Vanishing Properties of D'Arcais like Polynomials $P_n^g(x)$

— Part 2 —

1. D'Arcais Polynomials $P_n^{\sigma}(x)$

a) $P_n^{\sigma}(x) := \dfrac{x}{n} \sum_{k=1}^{n} \sigma(k) P_{n-k}^{\sigma}(x)$, with $P_0^{\sigma}(x) = 1$

by recursion

b)

$$\prod (1-q^n)^{-x} = \exp\left(x \sum_{n=1}^{\infty} \sigma(n) \frac{q^n}{n} \right)$$

$$=: \sum_{n=0}^{\infty} P_n^{\sigma}(x)\, q^n$$

c) Def. by Nekrasov–Okounkov

hook-length formula

Conclusions

a) • $P_n^{\sigma}(x) = \frac{1}{n!}\left(x^n + A^{\sigma}_{n,n-1}x^{n-1} + \dots + A^{\sigma}_{n,1}x\right)$

— $n \geq 1$ —

• Generalization: $g: \mathbb{N} \to \mathbb{N}_0$, $h_s(n): \mathbb{N} \to \mathbb{R}_{>0}$
with $g(1) = h(1) = 1$, $s \in [0,1]$, $h_s(n) = n^s$

$$P_n^{g,h_s}(x) := \frac{x}{h_s(n)} \sum_{k=1}^{n} g(k)\, P_{n-k}^{g,h_s}(x), \quad P_0(x) = 1$$

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$$A_n^G(x) := n! \cdot P_n^G(x)$$

$$= x^n + A_{n,n-1}^G x^{n-1} + \dots + A_{n,1}^G x$$

with $A_{n,m}^G \in \mathbb{N}$.

$\rightsquigarrow$ $P_n^G(\alpha) = 0 \Rightarrow \alpha \in \overline{\mathbb{Z}}$

and if $\alpha \in \mathbb{R}$, then $\alpha \leq 0$.

b) Captures the secret of

vanishing coefficients of

Dedekind-eta powers

$$P_n^6(-24) = 0$$

$$\Longleftrightarrow \tau(n+1) = 0$$

Ramanujan τ-function

$\rightarrow$ Lehmer's

conjecture: $\tau(n) \neq 0$

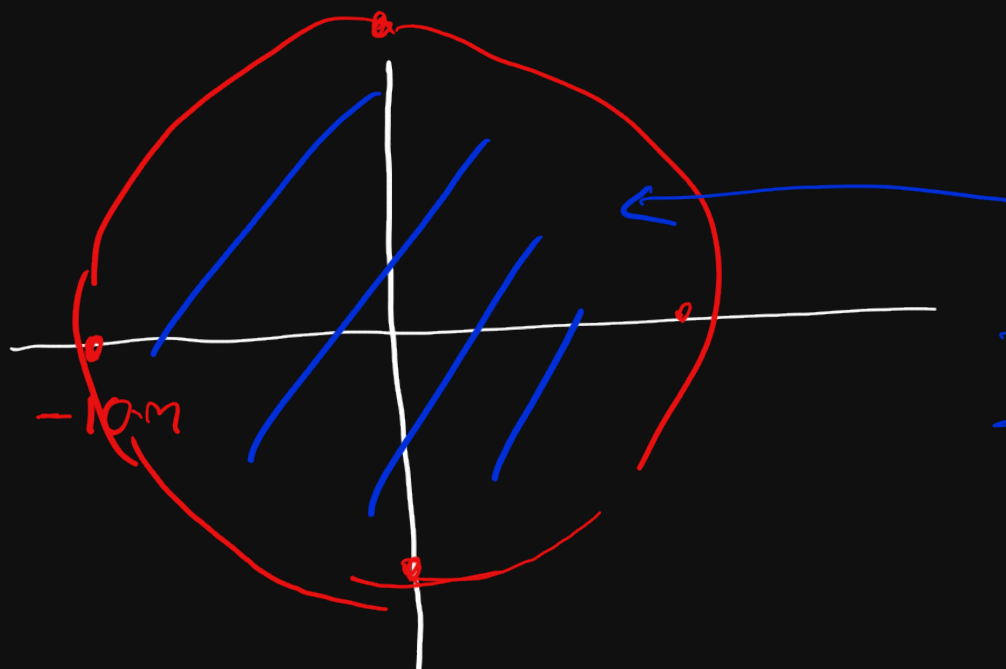
$\forall n \in \mathbb{N}$.

c)

- combinatorics
- statist. mechanic

2. Vanishing results

a)



Th: H. L. Nadaune

complex plane

$n \in \mathbb{N}$

b) $1 \leq n \leq 1500$

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all zeros α satisfy

$Re(\alpha) < 0$

for $\alpha \neq 0$

c) <u>Theorem (H., Luca, N.)</u>

Let $n = m \cdot p + r$, $p \in \mathbb{P}$
and $0 \leq r < p$.

Then

$$A_n(x) \equiv A_r(x) \cdot A_p(x)^m \quad (p)$$

- $A_1(x), A_2(x), A_3(x), A_4(x)$ linear
- $A_p(x) \ (p) \equiv x \cdot (x^{p-1} - 1) \quad (p)$

Theorem (H., Luca, N.)

Let $\alpha = \xi_m$ be a m-th root unity. Let $n \geq 1$ and

$$P_n^c(\alpha) = 0 \implies \alpha = \xi_2 = -1.$$

- 1. Proof: – brute force
  – analytic number theory.

- 2. Proof: Algebraic Number Theory
  – Dedekind sums

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