to Ramanujan's result whose special cases are two modular relations for $G(q)$ and $H(q)$ (when $a = b = 1$), or a result connecting the fifth order mock theta functions with $G(q)$ and $H(q)$ (when $a = b = -1$.
- The case $s = 1$ gives the Jacobi triple product identity (when $a = b = 1$), and a relation between the third order mock theta functions $\phi(q)$ and $\psi(q)$ (when $a = b = -1$).

# Concluding remarks

- Our main result shows that a theta function can be dissected as a sum of $s$ number of products of generalized Rogers-Ramanujan functions, where $s$ is *any* natural number.
- Its special case $s = 2$ corresponds to Ramanujan's result whose special cases are two modular relations for $G(q)$ and $H(q)$ (when $a = b = 1$), or a result connecting the fifth order mock theta functions with $G(q)$ and $H(q)$ (when $a = b = -1$.
- The case $s = 1$ gives the Jacobi triple product identity (when $a = b = 1$), and a relation between the third order mock theta functions $\phi(q)$ and $\psi(q)$ (when $a = b = -1$).
- Thus, in a sense, both the cases $s = 1, 2$ correspond to results which dwell in the theory of modular forms or mock modular forms.

# Guessing Rogers' first identity

- However, for $s > 2$, the identities seem to transcend the modular/mock modular worlds. We explain this now.

# Guessing Rogers' first identity

- However, for $s > 2$, the identities seem to transcend the modular/mock modular worlds. We explain this now.
- An asymptotic analysis of the series occurring in Rogers' identity $G(q) = (-q^2; q^2)_\infty \sum_{n=0}^{\infty} q^{n^2}/(q^4; q^4)_n$, yields, as $q \to 1^-$,

# Guessing Rogers' first identity

- However, for $s > 2$, the identities seem to transcend the modular/mock modular worlds. We explain this now.
- An asymptotic analysis of the series occurring in Rogers' identity $G(q) = (-q^2;q^2)_\infty \sum_{n=0}^{\infty} q^{n^2}/(q^4;q^4)_n$, yields, as $q \to 1^-$,

$$\frac{\sum_{m=0}^{\infty} \frac{q^{m^2}}{(q)_m}}{\sum_{n=0}^{\infty} \frac{q^{n^2}}{(q^4;q^4)_n}} \sim \frac{1}{\sqrt{2}} \exp\left\{-\frac{1}{\log(q)}\left(\mathrm{Li}_2(1-u) + \log^2(u) - \frac{\mathrm{Li}_2(u)}{4} - \frac{\log^2(1-u)}{16}\right)\right\},$$

where $u = (\sqrt{5}-1)/2$.

# Guessing Rogers' first identity

- However, for $s > 2$, the identities seem to transcend the modular/mock modular worlds. We explain this now.
- An asymptotic analysis of the series occurring in Rogers' identity $G(q) = (-q^2; q^2)_\infty \sum_{n=0}^{\infty} q^{n^2}/(q^4; q^4)_n$, yields, as $q \to 1^-$,

$$\frac{\sum_{m=0}^{\infty} \frac{q^{m^2}}{(q)_m}}{\sum_{n=0}^{\infty} \frac{q^{n^2}}{(q^4;q^4)_n}} \sim \frac{1}{\sqrt{2}} \exp\left\{-\frac{1}{\log(q)}\left(\mathrm{Li}_2(1-u) + \log^2(u) - \frac{\mathrm{Li}_2(u)}{4} - \frac{\log^2(1-u)}{16}\right)\right\},$$

where $u = (\sqrt{5}-1)/2$. But

$$\mathrm{Li}_2(1-u) - \frac{1}{4}\mathrm{Li}_2(u) + \log^2(u) - \frac{1}{16}\log^2(1-u) = \frac{\pi^2}{24},$$

$$(-q^2; q^2)_\infty = \frac{1}{(q^2; q^4)_\infty} \sim \frac{1}{\sqrt{2}} \exp\left\{-\frac{\pi^2}{24\log(q)}\right\},$$

so that the identity can be guessed.

# Unlikelihood of “higher analogues” of Rogers’ identity

Recall the $s = 3$ case of our main result with $a = b = 1$:

$$\sum_{m=0}^{\infty} \frac{q^{\frac{3m^2}{2}}}{(q)_m} \sum_{n=0}^{\infty} \frac{q^{\frac{n^2}{6}}}{(q)_n} + q^{\frac{1}{6}} \sum_{m=0}^{\infty} \frac{q^{\frac{3m^2}{2}+m}}{(q)_m} \sum_{n=0}^{\infty} \frac{q^{\frac{n(n+4)}{6}}}{(q)_n} + q^{\frac{2}{3}} \sum_{m=0}^{\infty} \frac{q^{\frac{3m^2}{2}+2m}}{(q)_m} \sum_{n=0}^{\infty} \frac{q^{\frac{n(n+2)}{6}}}{(q)_n}$$
$$= \frac{1}{(q)_\infty} \sum_{n=-\infty}^{\infty} q^{n^2/6}.$$

# Unlikelihood of “higher analogues” of Rogers’ identity

Recall the $s = 3$ case of our main result with $a = b = 1$:

$$\sum_{m=0}^{\infty}\frac{q^{\frac{3m^2}{2}}}{(q)_m}\sum_{n=0}^{\infty}\frac{q^{\frac{n^2}{6}}}{(q)_n}+q^{\frac{1}{6}}\sum_{m=0}^{\infty}\frac{q^{\frac{3m^2}{2}+m}}{(q)_m}\sum_{n=0}^{\infty}\frac{q^{\frac{n(n+4)}{6}}}{(q)_n}+q^{\frac{2}{3}}\sum_{m=0}^{\infty}\frac{q^{\frac{3m^2}{2}+2m}}{(q)_m}\sum_{n=0}^{\infty}\frac{q^{\frac{n(n+2)}{6}}}{(q)_n}$$
$$=\frac{1}{(q)_\infty}\sum_{n=-\infty}^{\infty}q^{n^2/6}.$$

Are $\sum_{m=0}^{\infty}\frac{q^{3m^2/2}}{(q)_m}$ and $\sum_{n=0}^{\infty}\frac{q^{n^2/6}}{(q)_n}$ related in the same way as the series in Rogers’ identity?

# Unlikelihood of "higher analogues" of Rogers' identity

Recall the $s=3$ case of our main result with $a=b=1$:

$$\sum_{m=0}^{\infty}\frac{q^{\frac{3m^2}{2}}}{(q)_m}\sum_{n=0}^{\infty}\frac{q^{\frac{n^2}{6}}}{(q)_n}+q^{\frac{1}{6}}\sum_{m=0}^{\infty}\frac{q^{\frac{3m^2}{2}+m}}{(q)_m}\sum_{n=0}^{\infty}\frac{q^{\frac{n(n+4)}{6}}}{(q)_n}+q^{\frac{2}{3}}\sum_{m=0}^{\infty}\frac{q^{\frac{3m^2}{2}+2m}}{(q)_m}\sum_{n=0}^{\infty}\frac{q^{\frac{n(n+2)}{6}}}{(q)_n}$$

$$=\frac{1}{(q)_\infty}\sum_{n=-\infty}^{\infty}q^{n^2/6}.$$

Are $\sum_{m=0}^{\infty}\frac{q^{3m^2/2}}{(q)_m}$ and $\sum_{n=0}^{\infty}\frac{q^{n^2/6}}{(q)_n}$ related in the same way as the series in Rogers' identity?

If $v$ is the positive root of $z^3+z-1=0$, then as $q\to 1^-$,

$$\frac{\sum_{m=0}^{\infty}q^{\frac{3}{2}m^2}/(q)_m}{\sum_{n=0}^{\infty}q^{n^2}/(q^6;q^6)_n}\sim\frac{1}{\sqrt{3}}\exp\left\{-\frac{1}{\log(q)}\left(\mathrm{Li}_2(1-v)+\frac{5}{4}\log^2(v)-\frac{1}{6}\mathrm{Li}_2(v)\right)\right\}.$$

# Unlikelihood of "higher analogues" of Rogers' identity

Recall the $s=3$ case of our main result with $a=b=1$:

$$\sum_{m=0}^{\infty}\frac{q^{\frac{3m^2}{2}}}{(q)_m}\sum_{n=0}^{\infty}\frac{q^{\frac{n^2}{6}}}{(q)_n}+q^{\frac{1}{6}}\sum_{m=0}^{\infty}\frac{q^{\frac{3m^2}{2}+m}}{(q)_m}\sum_{n=0}^{\infty}\frac{q^{\frac{n(n+4)}{6}}}{(q)_n}+q^{\frac{2}{3}}\sum_{m=0}^{\infty}\frac{q^{\frac{3m^2}{2}+2m}}{(q)_m}\sum_{n=0}^{\infty}\frac{q^{\frac{n(n+2)}{6}}}{(q)_n}$$

$$=\frac{1}{(q)_\infty}\sum_{n=-\infty}^{\infty}q^{n^2/6}.$$

Are $\sum_{m=0}^{\infty}\frac{q^{3m^2/2}}{(q)_m}$ and $\sum_{n=0}^{\infty}\frac{q^{n^2/6}}{(q)_n}$ related in the same way as the series in Rogers' identity?

If $v$ is the positive root of $z^3+z-1=0$, then as $q\to 1^-$,

$$\frac{\sum_{m=0}^{\infty}q^{\frac{3}{2}m^2}/(q)_m}{\sum_{n=0}^{\infty}q^{n^2}/(q^6;q^6)_n}\sim\frac{1}{\sqrt{3}}\exp\left\{-\frac{1}{\log(q)}\left(\mathrm{Li}_2(1-v)+\frac{5}{4}\log^2(v)-\frac{1}{6}\mathrm{Li}_2(v)\right)\right\}.$$

It is true that $6\mathrm{Li}_2(v)-30\mathrm{Li}_2(1-v)-36\mathrm{Li}_2(-v^2)=\pi^2$, but using this, we see that the expression in the parantheses still depends on $v$!

# Concluding remarks

Recall the identity on page 26 of the Lost Notebook:

$$\sum_{n=-\infty}^{\infty} a^n q^{n^2/4} \sum_{n=0}^{\infty} \frac{(-a)^n q^{n^2/4}}{(q)_n} - \sum_{n=-\infty}^{\infty} (-a)^n q^{n^2/4} \sum_{n=0}^{\infty} \frac{a^n q^{n^2/4}}{(q)_n}$$

$$= 2q^{1/4}(q)_\infty \sum_{m=0}^{\infty} \frac{a^{-2m-1} q^{m^2+m}}{(bq)_m}.$$

# Concluding remarks

Recall the identity on page 26 of the Lost Notebook:

$$\sum_{n=-\infty}^{\infty} a^n q^{n^2/4} \sum_{n=0}^{\infty} \frac{(-a)^n q^{n^2/4}}{(q)_n} - \sum_{n=-\infty}^{\infty} (-a)^n q^{n^2/4} \sum_{n=0}^{\infty} \frac{a^n q^{n^2/4}}{(q)_n}$$
$$= 2q^{1/4}(q)_\infty \sum_{m=0}^{\infty} \frac{a^{-2m-1} q^{m^2+m}}{(bq)_m}.$$

- Can this identity be generalized in a similar manner as (1) was generalized to our main result?

# Concluding remarks

Recall the identity on page 26 of the Lost Notebook:

$$\sum_{n=-\infty}^{\infty} a^n q^{n^2/4} \sum_{n=0}^{\infty} \frac{(-a)^n q^{n^2/4}}{(q)_n} - \sum_{n=-\infty}^{\infty} (-a)^n q^{n^2/4} \sum_{n=0}^{\infty} \frac{a^n q^{n^2/4}}{(q)_n}$$
$$= 2q^{1/4}(q)_\infty \sum_{m=0}^{\infty} \frac{a^{-2m-1} q^{m^2+m}}{(bq)_m}.$$

- Can this identity be generalized in a similar manner as (1) was generalized to our main result?
- Do there exist analogues of Ramanujan's result where one has sum of products of *three or more* generalized Rogers-Ramanujan functions?

# Concluding remarks

Recall the identity on page 26 of the Lost Notebook:

$$\sum_{n=-\infty}^{\infty} a^n q^{n^2/4} \sum_{n=0}^{\infty} \frac{(-a)^n q^{n^2/4}}{(q)_n} - \sum_{n=-\infty}^{\infty} (-a)^n q^{n^2/4} \sum_{n=0}^{\infty} \frac{a^n q^{n^2/4}}{(q)_n}$$
$$= 2q^{1/4}(q)_\infty \sum_{m=0}^{\infty} \frac{a^{-2m-1} q^{m^2+m}}{(bq)_m}.$$

- Can this identity be generalized in a similar manner as (1) was generalized to our main result?
- Do there exist analogues of Ramanujan's result where one has sum of products of *three or more* generalized Rogers-Ramanujan functions?
- This is because identities for sums or differences of products of quadruples of Rogers-Ramanujan functions $G(q)$, $H(q)$ *do* exist.

# Concluding remarks: The work of Bressoud, O. Santos and Mondek

- Let $a = b = 1$ in the series $\sum_{m=0}^{\infty} a^{-sm} q^{sm(m-1)/2+m(k+s/2)}/(bq)_m$ occurring in the main result, where $s \in \mathbb{N}$ and $0 \leq k \leq s-1$.

[7] D. M. Bressoud, J. P. O. Santos and P. Mondek, *A family of partition identities proved combinatorially*, Ramanujan J. **4** (2000), 311–315.

# Concluding remarks: The work of Bressoud, O. Santos and Mondek

- Let $a = b = 1$ in the series $\sum_{m=0}^{\infty} a^{-sm} q^{sm(m-1)/2+m(k+s/2)}/(bq)_m$ occurring in the main result, where $s \in \mathbb{N}$ and $0 \leq k \leq s-1$.
- Its special case when $s$ is even and $k \leq s/2$ is considered by Bressoud, O. Santos and Mondek[7] who showed that three different restricted partitions have it as their generating function.

[7] D. M. Bressoud, J. P. O. Santos and P. Mondek, *A family of partition identities proved combinatorially*, Ramanujan J. **4** (2000), 311–315.

# Concluding remarks: The work of Bressoud, O. Santos and Mondek

- Let $a = b = 1$ in the series $\sum_{m=0}^{\infty} a^{-sm} q^{sm(m-1)/2+m(k+s/2)}/(bq)_m$ occurring in the main result, where $s \in \mathbb{N}$ and $0 \leq k \leq s-1$.
- Its special case when $s$ is even and $k \leq s/2$ is considered by Bressoud, O. Santos and Mondek[7] who showed that three different restricted partitions have it as their generating function.
- This suggests the remaining cases may also be of interest.

[7] D. M. Bressoud, J. P. O. Santos and P. Mondek, *A family of partition identities proved combinatorially*, Ramanujan J. **4** (2000), 311–315.

Thank You!!