Mock Theta Transformations without Watson

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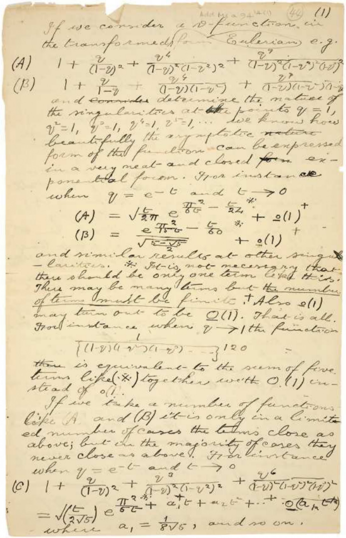
Joint Work with Dr. Frank Garvan

Ramanujan's Last Letter to Hardy (1920)



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If we consider a ϑ-function in the transformed Eulerian form e.g.

(A) $1 + \frac{q}{(1-q)^2} + \frac{q^4}{(1-q)^2(1-q^2)^2} + \frac{q^9}{(1-q)^2(1-q^2)^2(1-q^3)^2}$

(B) $1 + \frac{q}{1-q} + \frac{q^4}{(1-q)(1-q^2)} + \frac{q^9}{(1-q)(1-q^2)(1-q^3)}$

and ~~consider~~ determine the nature of the singularities at ~~the~~ the points $q = 1$, $q^2 = 1$, $q^3 = 1$, $q^4 = 1$, $q^5 = 1$, ... we know how beautifully the asymptotic ~~nature~~ form of the function can be expressed in a very neat and closed ~~form~~ exponential form. For instance when $q = e^{-t}$ and $t \to 0$

$$(A) = \sqrt{\tfrac{t}{2\pi}}\, e^{\frac{\pi^2}{6t} - \frac{t}{24}} {}^{*} + o(1)^{+}$$

$$(B) = \frac{e^{\frac{\pi^2}{15t} - \frac{t}{60}}}{\sqrt{\frac{5-\sqrt{5}}{2}}} {}^{*} + o(1)^{+}$$

and similar results at other singularities. * It is not necessary that there should be only one term like this. There may be many terms but the number of terms must be finite. † Also $o(1)$ may turn out to be $O(1)$. That is all. For instance when $q \to 1$ the function

$$\left\{\frac{1}{(1-q)(1-q^2)(1-q^3)\cdots}\right\}^{120}$$

is equivalent to the sum of five terms like (*) together with $O(1)$ instead of $o(1)$.

If we take a number of functions like (A) and (B) it is only in a limited number of cases the terms close as above; but in the majority of cases they never close as above. For instance when $q = e^{-t}$ and $t \to 0$

(C) $1 + \frac{q}{(1-q)^2} + \frac{q^3}{(1-q)^2(1-q^2)^2} + \frac{q^6}{(1-q)^2(1-q^2)^2(1-q^3)^2}$

$$= \sqrt{\left(\frac{t}{2\sqrt{5}}\right)}\, e^{\frac{\pi^2}{6t}} {}^{*} + a_1 t + a_2 t^2 + \cdots + O(a_k t^k)$$

where $a_1 = \frac{1}{8\sqrt{5}}$, and so on.

Ramanujan's Last Letter to Hardy (1920)

I am extremely sorry for not writing you a single letter up to now.... I discovered very interesting functions recently which I call "Mock" θ-functions. Unlike the "False" θ functions (studied partially by Prof. Rogers in his interesting paper) they enter into mathematics as beautifully as ordinary θ-functions. I am sending you this letter with some examples....

Mock Theta Functions

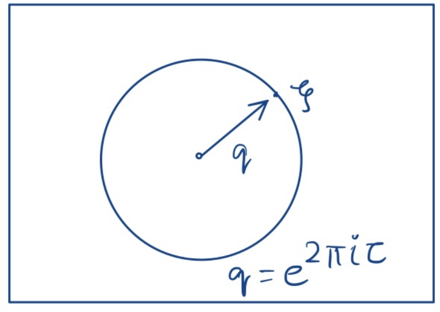
Mock Theta Functions

A mock ϑ-function is a function f of the complex variable q, defined by a q-series of a particular type (Ramanujan calls this the Eulerian form), which converges for $|q| < 1$ and satisfies the following conditions:

- infinitely many roots of unity are exponential singularities,
- for every root of unity ξ, there is a ϑ-function $\vartheta_\xi(q)$ such that the difference $f(q) - \vartheta_\xi(q)$ is bounded as $q \to \xi$ radially,
- There is no ϑ-function that works for all ξ, i.e., f is not the sum of two functions, one of which is a ϑ-function and the other which is bounded at all roots of unity.

It seems that by a ϑ-function Ramanujan means sums, products and quotients of series of the form $\sum_{n=-\infty}^{\infty} k^n q^{an^2+bn}$, where $k \in \{-1, 1\}$ and a, b are rational.

Radial Limits



Notations and Conventions

For any non-negative integer L, we define the conventional q-Pochammer symbol as

$$(a)_L = (a;q)_L := \prod_{k=0}^{L-1}(1-aq^k),$$
$$(a)_\infty = (a;q)_\infty := \lim_{L\to\infty}(a)_L \text{ where } |q|<1.$$

Next, we define

$$E(q) := (q)_\infty,$$
$$\eta(\tau) := q^{\frac{1}{24}}E(q) \text{ where } q = e^{2\pi i\tau}.$$

Examples of Mock Theta Functions

- $\phi(q) = 1 + \dfrac{q}{(1+q^2)} + \dfrac{q^4}{(1+q^2)(1+q^4)} + \ldots = \sum_{n=0}^{\infty} \dfrac{q^{n^2}}{(-q^2;q^2)_n}.$
- $\psi(q) = \dfrac{q}{(1-q)} + \dfrac{q^4}{(1-q)(1-q^2)} + \ldots = \sum_{n=1}^{\infty} \dfrac{q^{n^2}}{(q;q^2)_n}.$
- $f(q) = 1 + \dfrac{q}{(1+q)^2} + \dfrac{q^4}{(1+q)^2(1+q^2)^2} + \ldots = \sum_{n=0}^{\infty} \dfrac{q^{n^2}}{(-q;q)_n^2}$
- $\chi(q) = 1 + \dfrac{q}{(1-q+q^2)} + \ldots = \sum_{n=0}^{\infty} \dfrac{q^n(-q;q)_n}{(q^3;q^3)_n}.$

Mock Theta functions in Mathematics

Mock Theta functions in Mathematics

Recent Work!

(with A. Dhar and R. Sarma) *On new minimal excludants of overpartitions related to some q-series of Ramanujan*, accpted in *Bulletin of the Australian Mathematical Society*.

G.N.Watson

THE FINAL PROBLEM : AN ACCOUNT OF THE MOCK THETA FUNCTIONS

G. N. Watson‡.

New Mock Theta Functions

- $\nu(q) = 1 + \dfrac{q}{(1+q)} + \dfrac{q^2}{(1+q)(1+q^3)} + \dots = \displaystyle\sum_{n=0}^{\infty} \frac{q^{n^2+n}}{(-q;q^2)_{n+1}}.$
- $\omega(q) = \dfrac{1}{(1-q)^2} + \dfrac{q^4}{(1-q)^2(1-q^3)^2} + \dots = \displaystyle\sum_{n=0}^{\infty} \frac{q^{2n^2+2n}}{(q;q^2)_{n+1}^2}.$
- $\rho(q) = \dfrac{1}{(1+q+q^2)} + \dfrac{q^4}{(1+q+q^2)(1+q^3+q^6)} + \dots = \displaystyle\sum_{n=0}^{\infty} \frac{q^{2n(n+1)}(q;q^2)_n}{(q^3;q^6)_n}.$

Zwegers

Zwegers

Define $F = (f_0, f_1, f_2)^T$ by

$$f_0(q) := q^{-1/24} f(q),$$
$$f_1(q) := 2q^{1/3}\omega(q^{1/2}),$$
$$f_2(q) := 2q^{1/3}\omega(-q^{1/2}).$$

Zwegers

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$$f_2(q) := 2q^{1/3}\omega(-q^{1/2}).$$

Recall

- $f(q) = 1 + \frac{q}{(1+q)^2} + \frac{q^4}{(1+q)^2(1+q^2)^2} + \ldots = \sum_{n=0}^{\infty} \frac{q^{n^2}}{(-q;q)_n^2}.$
- $\omega(q) = \frac{1}{(1-q)^2} + \frac{q^4}{(1-q)^2(1-q^3)^2} + \ldots = \sum_{n=0}^{\infty} \frac{q^{2n^2+2n}}{(q;q^2)_{n+1}^2}.$

The G function

Define

$$g_0(z) := \sum_{n\in\mathbb{Z}} (-1)^n (n+1/3) e^{3\pi(n+1/3)^2 z},$$
$$g_1(z) := -\sum_{n\in\mathbb{Z}} (n+1/6) e^{3\pi(n+1/6)^2 z},$$
$$g_2(z) := \sum_{n\in\mathbb{Z}} (n+1/3) e^{3\pi(n+1/3)^2 z}.$$

We then define

$$G(\tau) := 2i\sqrt{3} \int_{-\tau}^{i\infty} \frac{(g_1(z), g_0(z), g_2(z))^T}{\sqrt{-i(z+\tau)}}\, dz.$$

Main Theorem

Theorem (Zwegers (2000))

The function H defined by $H(\tau) := F(\tau) - G(\tau)$ is a vector valued real analytic modular form of weight $1/2$ satisfying

- $H(\tau+1) = \begin{pmatrix} \zeta_{24}^{-1} & 0 & 0 \\ 0 & 0 & \zeta_3 \\ 0 & \zeta_3 & 0 \end{pmatrix} H(\tau)$,
- $\frac{1}{\sqrt{-i\tau}} H(\frac{-1}{\tau}) = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} H(\tau)$.

$$\begin{aligned} H(\tau) &= (h_0(\tau), h_1(\tau), h_2(\tau)) \\ &= (f_0(q), f_1(q), f_2(q))^T - 2i\sqrt{3} \int_{-\tau}^{i\infty} \frac{(g_1(z), g_0(z), g_2(z))^T}{\sqrt{-i(z+\tau)}}\, dz. \end{aligned}$$

Zwegers' Proof

Lemma (Watson (1936))

For $\tau \in \mathbb{H}$, we have

$$F(\tau+1) = \begin{pmatrix} \zeta_{24}^{-1} & 0 & 0 \\ 0 & 0 & \zeta_3 \\ 0 & \zeta_3 & 0 \end{pmatrix} F(\tau)$$

and $$\frac{1}{\sqrt{-i\tau}} F(-1/\tau) = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} F(\tau) + R(\tau)$$

with $\zeta_n = e^{2\pi i/n}$, $R(\tau) = 4\sqrt{3}\sqrt{-i\tau}(j_2(\tau)), -j_1(\tau)), j_3(\tau))^T$.

Zweger's Proof Continued

$$j_1(\tau) = \int_0^\infty e^{3\pi i x^2} \frac{\sin(2\pi\tau x)}{\sin(3\pi\tau x)} dx$$

$$j_2(\tau) = \int_0^\infty e^{3\pi i x^2} \frac{\cos(\pi\tau x)}{\cos(3\pi\tau x)} dx$$

$$j_3(\tau) = \int_0^\infty e^{3\pi i x^2} \frac{\sin(\pi\tau x)}{\sin(3\pi\tau x)} dx$$

Zwegers' Brilliance!

Lemma (Zwegers)

For $\tau \in \mathbb{H}$, we have

$$\mathrm{G}(\tau+1) = \begin{pmatrix} \zeta_{24}^{-1} & 0 & 0 \\ 0 & 0 & \zeta_3 \\ 0 & \zeta_3 & 0 \end{pmatrix} G(\tau)$$

and $\frac{1}{\sqrt{-i\tau}} G(-1/\tau) = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} G(\tau) + R(\tau).$

So, now when we work out $F(\tau) - G(\tau)$, the $R(\tau)$'s cancel and we get Zwegers' theorem!

To prove

To prove

- $$H(\tau+1) = \begin{pmatrix} \zeta_{24}^{-1} & 0 & 0 \\ 0 & 0 & \zeta_3 \\ 0 & \zeta_3 & 0 \end{pmatrix} H(\tau),$$
- $$\frac{1}{\sqrt{-i\tau}} H\left(\frac{-1}{\tau}\right) = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} H(\tau).$$

To prove

- $H(\tau+1) = \begin{pmatrix} \zeta_{24}^{-1} & 0 & 0 \\ 0 & 0 & \zeta_3 \\ 0 & \zeta_3 & 0 \end{pmatrix} H(\tau),$
- $\frac{1}{\sqrt{-i\tau}} H(\frac{-1}{\tau}) = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} H(\tau).$

$$\frac{1}{\sqrt{-i\tau}} h_0(-1/\tau) = h_1(\tau)$$
$$\frac{1}{\sqrt{-i\tau}} h_1(-1/\tau) = h_0(\tau)$$
$$\frac{1}{\sqrt{-i\tau}} h_2(-1/\tau) = -h_2(\tau)$$

To prove

- $H(\tau+1) = \begin{pmatrix} \zeta_{24}^{-1} & 0 & 0 \\ 0 & 0 & \zeta_3 \\ 0 & \zeta_3 & 0 \end{pmatrix} H(\tau)$,
- $\frac{1}{\sqrt{-i\tau}} H(\frac{-1}{\tau}) = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} H(\tau)$.

$$\frac{1}{\sqrt{-i\tau}} h_0(-1/\tau) = h_1(\tau)$$
$$\frac{1}{\sqrt{-i\tau}} h_1(-1/\tau) = h_0(\tau)$$
$$\frac{1}{\sqrt{-i\tau}} h_2(-1/\tau) = -h_2(\tau)$$

We try to find an alternative proof using Lerch sums.

A peek into Zwegers' Thesis (2002)

The classical Theta function

$$\vartheta(z,q) := \sum_{n\in\frac{1}{2}+\mathbb{Z}} e^{\pi i n^2\tau+2\pi i n(z+\frac{1}{2})} \text{ for all } z\in\mathbb{C}, \tau\in\mathbb{H}.$$

The μ-function

$$\mu(u,v;\tau) = \frac{e^{\pi i u}}{\vartheta(v;\tau)} \sum_{n\in\mathbb{Z}} \frac{(-1)^n e^{\pi i(n^2+n)\tau+2\pi i n v}}{1-e^{2\pi i n\tau+2\pi i u}}.$$

Continued

The R function of Zwegers

For $z \in \mathbb{C}$, we define

$$E(z) = 2\int_0^z e^{-\pi i u^2} du = \sum_{n=0}^{\infty} \frac{(-\pi)^n z^{2n+1}}{n!(n+1/2)}.$$

Now for $z \in \mathbb{R}$, we have

$$E(z) = sgn(z)(1 - \beta(z^2))$$

where, for $x \in \mathbb{R}_{\geq 0}$,

$$\beta(x) = \int_x^{\infty} u^{-1/2} e^{-\pi u} du$$

The R function and it's properties

Definition

For $u \in \mathbb{C}$ and τ in $\mathbb{H}$,

$$R(u;\tau) = \sum_{n \in 1/2+\mathbb{Z}} \{sgn(n) - E\left((n+a)\sqrt{2y}\right)\}(-1)^{n-1/2} e^{-\pi i n^2 \tau - 2\pi i n u}$$

where $y = Im(\tau)$ and $a = \dfrac{Im(u)}{Im(\tau)}$.

Properties

- $R(u+1) = -R(u)$
- $R(-u) = R(u)$

Continued

The $g_{a,b}$ function

$$g_{a,b}(\tau) := \sum_{n\in\mathbb{Z}} n e^{\pi i n^2 \tau + 2\pi i n b}.$$

Properties

- $g_{-a,-b}(\tau) = -g_{a,b}(\tau)$
- $g_{a,b}\left(\frac{-1}{\tau}\right) = i e^{2\pi i a b}(-i\tau)^{3/2} g_{b,-a}(\tau)$

The $\tilde{\mu}$ function

The $\tilde{\mu}$ function

$\tilde{\mu}(u, v; \tau) = \mu(u, v; \tau) + \frac{i}{2} R(u - v; \tau).$

Transformations of the $\tilde{\mu}$ function:

$$\tilde{\mu}(u + k\tau + l, v + m\tau + n) = (-1)^{k+l+m+n} e^{\pi i (k-m)^2) + 2\pi i (k-m)(u-v)} \tilde{\mu}(u, v). \quad (1)$$

$$\tilde{\mu}\left(\frac{u}{c\tau + d}, \frac{v}{c\tau + d}; \frac{a\tau + b}{c\tau + d}\right) = V(\gamma)^{-3}(c\tau + d)^{1/2} \exp\left(\frac{-\pi i c (u - v)^2}{c\tau + d}\right) \tilde{\mu}(u, v; \tau). \quad (2)$$

Thesis Continued

Zwegers

Let $\tau \in \mathbb{H}$ and for $a \in (-1/2, 1/2)$ and $b \in \mathbb{R}$

$$\int_{-\overline{\tau}}^{i\infty} \frac{g_{a+1/2,b+1/2}(z)\,dz}{\sqrt{i(z+\tau)}} = -e^{-\pi i a^2 \tau + 2\pi a(b+1/2)} R(a\tau - b; \tau).$$

Recall

We want to prove that:

$$\frac{1}{\sqrt{-i\tau}} h_2(-1/\tau) = -h_2(\tau)$$

where

$$h_2(\tau) = 2q^{1/3}\omega(-q^{1/2}) + 2i\sqrt{3}\int_{-\overline{\tau}}^{i\infty} \frac{g_2(z)\,dz}{\sqrt{i(z+\tau)}}$$

Structure of the proof

$$h_2(\tau) = 2q^{1/3}\omega(-q^{1/2}) + 2i\sqrt{3}\int_{-\overline{\tau}}^{i\infty} \frac{g_2(z)\,dz}{\sqrt{i(z+\tau)}}$$

- Step 1: Rewrite $h_2(\tau)$ in terms of the $\tilde{\mu}$ function.
- Step 2: Transformation of $h_2(\tau)$ as $\tau \to \frac{-1}{\tau}$ and arrive at the ω-identity.

$$\begin{aligned}\frac{-1}{\sqrt{-i\tau}} h_2\left(\frac{-1}{\tau}\right) = & -\frac{2i}{\sqrt{3}} - \frac{2}{3}\frac{\eta(\tau/6)^2\eta(2\tau/3)^2}{\eta(\tau/3)^2\eta(\tau)} \\ & - \frac{4}{\sqrt{3}}\exp\left(-\frac{\pi i\tau}{12} - \frac{\pi i}{6}\right)\mu(\tau/6+1/2, 1/3; \tau/3) \\ & + 2i\sqrt{3}\int_{-\overline{\tau}}^{i\infty} \frac{g_2(z)dz}{\sqrt{i(z+\tau)}}.\end{aligned}$$

Structure of the proof..

A new Omega identity

$$2q^{1/3}\omega(-\sqrt{q}) = -\frac{2i}{\sqrt{3}} - \frac{2}{3}\frac{\eta(\tau/6)^2\eta(2\tau/3)^2}{\eta(\tau/3)^2\eta(\tau)} - \frac{4}{\sqrt{3}}\exp\left(-\frac{\pi i\tau}{12} - \frac{\pi i}{6}\right)\mu(\tau/6+1/2,1/3;\tau/3).$$

- Step 3: Dissection of the ω-identity.
- Step 4: Bringing everything together.

Step 1

$$h_2(\tau) = 2q^{1/3}\omega(-q^{1/2}) + 2i\sqrt{3}\int_{-\overline{\tau}}^{i\infty} \frac{g_2(z)\,dz}{\sqrt{i(z+\tau)}}$$

- (1A) We write the integrand $g_2(z)$ in terms of the $g_{a,b}$ function.
- (1B) Write $g_2(z)$ in terms of the R function in Zwegers' thesis.
- (1C) Use Watson's identity for $\omega(q)$:

$$\omega(-q^{1/2}) = \frac{1}{(q;q)_\infty}\sum_{n=-\infty}^{\infty}\frac{(-1)^n q^{3n(n+1)/2}}{1+q^{n+1/2}}.$$

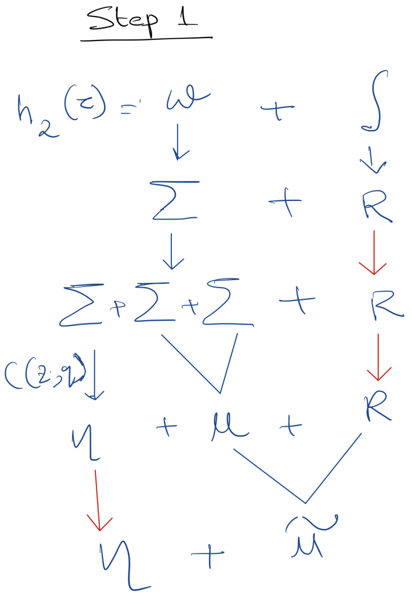
Step 1 continued..

- (1D) Use the Crank generating function $C(z;q)$ and work out the sums to obtain:

$$2q^{1/3}\omega(-q^{1/2}) = 2\frac{\eta(6\tau)^2\eta(3\tau/2)^2}{\eta(3\tau)^2\eta(\tau)} - 4q^{-1/24}\mu(3\tau/2+1/2,\tau;3\tau).$$

- (1E) We add the R function to the the above, we get

$$h_2(\tau) = 2\frac{\eta(6\tau)^2\eta(3\tau/2)^2}{\eta(3\tau)^2\eta(\tau)} - 4q^{-1/24}\tilde{\mu}(3\tau/2+1/2,-\tau;3\tau).$$


Step 1
h_2(τ) = W + ∫
Σ + R
Σ + Σ + Σ + R
C(z;q)
+ μ + R
+ μ̃

Recall

$$h_2(\tau) = 2q^{1/3}\omega(-q^{1/2}) - 2i\sqrt{3}\int_{-\overline{\tau}}^{i\infty} \frac{g_2(z)\,dz}{\sqrt{i(z+\tau)}}$$

$$g_2(z) = \sum_{n\in\mathbb{Z}} (n+1/3)e^{3\pi i(n+1/3)^2 z}$$

(1A)

$$g_2(\tau) = g_{1/3,0}(3\tau).$$

Thus the above integral becomes

$$\int_{-\overline{\tau}}^{i\infty} \frac{g_{1/3,0}(3z)\,dz}{\sqrt{i(z+\tau)}}$$

Treating G (continued..)

Substituting $3z$ to z we get,

$$\frac{1}{\sqrt{3}} \int_{-3\overline{\tau}}^{i\infty} \frac{g_{1/3,0}(z)\, dz}{\sqrt{i(z+3\tau)}}$$

Recall

Let $\tau \in \mathbb{H}$ and for $a \in (-1/2, 1/2)$ and $b \in \mathbb{R}$

$$\int_{-\overline{\tau}}^{i\infty} \frac{g_{a+1/2,b+1/2}(z)\, dz}{\sqrt{i(z+\tau)}} = -e^{-\pi i a^2 \tau + 2\pi a(b+1/2)} R(a\tau - b; \tau).$$

(1B) Letting $a = -1/6$ and $b = -1/2$ in the previous slide, the integral becomes

$$\frac{-1}{\sqrt{3}} \exp(\frac{-\pi i \tau}{12}) R(\frac{-\tau}{2} + \frac{1}{2}; 3\tau)$$

$$= \frac{-1}{\sqrt{3}} q^{-1/24} R(\frac{-\tau}{2} + \frac{1}{2}; 3\tau)$$

where $q = e^{2\pi i \tau}$.

Now the ω part(1C)

the ω function

$$\begin{aligned}
\omega(-q^{1/2}) &= \frac{1}{(q;q)_\infty}\sum_{n=-\infty}^{\infty}\frac{(-1)^n q^{3n(n+1)/2}}{1+q^{n+1/2}} \\
&= \frac{1}{(q;q)_\infty}\sum_{n=-\infty}^{\infty}\frac{(-1)^n q^{3n(n+1)/2}(1-q^{n+1/2}+q^{2n+1})}{1+q^{3n+3/2}} \\
&= \frac{1}{(q;q)_\infty}\sum_{n=-\infty}^{\infty}\frac{(-1)^n q^{3n(n+1)/2}}{1+q^{3n+3/2}} \\
&- \frac{q^{1/2}}{(q;q)_\infty}\sum_{n=-\infty}^{\infty}\frac{(-1)^n q^{3n(n+1)/2+n}}{1+q^{3n+3/2}} \\
&+ \frac{q}{(q;q)_\infty}\sum_{n=-\infty}^{\infty}\frac{(-1)^n q^{3n(n+1)/2+2n}}{1+q^{3n+3/2}}
\end{aligned}$$

Transforming the omega part

The Crank Generating function

$$C(z,q) = \frac{(q)_\infty}{(zq, z^{-1}q; q)_\infty} = \frac{(1-z)}{(q)_\infty} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{n(n+1)/2}}{1-zq^n}.$$

(1D) Using the Crank Generating function and letting $q \to q^3$ and $z \to q^{3/2}$ we get

$$\frac{(q^6;q^6)_\infty^2 (q^{3/2};q^{3/2})_\infty^2}{(q^3;q^3)_\infty^2 (q;q)_\infty}$$

Remaining sums

$$\frac{-q^{1/2}}{(q;q)_\infty}\sum_{n=-\infty}^{\infty}\frac{(-1)^n q^{3n(n+1)/2+n}}{1+q^{3n+3/2}}$$

let $n \to n-1$ becomes

$$\frac{1}{(q;q)_\infty}\sum_{n=-\infty}^{\infty}\frac{(-1)^n q^{3n(n+1)/2-n-1}}{1+q^{3n-3/2}}$$

Remaining sums

and when we let $n \to -n$ in

$$\frac{1}{(q;q)_\infty} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{3n(n+1)/2+2n}}{1+q^{3n+3/2}}$$

we get

$$\frac{1}{(q;q)_\infty} \sum_{-\infty}^{\infty} \frac{(-1)^n q^{3n(n+1)/2-n-1}}{1+q^{3n-3/2}}$$

Thus the two sums add to

$$\frac{-2}{(q)_\infty}\sum_{-\infty}^{\infty}\frac{(-1)^n q^{3n(n+1)/2-n}}{1+q^{3n-3/2}}$$

(1E).
Thus $2q^{1/3}\omega(-q^{1/2})$ becomes:

$$2\frac{\eta(6\tau)^2\eta(3\tau/2)^2}{\eta(3\tau)^2\eta(\tau)} - 4q^{-1/24}\mu(3\tau/2+1/2,\tau;3\tau)$$

thus $h_2(\tau)$ is equal to

$$\begin{aligned} &2\frac{\eta(6\tau)^2\eta(3\tau/2)^2}{\eta(3\tau)^2\eta(\tau)} \\ &- 4q^{-1/24}\mu(3\tau/2+1/2,2\tau;3\tau) \\ &- 2iq^{-1/24}R(-\tau/2+1/2;3\tau). \end{aligned}$$

Step 2: Transformations

- (2A) We use the fact that $\eta(-1/\tau)=\sqrt{-i\tau}\eta(\tau)$ to transform the Eta-quotient

$$2\frac{\eta(6\tau)^2\eta(3\tau/2)^2}{\eta(3\tau)^2\eta(\tau)}$$

transforms to

$$\frac{2\sqrt{-i\tau}(\eta(\tau/6)^2\eta(3\tau/2)^2}{3\eta(\tau/3)^2\eta(\tau)}.$$

- (2B) Use

$$\tilde{\mu}\left(\frac{u}{c\tau+d},\frac{v}{c\tau+d};\frac{a\tau+b}{c\tau+d}\right)=V(\gamma)^{-3}(c\tau+d)^{1/2}\exp\left(\frac{-\pi ic(u-v)^2}{c\tau+d}\right)\tilde{\mu}(u,v;\tau).$$

to transform

$$4q^{-1/24}\tilde{\mu}(3\tau/2+1/2,-\tau;3\tau)$$

to get

$$\frac{4\sqrt{-i\tau}}{3}\exp\left(\frac{-\pi i\tau}{12}-\frac{-\pi i}{6}\right)\tilde{\mu}(\tau/6+1/2,1/3;\tau/3).$$

Step 2 continued..

- (2C) Split the $\tilde{\mu}$ after the transformation as a sum of $\mu + \frac{i}{2}R$
- (2D) Treat the R function from the $\tilde{\mu}$ by converting it back into an integral using the following proposition

Proposition

let τ be in $\mathbb{H}$ and $b \in \mathbb{R}$, then,

$$R(-\tau/2 - b; \tau) = e^{\pi i\tau/4 - \pi(b+1/2) + \pi i/2} - e^{\pi i \tau/4 - \pi(b+1/2)} \int_{-\overline{\tau}}^{i\infty} \frac{g_{0,b+1/2}(z)dz}{-i\sqrt{(z+\tau)}}.$$

we get

$$2\sqrt{\frac{-i\tau}{3}} \left(\int_{-\overline{\tau}/3}^{i\infty} \frac{g_{0,2/3}(z)dz}{\sqrt{-i(z+\tau/3)}} + i \right).$$

Step 2 continued..

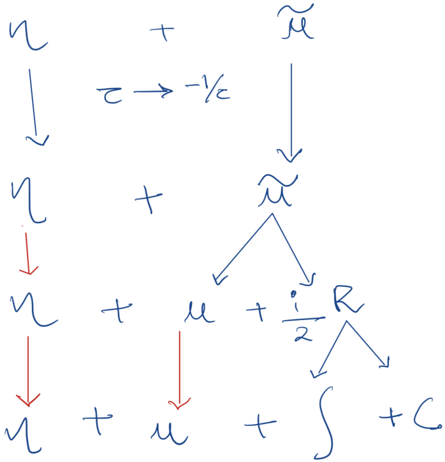
- (2E) Write $g_{a,b}$ in terms of g_2 using the definition. Then the previous integral becomes

$$-2i\sqrt{-i\tau}\sqrt{3}\int_{-\overline{\tau}}^{i\infty}\frac{g_2(z)dz}{\sqrt{i(z+\tau)}}.$$

- (2F) Bring everything together and multiply by $\frac{-1}{\sqrt{-i\tau}}$.

We arrive at the ω- identity which is equivalent to proving the transformation done by Zwegers.

Step 2



h_2 in terms of $\tilde{\mu}$ function

Using the following:

The $\tilde{\mu}$ function

$\tilde{\mu}(u,v;\tau) = \mu(u,v;\tau) + \frac{i}{2}R(u-v;\tau)$

$$h_2(\tau) = 2\frac{\eta(6\tau)^2\eta(3\tau/2)^2}{\eta(3\tau)^2\eta(\tau)} - 4q^{-1/24}\tilde{\mu}(3\tau/2+1/2,-\tau;3\tau).$$

We use the $\tilde{\mu}$ function since we know it's transformation properties.

First we transform the product part: $\tau \to \dfrac{-1}{\tau}$,

$$2\frac{\eta(-6/\tau)^2\eta(-3/2\tau)^2}{\eta(-3/\tau)^2\eta(-1/\tau)}$$

Transforming the product(2A)

$$= 2\frac{\eta(-1/(\tau/6))^2\eta(-1/(2\tau/3))^2}{\eta(-1/(\tau/3))^2\eta(-1/\tau)}$$

Using the fact that

$$\eta(-1/\tau) = \sqrt{-i\tau}\eta(\tau)$$

The above becomes:

$$\frac{2(\sqrt{-i\tau/6})^2\eta(\tau/6)^2(\sqrt{-i3\tau/2})^2\eta(3\tau/2)^2}{(\sqrt{-i\tau/3})^2\eta(\tau/3)^2\sqrt{-i\tau}\eta(\tau)} = \frac{2\sqrt{-i\tau}(\eta(\tau/6)^2\eta(3\tau/2)^2}{3\eta(\tau/3)^2\eta(\tau)}.$$

Transforming the sum

(2B)The sum part is as follows in terms of the $\tilde{\mu}$ function was :

$$4q^{-1/24}\tilde{\mu}(3\tau/2 + 1/2, -\tau; 3\tau).$$

Now let us look at the transformation properties of the $\tilde{\mu}$ function:

Recalling Zwegers again

$\tilde{\mu}\left(\frac{u}{c\tau+d}, \frac{v}{c\tau+d}; \frac{a\tau+b}{c\tau+d}\right) = V(\gamma)^{-3}(c\tau+d)^{1/2}\exp\left(\frac{-\pi i c(u-v)^2}{c\tau+d}\right)\tilde{\mu}(u, v; \tau).$

$\tau \to \dfrac{-1}{\tau}$, we get the following

$$\begin{aligned} & 4\exp(\pi i/12\tau)\tilde{\mu}(-3/2\tau + 1/2, 1/\tau; -3/\tau) \\ = {} & 4\exp(\pi i/12\tau)\tilde{\mu}\left(\frac{1/2+\tau/6}{\tau/3}, \frac{1/3}{\tau/3}; \frac{-1}{\tau/3}\right). \end{aligned}$$

Transforming the sum part continued..

$$4\exp\left(\frac{\pi i}{12\tau}\right)\sqrt{\frac{-i\tau}{3}}\exp\left(\frac{-\pi i}{12}(\tau+1/\tau+2)\right)\tilde{\mu}(\tau/6+1/2,1/3;\tau/3)$$
$$=\frac{4\sqrt{-i\tau}}{3}\exp\left(\frac{-\pi i\tau}{12}-\frac{-\pi i}{6}\right)\tilde{\mu}(\tau/6+1/2,1/3;\tau/3).$$

(2C) Now we rewrite $\tilde{\mu}$ as a combination of μ and the R function

$$\frac{4\sqrt{-i\tau}}{3}\exp\left(\frac{-\pi i\tau}{12}-\frac{-\pi i}{6}\right)\mu(\tau/6+1/2,1/3;\tau/3)$$
$$+i\frac{2\sqrt{-i\tau}}{3}\exp\left(\frac{-\pi i\tau}{12}-\frac{-\pi i}{6}\right)R(\tau/6+1/6;\tau/3)$$

Can something be done about the R function? (2D)

Recall

Let $\tau \in \mathbb{H}$ and for $a \in (-1/2, 1/2)$ and $b \in \mathbb{R}$

$$\int_{-\overline{\tau}}^{i\infty} \frac{g_{a+1/2,b+1/2}(z)\,dz}{\sqrt{i(z+\tau)}} = -e^{-\pi i a^2 \tau + 2\pi a(b+1/2)} R(a\tau - b; \tau)$$

$R(\tau/6 + 1/6; \tau/3)$ can be written as $R(\frac{1}{2}.\tau/3 - (-1/6); \tau/3)$

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$R(\tau/6 + 1/6; \tau/3)$ can be written as $R(\frac{1}{2}.\tau/3 - (-1/6); \tau/3)$

Proposition

let τ be in $\mathbb{H}$ and $b \in \mathbb{R}$, then

$$R(-\tau/2 - b; \tau) = e^{\pi i \tau/4 - \pi(b+1/2) + \pi i/2} - e^{\pi i \tau/4 - \pi(b+1/2)} \int_{-\overline{\tau}}^{i\infty} \frac{g_{0,b+1/2}(z) dz}{-i(z+\tau)}$$

Working with R Continued..

$$i\frac{2\sqrt{-i\tau}}{3}\exp\left(\frac{-\pi i\tau}{12}-\frac{-\pi i}{6}\right)R(\tau/6+1/6;\tau/3)$$
$$=2\sqrt{\frac{-i\tau}{3}}\left(\int_{-\overline{\tau}/3}^{i\infty}\frac{g_{0,2/3}(z)dz}{\sqrt{-i(z+\tau/3)}}+i\right)$$

(2E) However the integral

$$\int_{-\overline{\tau}/3}^{i\infty}\frac{g_{0,2/3}(z)dz}{\sqrt{-i(z+\tau)}}=-3i\int_{-\overline{\tau}}^{i\infty}\frac{g_2(z)dz}{\sqrt{-i(z+\tau)}}$$

Bringing Everything Together

(2F)

$$\frac{1}{\sqrt{-i\tau}} h_2(-1/\tau) = -h_2(\tau).$$

Everything on the left comes after the transformation $-\frac{1}{\sqrt{-i\tau}} h_2\left(-\frac{1}{\tau}\right)$ turns out to be

$$-\frac{2i}{\sqrt{3}} - \frac{2}{3}\frac{\eta(\tau/6)^2\eta(2\tau/3)^2}{\eta(\tau/3)^2\eta(\tau)} - \frac{4}{\sqrt{3}}\exp\left(-\frac{\pi i\tau}{12} - \frac{\pi i}{6}\right)\mu(\tau/6+1/2, 1/3; \tau/3) + 2i\sqrt{3}\int_{-\overline{\tau}}^{i\infty} \frac{g_2(z)dz}{\sqrt{i(z+\tau)}}.$$

And on the right, we have

$$2q^{1/3}\omega(-\sqrt{q}) + 2i\sqrt{3}\int_{-\overline{\tau}}^{i\infty} \frac{g_2(z)dz}{\sqrt{i(z+\tau)}}.$$

Omega identity

Theorem (Garvan-M, 2024)

$$2q^{1/3}\omega(-\sqrt{q}) = -\frac{2i}{\sqrt{3}} - \frac{2}{3}\frac{\eta(\tau/6)^2\eta(2\tau/3)^2}{\eta(\tau/3)^2\eta(\tau)} - \frac{4}{\sqrt{3}}\exp\left(-\frac{\pi i\tau}{12} - \frac{\pi i}{6}\right)\mu(\tau/6+1/2,1/3;\tau/3).$$

Let $\tau \to 6\tau$, we get

$$2q^2\omega(-q^3) = -\frac{2i}{\sqrt{3}} - \frac{2}{3}\frac{E(q)^2E(q^4)^2}{E(q^2)^2E(q^6)} - \frac{4}{\sqrt{3}}\exp\left(\frac{-\pi i\tau}{12} - \frac{\pi i}{6}\right)\mu(\tau+1/2,1/3;2\tau).$$

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we know

$$\omega(-q^3) = \frac{1}{(q^6;q^6)_\infty}\sum_{n=-\infty}^{\infty}\frac{(-1)^n q^{9n(n+1)}}{1+q^{6n+3}}.$$

Step 3: Dissecting the RHS

- We dissect the η part and the μ parts separately.
- We then prove the identities needed for the proof.

Dissections

$$\frac{E(q)^2E(q^4)^2}{E(q^2)^2E(q^6)} = \frac{1}{E(q^6)}\left(\mathsf{e}_0(q^3) - 2q\mathsf{e}_1(q^3) + q^2\mathsf{e}_2(q^3)\right).$$

Dissection of $\Delta^2(-q) := \dfrac{E(q)^2E(q^4)^2}{E(q^2)^2}$.

$$\Delta(-q) = \frac{E(q)E(q^4)}{E(q^2)} = \sum_{n\geq 0}(-q)^{n(n+1)/2},$$

$$\Delta(q) = \sum_{n\geq 0} q^{n(n+1)/2} = \sum_{n=-\infty}^{\infty} q^{n(2n+1)}$$

When $n \equiv 0 \pmod 3$, $n(2n+1) \equiv 0 \pmod 3$.
When $n \equiv 1 \pmod 3$, $n(2n+1) \equiv 0 \pmod 3$.
When $n \equiv 2 \pmod 3$, $n(2n+1) \equiv 1 \pmod 3$.

$$
\begin{aligned}
\Delta(q) &= P_0(q^3) + qP_1(q^3), \\
P_0(q) &= \sum_{n=-\infty}^{\infty} q^{6n^2+n} + \sum_{n=-\infty}^{\infty} q^{6n^2+5n}, \text{ and} \\
P_0(q) &= \sum_{n=-\infty}^{\infty} q^{\frac{3n^2+3n}{2}} \\
&= \frac{E(q^2)E(q^3)^2}{E(q^6)E(q)}.
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
P_1(q) &= \sum_{n=-\infty}^{\infty} q^{6n^2-3n} = \frac{E(q^2)^2}{E(q^3)}, \text{ and} \\
\Delta^2(q) &= (P_0(q^3) + qP_1(q^3))^2 = P_0^2(q^3) + 2qP_0P_1(q^3) + q^2P_2^2(q^3). \\
\Delta^2(-q) &= \mathsf{e}_0(q^3) - 2q\mathsf{e}_1(q^3) + q^2\mathsf{e}_2(q^3)
\end{aligned}
$$

Dissection of the μ function

Now we need to dissect the following:

$$\mu(\tau + 1/2, 1/3; 2\tau) = \frac{e^{\pi i(\tau+1/2)}}{\vartheta(1/3; 2\tau)} \sum_{n=-\infty}^{\infty} \frac{(-1)^n e^{2\pi i n/3} q^{n^2+n}}{1 + q^{2n+1}}.$$

One could easily calculate $\vartheta(1/3; 2\tau)$ using the Jacobi's Triple Product Identity. However finding the dissection of the sum takes some work.

$$\sum_{n=-\infty}^{\infty} \frac{(-1)^n e^{2\pi i n/3} q^{n^2+n}}{1 + q^{2n+1}} = Y_0 + \zeta Y_1 + \zeta^2 Y_2,$$

where $\zeta = e^{2\pi i/3}$.

$$Y_0 = \sum_{n \equiv 0 \pmod 3} \frac{(-1)^n q^{n^2+n}}{1+q^{2n+1}},$$

$$Y_1 = \sum_{n \equiv 1 \pmod 3} \frac{(-1)^n q^{n^2+n}}{1+q^{2n+1}},$$

$$Y_2 = \sum_{n \equiv 2 \pmod 3} \frac{(-1)^n q^{n^2+n}}{1+q^{2n+1}}.$$

Observe that

$$Y_0 = y_{00}(q^3) + q y_{01}(q^3) + q^2 y_{02}(q^3),$$
$$Y_1 = y_{10}(q^3) + q y_{11}(q^3) + q^2 y_{12}(q^3),$$
$$Y_2 = y_{20}(q^3) + q y_{21}(q^3) + q^2 y_{22}(q^3).$$

Results Needed

We need the following identities to hold:

- $Y_0 - Y_2 = E(q^6)$.
- $\dfrac{y_{00}(q)}{E(q^2)} = 1/2(e_0(q) + 1)$.
- $\dfrac{y_{01}(q)}{E(q^2)} = -e_1(q)$.
- $y_{10}(q) = y_{11}(q) = 0$.
- $\dfrac{y_{12}(q)}{E(q^2)} = -\omega(-q)$.
- $2\dfrac{y_{02}(q)}{E(q^2)} = \omega(-q) + e_2(q)$.

Tools Necessary

- The Crank generating function
- The Rank Generating funtion
- JTP
- Shifting n as per need and different congruence conditions

Identity

$$\sum_{n=-\infty}^{\infty}(-1)^n q^{n^2} z^n \frac{1-zq^{2n}}{1+zq^{2n}} = \frac{\Theta(z,q^2)\Theta(-zq,q^2)\Theta_3(q)}{\Theta(-z,q^2)}$$ where

$\Theta(z,q) = (z)_\infty(z^{-1}q)_\infty(q)_\infty$ and $\Theta_3(q) = \sum_{n=-\infty}^{\infty} q^{n^2}$.

The other Half

$$\frac{1}{\sqrt{-i\tau}} h_0(-1/\tau) = h_1(\tau)$$
$$\frac{1}{\sqrt{-i\tau}} h_1(-1/\tau) = h_0(\tau)$$

$$h_0(\tau) = q^{-1/24} f(q) - 2i\sqrt{3} \int_{-\overline{\tau}}^{i\infty} \frac{g_1(z)\, dz}{\sqrt{i(z+\tau)}}.$$

The Rank Generating function

$$\mathcal{R}(w;q) := \sum_{n=0}^{\infty} \frac{q^{n^2}}{\prod_{k=1}^{n} (1 - wq^k)(1 - w^{-1}q^k)}.$$

Zagier's Identity

$$\frac{q^{-1/24}\mathcal{R}(e^{2\pi i\alpha};q)}{e^{-\pi i\alpha}-e^{\pi i\alpha}} = \frac{\eta(3\tau)^3}{\eta(\tau)\vartheta(3\alpha;3\tau)} - q^{-1/6}exp\left(-\pi i\alpha\right)\mu(3\alpha,\tau;3\tau) + q^{-1/6}e^{-\pi i\alpha}\mu(3\alpha,-\tau;3\tau).$$

It is well known that $f(q)=\mathcal{R}(-1;q)$. Also, we have

$$f(q) = 4q^{-1/8}\mu(2\tau+1/2,\tau;3\tau) + q^{1/24}\frac{\eta(3\tau)^4}{\eta(\tau)\eta(6\tau)^2}.$$

The other half

After considering the transformations $\tau \to -\dfrac{1}{\tau}$ on h_0 and following the exact procedure as before, we reach to the following stage:

Corollary (Garvan-M, 2024)

$$2q^{1/3}\omega(\sqrt{q}) = -\frac{2i}{\sqrt{3}} + \frac{2}{3}\frac{\eta(\tau/3)^4}{\eta(\tau/6)^2\eta(\tau)} - \frac{4}{\sqrt{3}}\exp\left(\frac{-\pi i\tau}{12} + \frac{4\pi i}{3}\right)\mu(\tau/6 - 2/3, 1/3; \tau/3).$$

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Thank You!