Non-Rascoe partitions and a rank parity function associated with the Rogers-Ramanujan partitions

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Seminar in Partition Theory, q-Series and Related Topics

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- For example, the rank of the partition $6+5+5+4+2+1+1+1$ of 25 is $6-8=-2$.
- Enumeration of partitions belonging to a particular class based on the parity of their rank is often useful, for, their generating functions often turn out to be important in combinatorics and modular forms.

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where $(a)_0 := (a;q)_0 = 1,$

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- If we now let $z=-1$, we see that the generating function of the number of partitions of n with even rank minus those with odd rank is Ramanujan's third order mock theta function

$$f(q) := \sum_{n=0}^{\infty}\frac{q^{n^2}}{(-q;q)_n^2}.$$

Well-known rank-parity functions

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- $q^{1/24}\sigma(q)$ is a prototypical example of a quantum modular form.
- The coefficients of $\sigma(q)$ have properties governed by the arithmetic of $Q(\sqrt{6})$.

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- The rank parity function associated with partitions into part with gap at least 1 is rich in properties and has implications in algebraic number theory and quantum modular forms apart from theory of partitions and q-series.
- Thus, it makes sense to study the corresponding rank parity function associated to partitions into parts with gap at least 2, that is, the Rogers-Ramanujan partitions.

The rank parity function $\sigma_2(q)$

- Let $R(m,n)$ to be the number of Rogers–Ramanujan partitions of n with rank m. Our first result is[1]

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Theorem (D.-Kumar-Srivastava)

We have

$$\sum_{n=0}^{\infty}\sum_{m=-\infty}^{\infty} R(m,n)z^m q^n = 1 + \sum_{n=1}^{\infty} \frac{z^{n-1}q^{n^2}}{(zq)_n}.$$

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- Let $z=-1$ in this result shows that the generating function of the excess number of Rogers–Ramanujan partitions with *odd* rank over those with *even* rank is the function $-2+\sigma_2(q)$, where

$$\sigma_2(q) := \sum_{n=0}^{\infty}\frac{(-1)^n q^{n^2}}{(-q)_n} = 1-q+q^2-q^3+2q^4-2q^5+q^6-q^7+\cdots.$$

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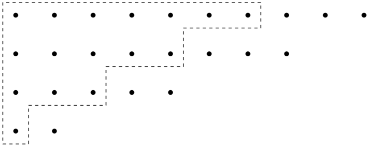
- Thus, $\frac{z^{n-1}q^{n^2}}{(zq)_n}$ generates a Rogers–Ramanujan partition where the number of parts is n and z keeps track of the rank $k-n$.



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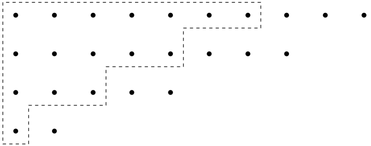

Another proof using MacMahon's Ω-operator method:

$$\sum_{n=0}^{\infty}\sum_{m=-\infty}^{\infty} R(m,n)z^m q^n = 1 + \sum_{j=1}^{\infty}\sum_{n_1=1}^{\infty}\sum_{n_2=n_1+2}^{\infty}\cdots\sum_{n_j=n_{j-1}+2}^{\infty} z^{n_j-j}q^{n_1+n_2+\cdots+n_j}.$$

Rascoe partitions

- John Tyler Rascoe[2] considered an interesting class of restricted partitions, namely, *the partitions of a positive integer into distinct parts in which the number of parts itself is a part of the partition.*

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- The Rascoe partitions of 11 are $9+2$, $7+3+1$ and $6+3+2$.
- Let $a(n)$ denote the number of Rascoe partitions of n. Then Rascoe gave the following generating function of $a(n)$:

$$\sum_{n=1}^{\infty} a(n)q^n = \sum_{n=1}^{\infty} q^{n(n+1)/2} \sum_{m=1}^{n} \begin{bmatrix} n-1 \\ m-1 \end{bmatrix} \frac{q^{m(m-1)}}{(q)_{m-1}}.$$

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Non-Rascoe partitions

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- Let $b(n)$ denote the number of non-Rascoe partitions of n.
- Clearly,

$$\sum_{n=0}^{\infty} b(n)q^n = (-q)_\infty - \sum_{n=1}^{\infty} q^{n(n+1)/2} \sum_{m=1}^{n} \begin{bmatrix} n-1 \\ m-1 \end{bmatrix} \frac{q^{m(m-1)}}{(q)_{m-1}}.$$

The main result

- The two objects introduced so far, namely, the rank parity function $\sigma_2(q) := \sum_{n=0}^{\infty} \frac{(-1)^n q^{n^2}}{(-q)_n}$ and the number of non-Rascoe partitions of n are intimately connected.

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Theorem (D.-Kumar-Srivastava)

$$\sum_{n=0}^{\infty} b(n)q^n = (-q)_\infty \sigma_2(q) = \sum_{n=0}^{\infty} (-1)^n q^{n^2} (-q^{n+1})_\infty.$$

In other words, the non-Rascoe partitions are generated by $(-q)_\infty \sigma_2(q)$.

Two applications of the main result

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The number $b(n)$ is odd if and only if $n = m(5m+1)/2$, where $m \in \mathbb{Z}$.

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Then an application of the first Rogers-Ramanujan identity followed by Jacobi triple product identity yields

$$(q)_\infty \sum_{n=0}^{\infty} \frac{q^{n^2}}{(q)_n} = \frac{(q)_\infty}{(q;q^5)_\infty (q^4;q^5)_\infty} = (q^2;q^5)_\infty (q^3;q^5)_\infty (q^5;q^5)_\infty$$
$$= \sum_{n=-\infty}^{\infty} (-1)^n q^{n(5n+1)/2}.$$

Two applications of the main result

Theorem

Let $P(j,n)$ be the number of partitions of n into distinct parts with the exception that the smallest part, say j, is allowed to repeat exactly j number of times. Then the excess number of such partitions with even smallest part over those with odd smallest part equals the number of non-Rascoe partitions of n, that is,

$$\sum_{j=0}^{n}(-1)^j P(j,n) = b(n),$$

where, $P(0,n)$ is, clearly, the number of partitions of n into distinct parts.

Example

- Let $n = 11$. We know that $b(11) = 9$.

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Example

- Let $n = 11$. We know that $b(11) = 9$.
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- $P(0, 11) = 12$ (the number of partitions of 11 into distinct parts)
- $P(1, 11) = 5$ as the number of admissible partitions are $10 + 1, 8 + 2 + 1, 7 + 3 + 1, 6 + 4 + 1$ and $5 + 3 + 2 + 1$.

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- $P(2, 11) = 2$, since only $7 + 2 + 2$ and $4 + 3 + 2 + 2$ are admissible.

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- Then $P(0, 11) - P(1, 11) + P(2, 11) = 12 - 5 + 2 = 9$, which agrees with $b(11)$.

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- Then $P(0, 11) - P(1, 11) + P(2, 11) = 12 - 5 + 2 = 9$, which agrees with $b(11)$.
- Andrews and El Bachraoui[3] have recently studied partitions where the smallest part appears exactly k times, where $k \in \mathbb{N}$ is fixed, and the remaining parts are distinct.

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Hecke-Rogers type representation for $\sigma_2(q)$

- An important feature of $\sigma(q)$ is that it has a Hecke-Rogers type representation due to Andrews, Dyson and Hickerson[4], namely,

$$\sigma(q) = \sum_{\substack{n\geq 0\\ |j|\leq n}}^{\infty} (-1)^{n+j} q^{n(3n+1)/2-j^2}(1-q^{2n+1}).$$

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- For $\sigma_2(q)$, we found the following Hecke-Rogers type representations:

Theorem

(i) $$\sigma_2(q) = \frac{1}{(-q)_\infty}\left\{1+\sum_{n=1}^{\infty}\frac{(-1)^n q^{n(5n-1)/2}(1+q^{2n})}{(1+q^n)}\sum_{j=0}^{n}\frac{(-1)_j}{(q)_j}\left(-q^{1-n}\right)^j\right\},$$

(ii) $$\sigma_2(q) = \frac{1}{(-q)_\infty}\left\{1-\sum_{n=1}^{\infty}\frac{(-1)^n q^{n(5n-1)/2}(1+q^{2n})}{(1-q^n)}\sum_{m=0}^{n-1}\frac{(-1)_m}{(q)_m}(-q^{-n})^m\right\}.$$

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Generating functions of Rascoe & non-Rascoe partitions

Theorem

$$\text{(i)} \sum_{n=1}^{\infty} a(n)q^n = \sum_{n=1}^{\infty} q^{n(n+1)/2} \sum_{m=1}^{n} \begin{bmatrix} n-1 \\ m-1 \end{bmatrix} \frac{q^{m(m-1)}}{(q)_{m-1}}, \tag{1}$$

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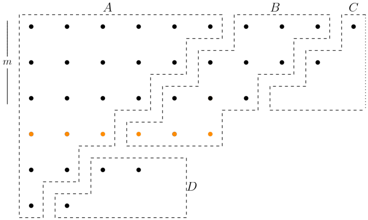
Proof: Assume n to be the number of parts in a partition, and m to be the number of parts greater than n. Clearly, $0 \le m \le n-1$.



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- Now sum over n from 1 to ∞ and replace m by $m-1$ to arrive at the required generating function.

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$$\text{To prove} \quad \sum_{n=1}^{\infty} b(n)q^n = \sum_{n=1}^{\infty} q^{n(n+1)/2} \sum_{m=1}^{n} \begin{bmatrix} n-1 \\ m-1 \end{bmatrix} \frac{q^{m^2}}{(q)_m},$$

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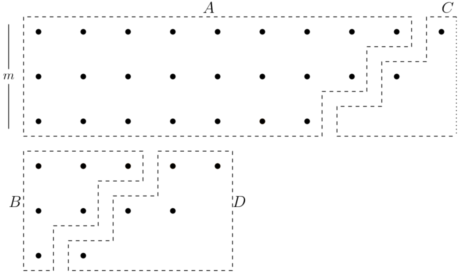

- The region A is formed from parts $> n$ which are also distinct, it contains exactly the parts $n+1, \cdots, n+m$. Therefore, it contributes $q^{nm+m(m+1)/2}$ towards the generating function.

Generating functions of Rascoe & non-Rascoe partitions

- Since the region B is formed by taking k nodes from k^{th} part of the partition, where $1 \leq k \leq n - m$, it will contribute $q^{(n-m)(n-m+1)/2}$.

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- Summing over n yields the required generating function.

Alternate representation of $\sigma_2(q)$ using conjugation

Theorem

$$\sum_{n=0}^{\infty} \frac{z^n q^{n^2}}{(-q)_n} = \frac{1}{(-q)_\infty} \sum_{n=0}^{\infty} q^{n(n+1)/2} \sum_{i=0}^{n} \frac{z^i q^{i(i-1)/2}}{(q)_{n-i}},$$

In particular, $$\sigma_2(q) = \frac{1}{(-q)_\infty} \sum_{n=0}^{\infty} q^{n(n+1)/2} \sum_{i=0}^{n} \frac{(-1)^i q^{i(i-1)/2}}{(q)_{n-i}}.$$

Proof: Let $P(i, N)$ denote the number of partitions of N into distinct parts, except the smallest part, say i, which repeats exactly i number of times.

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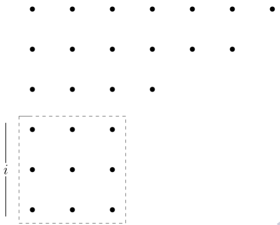

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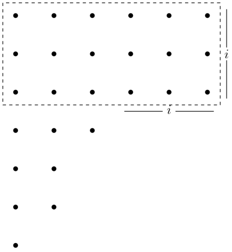
Take the conjugate of the above partition.



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The corresponding generating function of the number of such conjugate partitions, where z now keeps track of the number of times the largest part repeats, is given by

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The result now follows from (3) and (4).

A finite analogue of an identity of Bhargava and Adiga

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[5] H. M. Srivastava, *A note on a generalization of a q-series transformation of Ramanujan*, Proc. Japan Acad. Ser. A **63** (1987), 143–145.

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Theorem

$$\sum_{m=0}^{n} \begin{bmatrix} n \\ m \end{bmatrix} \frac{(-\lambda/a)_m a^m q^{m(m+1)/2}}{(-bq)_m} = \frac{(-aq)_n}{(-bq)_n} \sum_{m=0}^{n} \begin{bmatrix} n \\ m \end{bmatrix} \frac{(-\lambda/b)_m b^m q^{m(m+1)/2}}{(-aq)_m}.$$

We derived this result using Andrews' finite Heine transformation.

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Final step towards the proof of the main result

Letting $a \to 0$ in the above identity, we obtain

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$$\sigma_2(q) = \frac{1}{(-q)_\infty} \sum_{n=0}^{\infty} q^{n(n+1)/2} \sum_{i=0}^{n} \frac{(-1)^i q^{i(i-1)/2}}{(q)_{n-i}}.$$

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which proves the result.

Rascoe partitions and “almost” distinct partitions

Theorem

Let $P(j,n)$ be the number of partitions of n into distinct parts with the exception that the smallest part, say j, is allowed to repeat exactly j number of times. Then the excess number of such partitions with even smallest part over those with odd smallest part equals the number of non-Rascoe partitions of n, that is,

$$\sum_{j=0}^{n}(-1)^j P(j,n) = b(n),$$

where, $P(0,n)$ is, clearly, the number of partitions of n into distinct parts.

Proof

- Recall that we have proved

$$\sum_{N=0}^{\infty}\sum_{i=0}^{N} P(i,N) z^i q^N = \sum_{i=0}^{\infty} z^i q^{i^2} (-q^{i+1})_\infty.$$

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- The result simply follows by letting $z=-1$ in the above identity and using the main result

$$\sum_{n=0}^{\infty} b(n) q^n = (-q)_\infty \sigma_2(q) = \sum_{n=0}^{\infty} (-1)^n q^{n^2}(-q^{n+1})_\infty.$$

- Let $\ell \in \mathbb{N} \cup \{0\}$. We define a *generalized Rogers–Ramanujan partition* of a number N to be a partition of N into parts $> \ell$ and in which the difference between any two parts is greater than or equal to 2.

[6] K. Garrett, M. E. H. Ismail and D. Stanton, *Variants of Rogers–Ramanujan identities*, Adv. Appl. Math. **23** (1999), 274–299.

Generalized Rogers-Ramanujan partitions and identities

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- Garrett, Ismail and Stanton[6] showed that the generating function of the number of generalized Rogers–Ramanujan partitions satisfies the generalized Rogers–Ramanujan identity

$$\sum_{n=0}^{\infty} \frac{q^{n^2+\ell n}}{(q)_n} = \frac{(-1)^\ell q^{-\ell(\ell-1)/2} c_\ell(q)}{(q;q^5)_\infty (q^4;q^5)_\infty} - \frac{(-1)^\ell q^{-\ell(\ell-1)/2} d_\ell(q)}{(q^2;q^5)_\infty (q^3;q^5)_\infty},$$

where $c_\ell(q)$ and $d_\ell(q)$ are Laurent polynomials in q defined by

$$c_\ell(q) := \sum_j (-1)^j q^{\frac{j(5j-3)}{2}} \begin{bmatrix} \ell-1 \\ \left\lfloor \frac{\ell+1-5j}{2} \right\rfloor \end{bmatrix}, \quad d_\ell(q) := \sum_j (-1)^j q^{\frac{j(5j+1)}{2}} \begin{bmatrix} \ell-1 \\ \left\lfloor \frac{\ell-1-5j}{2} \right\rfloor \end{bmatrix}.$$

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Generalizations of the previous results

Throughout the sequel, we consider $\ell \in \mathbb{N} \cup \{0\}$.

Theorem

$$\sum_{N=0}^{\infty} \sum_{m=-\infty}^{\infty} R_{\ell}(m, N) z^m q^N = 1 + \sum_{n=1}^{\infty} \frac{z^{n+\ell-1} q^{n^2+\ell n}}{(zq)_n}.$$

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A generalized rank-parity function

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$$\sigma_{2,\ell}(q) := \sum_{n=0}^{\infty} \frac{(-1)^n q^{n^2+\ell n}}{(-q)_n}.$$

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Theorem

$$\sum_{n=0}^{\infty} b_\ell(n) q^n = (-q)_\infty \sigma_{2,\ell}(q).$$

Parity of $b_\ell(n)$

Theorem

Let $\ell \in \mathbb{N} \cup \{0\}$ and let $b_\ell(n)$ denote the generalized non-Rascoe partitions of n. Also, let $p(N, M, k)$ denote the number of partitions of k into at most M parts, each $\leq N$. Then,

$$\sum_{n=0}^{\infty} b_\ell(n) q^n$$
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Parity of $b_\ell(n)$

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Corollary

(i) $b_1(n)$ *is odd if and only if* $n = \frac{m(5m-3)}{2}$, *where* $m \in \mathbb{Z}$,
(ii) $b_2(n)$ *is odd if and only if* $n = \frac{m(5m+1)}{2} - 1$ *or* $n = \frac{m(5m-3)}{2} - 1$, *where* $m \in \mathbb{Z}$.

Hecke-Rogers type representations for $\sigma_{2,\ell}(q)$

Theorem

$$\sigma_{2,\ell}(q) = \frac{1}{(-q)_\infty}\left\{(-q)_\ell - \sum_{n=1}^{\infty}(-1)^{n+\ell} q^{\frac{n(5n-1)}{2}+2\ell n}(1+q^{\ell+2n})(q^{n+1})_{\ell-1} \times \sum_{m=0}^{n+\ell-1}\frac{(-1)_m}{(q)_m}(-q^{-n})^m\right\},$$

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A generalized modular relation of Ramanujan

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This identity can be used to relate the generating functions of $b_\ell(n)$ and $b_{\ell+1}(n)$.

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A relation between the generating functions of $b_\ell(n)$ and $b_{\ell+1}(n)$

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Generating functions of $b_1(n), b_2(n)$ and tenth order mock theta functions

Corollary

(i) $$\sum_{m=0}^{\infty}\frac{(-1)^m q^{m^2}}{(-q)_m}\sum_{n=0}^{\infty}\frac{(-1)^n q^{n^2}}{(q)_{2n}}-\sum_{m=0}^{\infty}\frac{(-1)^m q^{m^2+m}}{(-q)_m}\sum_{n=0}^{\infty}\frac{(-1)^n q^{(n+1)^2}}{(q)_{2n+1}}$$
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Tenth order mock theta functions:

$$X(q):=\sum_{n=0}^{\infty}\frac{(-1)^n q^{n^2}}{(-q)_{2n}}, \textit{ and } \chi(q):=\sum_{n=0}^{\infty}\frac{(-1)^n q^{(n+1)^2}}{(-q)_{2n+1}}.$$

A congruence

Theorem

Let $c_X(n)$ *and* $c_\chi(n)$ *be the coefficients of the tenth order mock theta functions* $X(q)$ *and* $\chi(q)$. *Then, for* $k \in \mathbb{N}$,

$$\sum_{n=0}^{k} b(n)c_X(k-n) + b_1(n)c_\chi(k-n) \equiv 0 \pmod 2.$$

[8] S.-Y. Kang, S. Kim, T. Matsusaka and J. Yoo, *Hauptmoduln and even-order mock theta functions modulo* 2, J. Korean Math. Soc., **62** No. 5 (2025), 1297–1312.

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Kang, Kim, Matsusaka and Yoo[8] proved $c_5(40n-1) \equiv c_{X_{10}}(n) \pmod 2$ and $c_5(40n-9) \equiv c_{\chi_{10}}(n) \pmod 2$, where $c_5(n)$ is the Fourier coefficient of $j_5(\tau) := \frac{1}{q}\frac{(q;q)_\infty^6}{(q^5;q^5)_\infty^6} = \frac{1}{q} - 6 + \sum_{n=1}^\infty c_5(n)q^n$. Hence, we have

$$\sum_{n=0}^{k} b(n)c_5(40k-40n-1) + b_1(n)c_5(40k-40n-9) \equiv 0 \pmod 2.$$

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Unrestricted Rascoe and non-Rascoe partitions

- Let $c(n)$ denote the partitions of n in which the number of parts is a part, and let $e(n)$ denote the partitions of n in which the number of parts is not a part.

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George Beck[9] has conjectured that $e(n)$ is the total number of distinct parts of each partition of $2n+2$ with rank $n+1$.

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Concluding remarks

- While $\sigma_2(q)$ is intimately connected with the generating function of restricted partition function $b(n)$,

[10] P. S. Kaur, S. C. Bhoria, P. Eyyunni and B. Maji, *Minimal excludant over partitions into distinct parts*, Int. J. Number Theory **18** no. 9, (2022), 2015–2028.

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$$(-q)_\infty \sigma(q) = \sum_{n=0}^{\infty} \sigma_d \mathrm{mex}(n) q^n,$$

where $\sigma_d\mathrm{mex}(n) := \sum_{\pi \in \mathcal{D}(n)} \mathrm{mex}(\pi)$ with $\mathrm{mex}(\pi)$ denotes the *minimal excludant* of the partition π, that is, the smallest positive integer that is not a part of π, and $\mathcal{D}(n)$ denotes the set of partitions of n into distinct parts.

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Concluding remarks: Asymptotics of $b(n)$?

- Ramanujan's Lost Notebook:

$$\sigma(q) = 1 + q\sum_{n=0}^{\infty}(q)_n(-q)^n. \tag{6}$$

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- Thus, if there exists a representation for $\sigma_2(q)$ analogous to (6) and one is able to proceed in a similar manner as that for $\sigma(q)$ it may allow us to find an asymptotic formula of $b(n)$ as $n \to \infty$.

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- For eg., for $0 \leq m \leq 118$, $b(29 \cdot 29m + 21)$ is divisible by 4 when $m = 0, 1, 7, 8, 13, 19, 22, 27, 28, 29, 32, 37, 41, 44, 47, 48, 49, 50, 51, 52, 53, 57, 64, 67, 69, 74, 75, 76, 77, 78, 79, 81, 82, 83, 84, 85, 89, 95, 100, 102, 104, 106, 108, 109, 115, 116, 117$ and 118.

A mod 4 congruence conjecture

- Numerical calculations led us to the following exotic conjecture.
- **Conjecture:** Let $b(n)$ denote the number of non-Rascoe partitions of an integer n. For $k \geq 1$ and k not a multiple of 29, the following congruence holds:

$$b(29k+21) \equiv 0 \pmod 4.$$

- This conjecture has been verified for the first 10^5 values in $\{b(n)\}_{n=0}^{\infty}$.
- If k is a multiple of 29, say, $k = 29m$, then $b(29k+21)$ may or may not be divisible by 4.
- For eg., for $0 \leq m \leq 118$, $b(29 \cdot 29m + 21)$ is divisible by 4 when $m = 0, 1, 7, 8, 13, 19, 22, 27, 28, 29, 32, 37, 41, 44, 47, 48, 49, 50, 51, 52, 53, 57, 64, 67, 69, 74, 75, 76, 77, 78, 79, 81, 82, 83, 84, 85, 89, 95, 100, 102, 104, 106, 108, 109, 115, 116, 117$ and 118.
- It is not divisible by 4 for the remaining values of m for $0 \leq m \leq 118$.

A mod 4 congruence conjecture

```
In[42]:= For[i = 1, i ≤ 3447, i++,
          Print["i=", i, ", v[", 29 * i + 22, "]/4->", Part[v / 4, 29 * i + 22]]]

i=1, v[51]/4->610
i=2, v[80]/4->11 752
i=3, v[109]/4->142 685
i=4, v[138]/4->1 292 884
i=5, v[167]/4->9 520 703
i=6, v[196]/4->59 915 930
i=7, v[225]/4->332 934 022
i=8, v[254]/4->1 670 900 892
i=9, v[283]/4->7 699 747 703
i=10, v[312]/4->32 986 689 459
i=11, v[341]/4->132 659 038 514
i=12, v[370]/4->504 675 790 149
i=13, v[399]/4->1 827 582 396 612
i=14, v[428]/4->6 332 375 280 317
i=15, v[457]/4->21 083 951 124 723
i=16, v[486]/4->67 704 349 034 656
i=17, v[515]/4->210 336 744 474 961
i=18, v[544]/4->633 895 496 144 200
i=19, v[573]/4->1 857 559 541 768 440
```

A mod 4 congruence conjecture

```
i=20, v[602]/4->5 303 759 392 595 666
i=21, v[631]/4->14 781 881 135 541 676
i=22, v[660]/4->40 279 220 804 471 241
i=23, v[689]/4->107 464 142 278 707 666
i=24, v[718]/4->281 085 728 617 730 780
i=25, v[747]/4->721 627 327 214 380 656
i=26, v[776]/4->1 820 306 056 914 286 117
i=27, v[805]/4->4 515 955 297 041 751 515
i=28, v[834]/4->11 028 298 139 173 296 985
i=29, v[863]/4->26 531 915 178 401 943 905
i=30, v[892]/4->62 928 731 974 381 123 826
i=31, v[921]/4->147 245 934 756 486 123 493
i=32, v[950]/4->340 112 541 538 682 434 878
i=33, v[979]/4->775 957 215 788 017 863 330
i=34, v[1008]/4->1 749 529 642 389 000 991 807
i=35, v[1037]/4->3 900 223 048 751 474 083 499
i=36, v[1066]/4->8 600 893 847 126 824 048 132
i=37, v[1095]/4->18 770 320 895 889 141 280 840
i=38, v[1124]/4->40 555 478 458 970 247 818 128
i=39, v[1153]/4->86 784 410 177 740 670 779 704
i=40, v[1182]/4->183 993 605 392 779 312 110 388
i=41, v[1211]/4->386 614 145 041 143 360 535 458
i=42, v[1240]/4->805 384 710 731 945 224 993 991
```

A mod 4 congruence conjecture

```
i=43,  v[1269]/4->1 663 828 046 866 705 733 662 796
i=44,  v[1298]/4->3 409 686 715 761 266 480 881 367
i=45,  v[1327]/4->6 933 242 266 107 579 247 137 022
i=46,  v[1356]/4->13 992 098 393 915 947 106 757 979
i=47,  v[1385]/4->28 032 181 780 224 036 688 022 552
i=48,  v[1414]/4->55 764 295 956 197 855 156 468 750
i=49,  v[1443]/4->110 172 597 235 272 160 058 211 254
i=50,  v[1472]/4->216 220 654 083 780 952 074 672 305
i=51,  v[1501]/4->421 609 996 593 875 079 031 862 795
i=52,  v[1530]/4->816 948 493 306 785 102 876 551 773
i=53,  v[1559]/4->1 573 347 656 836 679 488 992 878 101
i=54,  v[1588]/4->3 012 129 690 212 181 557 085 158 048
i=55,  v[1617]/4->5 733 383 834 212 005 676 222 597 301
i=56,  v[1646]/4->10 851 833 379 685 187 366 607 353 378
i=57,  v[1675]/4->20 427 416 900 015 535 853 888 124 274
```

i=58, v[1704]/4-> $\dfrac{152\,989\,912\,130\,680\,123\,609\,695\,048\,519}{4}$

A mod 4 congruence conjecture

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- Probably its appearance can be better understood once the modularity of $\sigma_2(q)$ is established.
- What can be said about the exceptions to the condition $k \neq 29m$?

Concluding remarks

- In this work, we have concentrated only on the number of non-Rascoe partitions, that is, $b(n)$, or its generalization $b_\ell(n)$.

Concluding remarks

- In this work, we have concentrated only on the number of non-Rascoe partitions, that is, $b(n)$, or its generalization $b_\ell(n)$.
- The number of Rascoe partitions $a(n)$ and its generalization $a_\ell(n)$ also warrant a serious study.

Thank You!!