Double and Triple Perfect Partitions

Augustine O. Munagi

Seminar in Partition Theory, q-Series and Related Topics
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Presentation Order

- Background and Definitions
- Perfect Partitions
- Double Perfect Partitions
- New Proofs and *p*-Perfect Partitions
- Triple Perfect Partitions
- Triple4 Perfect Partitions
- Partition Identities

What is a partition?

A partition λ of n is any *weakly increasing* sequence of positive integers $(\lambda_1, \lambda_2, \ldots, \lambda_k)$ such that $\lambda_1 + \cdots + \lambda_k = n$.

For example 4 has five partitions namely

$$(4), (1, 3), (2, 2), (1, 1, 2), (1, 1, 1, 1).$$

Alternative Notation:

$(\lambda_1^{v_1}, \lambda_2^{v_2}, \ldots, \lambda_t^{v_t}),\ 0 < \lambda_1 < \lambda_2 < \cdots < \lambda_t,\ v_i > 0,\ t \leq k$:

$$(4), (1, 3), (2^2), (1^2, 2), (1^4).$$

Basic Objects

Let λ be a partition of n, denoted by $\lambda \vdash n$.

How many nonempty partitions of $m \leq n$ are contained in λ?

Let $G(\lambda) :=$ set of nonempty partitions contained in λ.

E. g.

$$G((1^3, 2^2)) : (1), (1^2), (2), (1^3), (1, 2), (1^2, 2), (2^2), (1^3, 2),$$
$$(1, 2^2), (1^2, 2^2), (1^3, 2^2).$$

Note that

$$|G((\lambda_1^{v_1}, \lambda_2^{v_2}, \ldots, \lambda_k^{v_k}))| = (v_1 + 1)(v_2 + 1) \cdots (v_k + 1) - 1.$$

Perfect Partitions

Definition. (MacMahon)
A perfect partition of n is a partition that contains exactly one partition of every positive integer less than or equal to n.

A perfect partition $\lambda \vdash n$ satisfies $|G(\lambda)| = n$.

Notation: $\mathrm{Per}(n) :=$ set of perfect partitions of n.
$\mathrm{per}(n) := |\,\mathrm{Per}(n)|$, number of perfect partitions of n.

E.g., $\mathrm{Per}(5) = \{(1^5),\ (1, 2^2),\ (1^2, 3)\}$. So $\mathrm{per}(5) = 3$.

Thus, for insatnce,

$$G((1^2, 3)) : (1), (1^2), (3), (1, 3), (1^2, 3).$$

Perfect Partitions and Ordered Factorizations

Perfect partitions are most easily found using ordered factorizations.

Theorem 1

The number of perfect partitions of n is equal to the number of ordered factorizations of $n+1$ without unit factors.

Bijection. An ordered factorization $n+1 = q_1 q_2 \cdots q_r$, $q_i > 1$, corresponds to the perfect partition

$$\lambda = (1^{q_1 - 1}, q_1^{q_2 - 1}, (q_1 q_2)^{q_3 - 1}, \ldots, (q_1 q_2 \cdots q_{r-1})^{q_r - 1}).$$

This image is a partition of n and contains a unique partition of each m, $1 \leq m \leq n$.

Ordered Factorization of 6	6	$2 \cdot 3$	$3 \cdot 2$
Perfect Partition of 5	(1^5)	$(1, 2^2)$	$(1^2, 3)$

Let $f(n,k) :=$ number of ordered factorizations of n into k factors; $f(n) := f(n,1) + f(n,2) + \cdots$. Then

Theorem 2 (MacMahon, Andrews)

$$f(n,k) = \sum_{i=0}^{k} (-1)^i \binom{k}{i} \prod_{j=1}^{r} \binom{\alpha_j + k - i - 1}{\alpha_j},$$

where $n+1 = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_r^{\alpha_r}$ is the prime factorization and $1 \le k \le \alpha_1 + \alpha_2 + \cdots + \alpha_r$.

$f(n,k)$ counts perfect partitions of $n-1$ with k blocks (or runs) of equal parts, and $f(n) = \mathrm{per}(n-1)$:

$$\mathrm{per}(n) = f(n+1).$$

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Generalizations and Extensions of Perfect Partitions

- Complete Partitions (Park 1998): partitions $\lambda \vdash n$ that contain at least one partition of every positive integer $< n$.
- M-Partitions (O'Shea 2004): complete partitions with minimal lengths.
- Double Perfect Partitions (Lee 2006): ... discussed below ...
- n-Color Perfect Partitions (Agarwal and Sachdeva 2018): n-color partitions $\lambda \vdash n$ that contain one n-color partition of every positive integer $< n$.
- Perfect Compositions (M. 2020): compositions of n that contain one composition of every positive integer $< n$.
- Full K-Complete Partitions (M. and Takalani 2022): partitions that contain all partitions of every positive integer up to k.
- ... and so forth.

Double Perfect Partitions

Is there a partition of n that contains $t \geq 1$ partitions of each m, $t \leq m \leq n - t$ and one partition of every other integer not exceeding n?

Lee (2006) showed that the answer is 'yes' only when $t = 1$ or 2.

The case $t = 1$ gives perfect partitions!

He decided to study the seemingly overlooked case of $t = 2$.

Double Perfect Partitions (Lee 2006)

A double-perfect partition is a partition $(\lambda_1, \lambda_2, \ldots, \lambda_k) \vdash n$ such that each integer m, $2 \leq m \leq n - 2$ can be represented exactly twice as $m = \sum_{i=1}^{k} \alpha_i \lambda_i$, where $\alpha_i \in \{0, 1\}$.

E. g. $\lambda = (1^5, 2) \vdash 7$ is double-perfect:

$G(\lambda) : (1), (1^2), (2), (1^3), (1, 2), (1^4), (1^2, 2), (1^5), (1^3, 2), (1^4, 2), (1^5, 2).$

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$G(\lambda) : (1), (1^2), (2), (1^3), (1, 2), (1^4), (1^2, 2), (1^5), (1^3, 2), (1^4, 2), (1^5, 2).$

General Form (Lee 2006)

A double-perfect partition $\lambda \vdash n$ has the following form

$$(1^{a_1}, 2^{a_2}, (a_1 + 2a_2 - 1)^{a_3}, ((a_1 + 2a_2 - 1)(a_3 + 1))^{a_4}, \\ ((a_1 + 2a_2 - 1)(a_3 + 1)(a_4 + 1))^{a_5}, \ldots),$$

where $a_1 \geq 2$ and $a_2, a_3, \ldots$ are positive integers such that if $a_1 \neq 3$ then $a_2 = 1$.

Theorem 3

The number $d(n)$ of double-perfect partitions of n is given by

$$d(n) = \begin{cases} f(n-1) & \text{if } n \not\equiv 1 \pmod 4, \\ f(n-1) - f\left(\frac{n-1}{4}\right) & \text{if } n \equiv 1 \pmod 4. \end{cases}$$

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In the course of proving Theorem 3,

Lee **separated the general form into two types** of double-perfect partitions:

$$(1^3, 2^{a_2}, (2(a_2+1))^{a_3}, (2(a_2+1)(a_3+1))^{a_4}, \ldots, \\ (2(a_2+1)(a_3+1)\cdots(a_{r-1}+1))^{a_r}), \;\; a_2 \geq 2, \qquad (1)$$

$$(1^{a_1}, 2, (a_1+1)^{a_2}, ((a_1+1)(a_2+1))^{a_3}, \ldots, \\ ((a_1+1)(a_2+1)\cdots(a_{r-1}+1))^{a_r}), \;\; a_1 \geq 2. \qquad (2)$$

These then **imply** the following ordered factorizations:

$$n-1 = 2(a_2+1)(a_3+1)\cdots(a_{r-1}+1)(a_r+1), \;\; a_2 \geq 2 \qquad (1a)$$

$$n-1 = (a_1+1)(a_2+1)\cdots(a_{r-1}+1)(a_r+1), \;\; a_1 \geq 2. \qquad (2a)$$

(excluding the factorizations $n-1 = 2 \cdot 2 \cdot (a_3+1)\cdots$).

Observation for **New Proofs**

The foregoing results on double-perfect partitions can be obtained by starting out with perfect partitions.

The $d(n)$-formula shows that $d(n) \longleftarrow f(n-1) \longrightarrow \text{per}(n-2)$.

So the set $D(n)$ of double-perfect partitions of n can be found from perfect partitions by **inserting parts of total weight 2**.

Recall:

$$q_1 q_2 \cdots q_r \longleftrightarrow (1^{q_1-1}, q_1^{q_2-1}, (q_1 q_2)^{q_3-1}, \ldots, (q_1 q_2 \cdots q_{r-1})^{q_r-1})$$

Double Perfect Partitions from Ordered Factorizations

Proposition 1

Every double-perfect partition $\lambda \vdash N > 3$ may be obtained from a perfect partition $\beta \vdash N-2$ in two ways:

I. If the multiplicity of 1 in β is 1, then insert 1^2 into β. Denote the resulting set by $A(1^2)$.

II. If β does not contain 2 as a part, insert 2 into β. Denote the resulting set by $B(2)$.

Then

$$D(N) = A(1^2) \cup B(2).$$

Proof. Let $h(m) \in G(\beta)$, $1 \leq m \leq n$ and write (ρ, γ) for $\rho \cup \gamma$.
Assume that $\lambda \vdash N$ is obtained from $\beta \in \mathrm{Per}(N-2)$ by insertion of 1^2 or 2 according to I or II respectively.
We find one additional partition of each $j \in \{2, 3, \ldots, N-2\}$, namely $((1^2), h(j-2))$ or $((2), h(j-2))$.
Then one new partition of each of $N-1$ and N appears, that is, $((1^2), h(N-3))$ or $((2), h(N-3))$ and $((1^2), h(N-2))$ or $((2), h(N-2))$.
Thus the resulting partition λ is double-perfect.

E.g., let $\beta = (1^2, 3) \in \mathrm{Per}(5)$. Then from II,
$\lambda = ((2), \beta) = (1^2, 2, 3) \in D(7)$, and our proof runs as follows:

j	$h(j) \in G(\beta)$	$\gamma \in G(\lambda) \setminus G(\beta)$
1	(1)	$-$
2	(1^2)	$((2), h(0)) = (2)$
3	(3)	$((2), h(1)) = (1, 2)$
4	$(1, 3)$	$((2), h(2)) = (1^2, 2)$
5	$(1^2, 3)$	$((2), h(3)) = (2, 3)$
6	$-$	$((2), h(4)) = (1, 2, 3)$
7	$-$	$((2), h(5)) = (1^2, 2, 3)$

All members of $D(7)$ are obtained as follows:

Ordered Factorization of 6	6	$2 \cdot 3$	$3 \cdot 2$
Perfect Partitions of 5	(1^5)	$(1, 2^2)$	$(1^2, 3)$
Insert Parts	2	1^2	2
Double-Perfect Partition of 7	$(1^5, 2)$	$(1^3, 2^2)$	$(1^2, 2, 3)$

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3	(3)	$((2), h(1)) = (1, 2)$
4	$(1, 3)$	$((2), h(2)) = (1^2, 2)$
5	$(1^2, 3)$	$((2), h(3)) = (2, 3)$
6	$-$	$((2), h(4)) = (1, 2, 3)$
7	$-$	$((2), h(5)) = (1^2, 2, 3)$

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Shape and Enumeration

• Obtain the perfect partitions β corresponding to factorizations of the forms $N-1 = 2q_2q_3\cdots q_k$, $q_2 > 2$ and $N-1 = q_1q_2q_3\cdots q_k$, $q_1 > 2$:

I. $\beta = (1, 2^{q_2-1}, (2q_2)^{q_3-1}, (2q_2q_3)^{q_4-1}, \ldots, (2q_2q_3\cdots q_{k-1})^{q_k-1})$.

II. $\beta = (1^{q_1-1}, q_1^{q_2-1}, (q_1q_2)^{q_3-1}, \ldots, (q_1q_2q_3\cdots q_{k-1})^{q_k-1})$.

Insert 1^2 and 2 (respectively) to obtain the desired shapes of $\lambda \in D(N)$.

• Proposition 1 implies that $d(N) = f(N-1)$, with the exception of certain duplicates.

The factorizations $N-1 = 2\cdot 2\cdot m$ and $N-1 = 4\cdot m$ (m fixed) produce the same double-perfect partitions:

$N-1 = 2\cdot 2\cdot m \longmapsto (1, 2, 4^{m-1}) \longrightarrow (1^3, 2, 4^{m-1}) \in A(1^2)$;

$N-1 = 4\cdot m \longmapsto (1^3, 4^{m-1}) \longrightarrow (1^3, 2, 4^{m-1}) \in B(2)$.

So if $4|(N-1)$ we remove factorizations of the form $N-1 = 2\cdot 2\cdot m$ and get $d(N) = f(N-1) - f(\frac{N-1}{4})$. This completes the proof of Theorem 3.

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II. $\beta = (1^{q_1-1}, q_1^{q_2-1}, (q_1q_2)^{q_3-1}, \ldots, (q_1q_2q_3\cdots q_{k-1})^{q_k-1})$.

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Superset of p-Perfect Partitions

Let $V(\lambda)$ denote the sequence of multiplicities of members of $G(\lambda)$ when arranged by increasing weights. For example, the weights of partitions in $G((1^5, 2))$ are $1, 2, 2, 3, 3, 4, 4, 5, 5, 6, 7$, or $1, 2^2, 3^2, 4^2, 5^2, 6, 7$. Thus $V((1^5, 2)) = 1, 2, 2, 2, 2, 1, 1$.

A **perfect partition** $\lambda \vdash n$ contains partitions of $1^1, 2^1, 3^1, \ldots, (n-1)^1, n^1$. So

$$V(\lambda) = 1, 1, 1, \ldots, 1, 1. \tag{3}$$

A **double-perfect partition** $\lambda \vdash n$ follows the representation scheme $1, 2^2, 3^2, \ldots, (n-2)^2, n-1, n$ which gives

$$V(\lambda) = 1, 2, 2, \ldots, 2, 1, 1. \tag{4}$$

Notice that the sequences (3) and (4) are weakly **unimodal**.

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Generally consider any $\lambda \vdash n$ and make the following assumptions.

(Q1) λ is a complete partition, that is, $G(\lambda)$ contains at least one partition of every m, $1 \leq m \leq n$.

So the sequence of weights of members of $G(\lambda)$ has the form

$$1^{s_1}, 2^{s_2}, 3^{s_3}, \ldots, (n-1)^{s_{n-1}}, n^{s_n}, \ s_i > 0.$$

(Q2) The sequence $V(\lambda) = s_1, s_2, s_3, \ldots, s_n$ is weakly unimodal.

Let $S_p(n)$ be the set of partitions $\lambda \vdash n$ that satisfy properties (Q1) and (Q2) such that $\max(V(\lambda)) = p$:

$$S_p(n) = \{\lambda \vdash n \mid \lambda \text{ is complete}, V(\lambda) \text{ is unimodal and } \max(V(\lambda)) = p\}.$$

Then $S_1(n) = \text{Per}(n)$. However, $D(n) \subsetneq S_2(n)$ in general.
E.g., $(1^4, 3) \in S_2(7) \setminus D(7)$. (Note: $V((1^4, 3)) = 1, 1, 2, 2, 1, 1, 1$).

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$S_p(n)$ is a **superset** of p-perfect partitions of n.

Definition

Given an integer $p > 1$, a p-perfect partition of n is any $\lambda \in S_p(n)$ for which the sequence $V(\lambda)$ is '**minimal**' in the sense that $V(\lambda)$ starts with a single 1, ends with two 1's and has p distinct terms:

$$V(\lambda) = 1, s_2, s_3, \ldots, s_{n-2}, 1, 1, \ s_i > 1,$$
$$\max(s_2, s_3, \ldots, s_{n-2}) = p,$$
$$\left|\{1, s_2, s_3, \ldots, s_{n-2}, 1, 1\}\right| = p.$$

$\Longrightarrow$ triple-perfect partitions $\lambda \in S_3(n)$ ....?

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$\Longrightarrow$ triple-perfect partitions $\lambda \in S_3(n)$ ...... ?

There are Two Classes of Triple Perfect partitions λ

First: the sequence of weights of members of $G(\lambda)$ has the form

$$1, \underbrace{2, 3, \ldots, u}_{2 \text{ times}}, \underbrace{u+1, u+2, u+3, u+4}_{3 \text{ times}}, \underbrace{u+5, \ldots, n-2}_{2 \text{ times}}, n-1, n.$$

By symmetry $u = \frac{n-5}{2}$, so n is odd. Refer to such partitions as **triple4-perfect partitions**, denoted by $T_4(n)$. So any $\lambda \in T_4(n)$ satisfies

$$V(\lambda) = 1, \underbrace{2, 2, \ldots, 2}_{(n-7)/2 \text{ times}}, 3, 3, 3, 3, \underbrace{2, 2, \ldots, 2}_{(n-7)/2 \text{ times}}, 1, 1.$$

$$V(\lambda) = 1, 2^{(n-7)/2}, 3^4, 2^{(n-7)/2}, 1^2, \ n > 8.$$

Second: the sequence of weights of members of $G(\lambda)$ has the form

$$1, \underbrace{2, 3}_{2\ \text{times}}, \underbrace{4, 5, \ldots, n-4}_{3\ \text{times}}, \underbrace{n-3, n-2}_{2\ \text{times}}, n-1, n.$$

These constitute the (main) set of **triple-perfect partitions**, denoted by $T(n)$. So each $\lambda \in T(n)$ satisfies

$$V(\lambda) = 1, 2, 2, \underbrace{3, 3, \ldots, 3}_{N-7\ \text{times}}, 2, 2, 1, 1.$$

$$V(\lambda) = 1, 2^2, 3^{N-7}, 2^2, 1^2, \quad N > 7.$$

Minimality Statement

Theorem 4

Let $\lambda \in S_3(n)$. Then $V(\lambda)$ is minimal if and only if

1. $\lambda \in T_4(n)$:

$$V(\lambda) = 1, 2^{(n-7)/2}, 3^4, 2^{(n-7)/2}, 1^2, \ \ n > 8.$$

2. or $\lambda \in T(n)$:

$$V(\lambda) = 1, 2^2, 3^{n-7}, 2^2, 1^2, \ \ n > 7.$$

((The proof of Theorem 4 is under construction, still needs to be perfected!))

We will discuss Triple-Perfect Partitions first; then Triple4-Perfect Partitions.

Triple Perfect Partitions

If $\lambda \vdash N$ is triple-perfect, then $G(\lambda)$ contains partitions of

$$1, 2^2, 3^2, 4^3, 5^3 \ldots, (N-5)^3, (N-4)^3, (N-3)^2, (N-2)^2, (N-1), N.$$

$$\implies V(\lambda) = 1, 2^2, 3^{N-7}, 2^2, 1^2, \quad N > 7.$$

Definition

A *triple-perfect partition* is a partition $\lambda = (\lambda_1, \ldots, \lambda_k) \vdash N$ such that each integer m with $4 \le m \le N-4$ can be represented **three times** as $m = \sum_{i=1}^{k} \alpha_i \lambda_i$, $\alpha_i \in \{0, 1\}$ and each integer m with $m \in \{2, 3\} \cup \{N-3, N-2\}$ can be represented two times.

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Definition

A *triple-perfect partition* is a partition $\lambda = (\lambda_1, \ldots, \lambda_k) \vdash N$ such that each integer m with $4 \leq m \leq N-4$ can be represented **three times** as $m = \sum_{i=1}^{k} \alpha_i \lambda_i$, $\alpha_i \in \{0, 1\}$ and each integer m with $m \in \{2, 3\} \cup \{N-3, N-2\}$ can be represented two times.

Triple Perfect Partitions from Perfect Partitions

Theorem 5

Every triple-perfect partition $\lambda \vdash N > 7$ may be obtained from a perfect partition $\beta \vdash N - 4$ in three ways:

I. If the multiplicity of 1 in β is 1, then insert 1^4 into $\beta : \ A(1^4)$.

II. If the multiplicity of 1 in β is 1, then insert $1, 3$ into $\beta : \ B(1, 3)$.

III. If β does not contain 2 as a part, insert 2^2 into $\beta : \ C(2^2)$.

Then

$$T(N) = A(1^4) \cup B(1, 3) \cup C(2^2).$$

Proof. Assume that $\lambda \vdash N$ is obtained from $\beta \in \mathrm{Per}(N-4)$ using I, II or III. We claim that $\lambda \in T(N)$.

On insertion of parts into β, we find one additional partition of each of $2, 3$: $(1^2), (1^3)$ or $(1^2), (3)$ or $(2), (1, 2)$ with respect to I, II or III, respectively.

Then two additional partitions of each $m \in [4, N-4]$ follow: $(1^3, h(m-3)), (1^2, h(m-2))$ or $(1^2, h(m-2)), (3, h(m-3))$ or $(2, h(m-2)), (2^2, h(m-4))$.

Then we obtain two new partitions of both $N-3, N-2$ by symmetry, followed by one new partition of $N-1$ and N.

Thus $G(\lambda)$ matches the desired scheme for triple-perfect partitions.

Lastly, it can be proved that inserting the remaining partitions of 4 into β, namely, (4) and $(1^2, 2)$, does not affect the foregoing results.

□

Example

$\beta = (1^2, 3^2) \in \mathrm{Per}(8) \implies \lambda = (1^2, 2^2, 3^2) \in T(12)$, from $C(2^2)$:

weight	$G(\beta)$	$G(\lambda) \setminus G(\beta)$	
1	(1)	–	–
2	(1^2)	(2)	–
3	(3)	$(1, 2)$	–
4	$(1, 3)$	$(1^2, 2)$	(2^2)
5	$(1^2, 3)$	$(2, 3)$	$(1, 2^2)$
6	(3^2)	$(1, 2, 3)$	$(1^2, 2^2)$
7	$(1, 3^2)$	$(1^2, 2, 3)$	$(2^2, 3)$
8	$(1^2, 3^2)$	$(2, 3^2)$	$(1, 2^2, 3)$
9	–	$(1, 2, 3^2)$	$(1^2, 2^2, 3)$
10	–	$(1^2, 2, 3^2)$	$(2^2, 3^2)$
11	–	–	$(1, 2^2, 3^2)$
12	–	–	$(1^2, 2^2, 3^2)$

Corollary

A triple-perfect partition $\lambda \vdash N$ has one of the following forms:

$$\lambda = (1^5, 2^{q_2-1}, (2q_2)^{q_3-1}, (2q_2q_3)^{q_4-1}, \ldots, (2q_2q_3\cdots q_{k-1})^{q_k-1}), \tag{5}$$

$$\lambda = (1^2, 2^{q_2-1}, 3, (2q_2)^{q_3-1}, (2q_2q_3)^{q_4-1}, \ldots, (2q_2q_3\cdots q_{k-1})^{q_k-1}), \tag{6}$$

$$\lambda = (1^{q_1-1}, 2^2, q_1^{q_2-1}, (q_1q_2)^{q_3-1}, \ldots, (q_1q_2\cdots q_{k-1})^{q_k-1}), \ q_1 > 2. \tag{7}$$

Proof. Convert the following factorizations to perfect partitions, and then insert $(1^4), (1,3)$ and (2^2) respectively.

$$N - 3 = 2q_2q_3\cdots q_k, \ q_i > 1 \ \forall \ i, \tag{5a}$$

$$N - 3 = 2q_2q_3\cdots q_k, \tag{6a}$$

$$N - 3 = q_1q_2q_3\cdots q_k, \ q_1 > 2. \tag{7a}$$

□

The Counting Formula for $T(N)$

SN	Factorization	Set	Count
(1)	$N-3 = 2q_2q_3\cdots q_k$	$A(1^4)$	$f(\frac{N-3}{2})$
(2)	$N-3 = 2q_2q_3\cdots q_k$	$B(1,3)$	$f(\frac{N-3}{2})$
(3)	$N-3 = q_1q_2q_3\cdots q_k,\ q_1 > 2$	$C(2^2)$	$f(N-3) - f(\frac{N-3}{2})$

Duplicated partitions arise from factorizations of the form $2\cdot 3\cdot m$:

$$2\cdot 3\cdot m \longmapsto (1, 2^2, 6^{m-1}) \longrightarrow \begin{cases}(1^5, 2^2, 6^{m-1}) \in A(1^4)\\ (1^2, 2^2, 3, 6^{m-1}) \in B(1,3).\end{cases}$$

However, these two partitions belong uniquely to the set $C(2^2)$:

$6\cdot m \longmapsto (1^5, 6^{m-1}) \longrightarrow (1^5, 2^2, 6^{m-1}) \in C(2^2)$,

$3\cdot 2\cdot m \longmapsto (1^2, 3, 6^{m-1}) \longrightarrow (1^2, 2^2, 3, 6^{m-1}) \in C(2^2)$.

So when $N-3 \equiv 0 \pmod 6$, the number of duplicated partitions is $f(\frac{N-3}{6})$, and should be subtracted from $|A(1^4)|$ and $|B(1,3)|$.

Hence the number $t(N)$ of triple-perfect partitions of N is given by

$$\begin{cases} 2(f(\frac{N-3}{2}) - f(\frac{N-3}{6})) + f(N-3) - f(\frac{N-3}{2}), & N-3 \equiv 0 \pmod 6, \\ 2f(\frac{N-3}{2}) + f(N-3) - f(\frac{N-3}{2}), & N-3 \equiv 2, 4 \pmod 6, \\ f(N-3), & \text{otherwise.} \end{cases} \qquad \square$$

That is,

Theorem 6

The number $t(N)$ of triple-perfect partitions of N is given by

$$t(N) = \begin{cases} f(N-3) & \text{if } N \equiv 0 \pmod 2, \\ f(N-3) + f(\frac{N-3}{2}) & \text{if } N \equiv \pm 1 \pmod 6, \\ f(N-3) + f(\frac{N-3}{2}) - 2f(\frac{N-3}{6})) & \text{if } N \equiv 3 \pmod 6. \end{cases}$$

So when $N-3 \equiv 0 \pmod 6$, the number of duplicated partitions is $f(\frac{N-3}{6})$, and should be subtracted from $|A(1^4)|$ and $|B(1,3)|$.

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Example: $T(33) = \{\,?\,\}$

From the formula $t(33) = f(30) + f(15) - 2 \cdot f(5) = 13 + 3 - 2 = 14$.
But $T(33)$ is determined by Per(29) via the set $F(30)$ of factorizations:

SN	$F(30)$	Per(29)	Insert	$T(33)$	Type
1	30	(1^{29})	2^2	$(1^{29}, 2^2)$	III
2	$2 \cdot 15$	$(1, 2^{14})$	1^4	$(1^5, 2^{14})$	I
3	$2 \cdot 15$	$(1, 2^{14})$	$1, 3$	$(1^2, 2^{14}, 3)$	II
4	$15 \cdot 2$	$(1^{14}, 15)$	2^2	$(1^{14}, 2^2, 15)$	III
5	$3 \cdot 10$	$(1^2, 3^9)$	2^2	$(1^2, 2^2, 3^9)$	III
6	$10 \cdot 3$	$(1^9, 10^2)$	2^2	$(1^9, 2^2, 10^2)$	III
7	$5 \cdot 6$	$(1^4, 5^5)$	2^2	$(1^4, 2^2, 5^5)$	III
8	$6 \cdot 5$	$(1^5, 6^4)$	2^2	$(1^5, 2^2, 6^4)$	III
9	$2 \cdot 5 \cdot 3$	$(1, 2^4, 10^2)$	1^4	$(1^5, 2^4, 10^2)$	I
10	$2 \cdot 5 \cdot 3$	$(1, 2^4, 10^2)$	$1, 3$	$(1^2, 2^4, 3, 10^2)$	II
11	$3 \cdot 2 \cdot 5$	$(1^2, 3, 6^4)$	2^2	$(1^2, 2^2, 3, 6^4)$	III
12	$3 \cdot 5 \cdot 2$	$(1^2, 3^4, 15)$	2^2	$(1^2, 2^2, 3^4, 15)$	III
13	$5 \cdot 2 \cdot 3$	$(1^4, 5, 10^2)$	2^2	$(1^4, 2^2, 5, 10^2)$	III
14	$5 \cdot 3 \cdot 2$	$(1^4, 5^2, 15)$	2^2	$(1^4, 2^2, 5^2, 15)$	III
	$2 \cdot 3 \cdot 5$	$(1, 2^2, 6^4)$	–	$\emptyset$	none

Triple4 Perfect Partitions

If $\lambda \vdash N$ is triple4-perfect, then $G(\lambda)$ contains partitions of

$$1, \underbrace{2, 3, \ldots, \frac{n-5}{2}}_{2 \text{ times}}, \underbrace{\frac{n-3}{2}, \frac{n-1}{2}, \frac{n+1}{2}, \frac{n+3}{2}}_{3 \text{ times}}, \underbrace{\frac{n+5}{2}, \ldots, n-3, n-2}_{2 \text{ times}}, n-1, n.$$

They are defined only for odd weights $n = 2m - 1 > 8$. Therefore,

$$V(\lambda) = 1, 2^{(n-7)/2}, 3^4, 2^{(n-7)/2}, 1^2, \quad n > 8.$$

Definition

A *triple4 perfect partition* is a partition $\lambda = (\lambda_1, \ldots, \lambda_k) \vdash 2m - 1$, that contains three partitions of each of the **four integers** $m-2, m-1, m, m+1$, and two partitions of each member of $\{2, \ldots, m-3\} \cup \{m+2, \ldots, 2m-3\}$, $m > 4$.

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Triple4 Perfect Partitions from Ordered Factorizations

The following assertion may be established analogously like before.

Theorem 7

Every triple4-perfect partition $\lambda \vdash 2m-1$, $m \neq 6$ may be obtained from a partition $\beta \in \mathrm{Per}(m-1)$ in two ways:

I. If the multiplicity of 1 in β is 1, then insert $1^2, m-2$ into β.
$\implies E(1^2, m-2)$.

II. If β does not contain 2 as a part, insert $2, m-2$ into β.
$\implies W(2, m-2)$.

Then

$$T_4(2m-1) = E(1^2, m-2) \cup W(2, m-2).$$

Note: this is the same as the construction of double-perfect partitions in Proposition 1 except for the additional part $m-2$.

Corollary

A triple4-perfect partition $\lambda \vdash 2m-1$ has either of the forms ($q_i > 1$):

$$\lambda = (1^3, 2^{q_2-1}, (2q_2)^{q_3-1}, \dots, (2q_2q_3\cdots q_{k-1})^{q_k-1}, m-2),\ q_2 > 2;$$
$$\lambda = (1^{q_1-1}, 2, q_1^{q_3-1}, \dots, (q_1q_3q_4\cdots q_{k-1})^{q_k-1}, m-2),\ \ q_1 > 2.$$

Theorem 8

The number $t_4(2m-1)$ is given by

$$t_4(2m-1) = 0,\ m < 5,\ \ t_4(9) = 1,\ \ t_4(11) = 6$$

and if $m \geq 7$, then

$$t_4(2m-1) = \begin{cases} f(m) & \text{if } m \not\equiv 0 \pmod 4, \\ f(m) - f(\frac{m}{4}) & \text{if } m \equiv 0 \pmod 4. \end{cases}$$

Corollary

A triple4-perfect partition $\lambda \vdash 2m-1$ has either of the forms $(q_i > 1)$:

$$\lambda = (1^3, 2^{q_2-1}, (2q_2)^{q_3-1}, \ldots, (2q_2q_3\cdots q_{k-1})^{q_k-1}, m-2),\ q_2 > 2;$$
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Examples

- From the $t(n)$-formula, $t(11) = f(8) + f(4) = 6$:

$$T(11) = \{(1^7, 2^2), (1^5, 2^3), (1^2, 2^3, 3), (1^5, 2, 4), (1^3, 2^2, 4), (1^2, 2, 3, 4)\};$$

every $\lambda \in T(11)$ is both triple- and triple4-perfect because $V(\lambda) = 1, 2^2, 3^4, 2^2, 1^2$. However, the first three members cannot be obtained from Theorem 7 since none of them contains $m - 2 = 4$.

- $t_4(15) = t_4(2 \cdot 8 - 1) = f(8) - f(2) = 4 - 1 = 3$:

Factorization of $m = 8$	8	$2 \cdot 4$	$4 \cdot 2$	$2 \cdot 2 \cdot 2$
Perfect Partitions of 7	(1^7)	$(1, 2^3)$	$(1^3, 4)$	$\emptyset$
Insert Parts	2,6	$1^2, 6$	2,6	–
$T_4(15)$	$(1^7, 2, 6)$	$(1^3, 2^3, 6)$	$(1^3, 2, 4, 6)$	–

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$T_4(15)$	$(1^7, 2, 6)$	$(1^3, 2^3, 6)$	$(1^3, 2, 4, 6)$	–

A Partition Identity I

By comparison of the formulas for $d(N)$ and $t_4(N)$ we obtain

$$t_4(2m-1) = d(m+1), \;\; m \geq 5.$$

Proof. Define a bijection $D(m+1) \longrightarrow T_4(2m-1)$ by

$$\lambda \longmapsto (\lambda, (m-2)).$$

□

E.g. $d(9) = t_4(15) = 3 \;\; (m-2=6)$:

$D(9)$	$(1^7, 2)$	$(1^3, 2^3)$	$(1^3, 2, 4)$
$T_4(15)$	$(1^7, 2, 6)$	$(1^3, 2^3, 6)$	$(1^3, 2, 4, 6)$

Note that weights of partitions in any $\lambda \in T_4(15)$ fulfill the scheme

$$1, 2^2, 3^2, 4^2, 5^2, 6^3, 7^3, 8^3, 9^3, 10^2, 11^2, 12^2, 13^2, 14, 15 \ :$$

E.g. $\lambda = (1^7, 2, 6)$ contains, i.e., members of $G(\lambda)$:

$$
\begin{gathered}
(1) \\
(1^2), (2) \| (1^3), (1, 2) \| (1^4), (1^2, 2) \| (1^5), (1^3, 2) \\
(1^6), (1^4, 2), (6) \| (1^7), (1^5, 2), (1, 6) \| (1^6, 2), (1^2, 6), (2, 6) \| (1^7, 2), (1^3, 6), (1, 2, 6) \\
(1^4, 6), (1^2, 2, 6) \| (1^5, 6), (1^3, 2, 6) \| (1^6, 6), (1^4, 2, 6) \| (1^7, 6), (1^5, 2, 6) \\
(1^6, 2, 6) \| (1^7, 2, 6).
\end{gathered}
$$

A Partition Identity II

By comparison of the formulas for $d(N)$ and $t(N)$ we obtain

Given an odd integer $N > 8$, then

$$d(N+1) = t(N+3) \quad (= f(N)).$$

Proof. Define a bijection $D(N+1) \longrightarrow T(N+3)$ by

$$\lambda \longmapsto ((2), \lambda).$$

□

Hence

If $N > 8$ is any Odd integer, then

$$d(N+1) = t(N+3) = t_4(2N-1).$$

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Thank you for your attention!