An Orthogonal View of Gaussian Polynomials and Unimodality

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Specialty Seminar in Partition Theory, q-Series and Related Topics
Host: Michigan Technological University
Master of Ceremonies: William Keith
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Intro

For $a \geq 0$, the coefficients of Gaussian polynomials $\begin{bmatrix} N+3 \\ 3 \end{bmatrix}$ are described by the series expansions of the following three rational functions.

$$\frac{z^{2a}\left(1+z^2+z^3-z^{4a+2}\right)}{(1-z)(1-z^2)(1-z^4)}$$

$$\frac{z^{2a+1}\left(1+z+z^3-z^{4a+3}\right)}{(1-z)(1-z^2)(1-z^4)}$$

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Details to follow.

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(We'll talk more about partitions later in the semester!)

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Definition 1

A *partition* of a positive integer n is a finite non-increasing sequence of positive integers $\lambda_1, \lambda_2, \ldots \lambda_m$ such that

$$\sum_{i=1}^{m} \lambda_i = n$$

The λ_i are called the *parts* of the partition.

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$$
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Partitions: Adding and Counting

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The formulas we'll see today are often referred to as *constituents of a quasipolynomial*.

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Definition 2

A function $f(n)$ is a *quasipolynomial* if there exist d polynomials $f_0(n),\dots,f_{d-1}(n)$ such that

$$f(n) = \begin{cases} f_0(n) & \text{if } n \equiv 0 \pmod{d} \\ f_1(n) & \text{if } n \equiv 1 \pmod{d} \\ \vdots & \vdots \\ f_{d-1}(n) & \text{if } n \equiv d-1 \pmod{d} \end{cases}$$

for all $n \in \mathbb{Z}$. The polynomials f_i are called the *constituents* of f and the number of them, d, is the *period* of f.

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$$= \frac{1+q+2q^2+3q^3+4q^4+5q^5+4q^6+5q^7+4q^8+3q^9+2q^{10}+q^{11}+q^{12}}{(1-q^6)^3}$$

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Hence, $p(6k+1,3)=1\binom{k+2}{2}+5\binom{k+1}{2}=3k^2+4k+1.$

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Multiplying and collecting like terms five more times gives us a complete description of $p(n,3)$.

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In general

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$$\sum_{n=0}^{\infty} p(n,m)q^n = \frac{1}{(q;q)_m} = \frac{1}{(q;q)_m} \times \frac{\left(\frac{(1-q^{\mathrm{lcm}(m)})^m}{(q;q)_m}\right)}{\left(\frac{(1-q^{\mathrm{lcm}(m)})^m}{(q;q)_m}\right)} = \frac{\prod_{j=1}^{m} \sum_{i=0}^{\frac{\mathrm{lcm}(m)-j}{j}} q^{ij}}{(1-q^{\mathrm{lcm}(m)})^m}$$

$$= \prod_{j=1}^{m} \sum_{i=0}^{\frac{\mathrm{lcm}(m)-j}{j}} q^{ij} \times \sum_{k\geq 0} \binom{k+m-1}{m-1} q^{\mathrm{lcm}(m)k}$$

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$$= (q^{N+1};q)_3 \times \big(1 + q + 2q^2 + 3q^3 + 4q^4 + 5q^5 + 4q^6 + 5q^7 + 4q^8 + 3q^9$$

$$+2q^{10} + q^{11} + q^{12}\big) \times \sum_{k\geq 0} \binom{k+2}{2} q^{6k}$$

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$$= \Big(1 + q + 2q^2 + 3q^3 + 4q^4 + 5q^5 + 4q^6 + 5q^7 + 4q^8 + 3q^9 + 2q^{10} + q^{11} + q^{12} - q^{1+N} - 2q^{2+N} - 4q^{3+N} - 6q^{4+N} - 9q^{5+N} - 12q^{6+N} - 13q^{7+N} - 14q^{8+N} - 13q^{9+N} - 12q^{10+N} - 9q^{11+N} - 6q^{12+N} - 4q^{13+N} - 2q^{14+N} - q^{15+N} + q^{3+2N} + 2q^{4+2N} + 4q^{5+2N} + 6q^{6+2N} + 9q^{7+2N} + 12q^{8+2N} + 13q^{9+2N} + 14q^{10+2N} + 13q^{11+2N} + 12q^{12+2N} + 9q^{13+2N} + 6q^{14+2N} + 4q^{15+2N} + 2q^{16+2N} + q^{17+2N} - q^{6+3N} - q^{7+3N} - 2q^{8+3N} - 3q^{9+3N} - 4q^{10+3N} - 5q^{11+3N} - 4q^{12+3N} - 5q^{13+3N} - 4q^{14+3N} - 3q^{15+3N} - 2q^{16+3N} - q^{17+3N} - q^{18+3N}\Big) \times \sum_{k\geq 0} \binom{k+2}{2} q^{6k}.$$

Now we multiply and collect like terms...

The formulas for $p(n,3,N)$ are expressed as $p(6k+x,3,6j+y)$:

$p(6k+x,3,6j+y) \rightarrow 36$ constituents for $p(n,3,N)$.

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The formulas for $p(n,3,N)$ are expressed as $p(6k+x,3,6j+y)$: There are six cases for $n = 6k+x$ and six cases for $N = 6j+y$. This means there are 36 constituents in the quasipolynomial for $p(n,3,N)$.

36 Constituents of the quasipolynomial for $p(n,3,N)$

$$p\left(n,3,6j\right)$$

$$p(6k,3,6j) = \begin{cases} \binom{k+2}{2} + 4\binom{k+1}{2} + \binom{k}{2} - 12\binom{k+1-j}{2} - 6\binom{k-j}{2} \\ +6\binom{k+1-2j}{2} + 12\binom{k-2j}{2} - \binom{k+1-3j}{2} - 4\binom{k-3j}{2} - \binom{k-1-3j}{2} \end{cases}$$

$$p(6k+1,3,6j) = \begin{cases} \binom{k+2}{2} + 5\binom{k+1}{2} - \binom{k+2-j}{2} - 13\binom{k+1-j}{2} - 4\binom{k-j}{2} \\ +9\binom{k+1-2j}{2} + 9\binom{k-2j}{2} - \binom{k+1-3j}{2} - 5\binom{k-3j}{2} \end{cases}$$

$$p(6k+2,3,6j) = \begin{cases} 2\binom{k+2}{2} + 4\binom{k+1}{2} - 2\binom{k+2-j}{2} - 14\binom{k+1-j}{2} - 2\binom{k-j}{2} \\ +12\binom{k+1-2j}{2} + 6\binom{k-2j}{2} - 2\binom{k+1-3j}{2} - 4\binom{k-3j}{2} \end{cases}$$

$$p(6k+3,3,6j) = \begin{cases} 3\binom{k+2}{2} + 3\binom{k+1}{2} - 4\binom{k+2-j}{2} - 13\binom{k+1-j}{2} - \binom{k-j}{2} \\ +\binom{k+2-2j}{2} + 13\binom{k+1-2j}{2} + 4\binom{k-2j}{2} - 3\binom{k+1-3j}{2} - 3\binom{k-3j}{2} \end{cases}$$

$$p(6k+4,3,6j) = \begin{cases} 4\binom{k+2}{2} + 2\binom{k+1}{2} - 6\binom{k+2-j}{2} - 12\binom{k+1-j}{2} + 2\binom{k+2-2j}{2} \\ +14\binom{k+1-2j}{2} + 2\binom{k-2j}{2} - 4\binom{k+1-3j}{2} - 2\binom{k-3j}{2} \end{cases}$$

$$p(6k+5,3,6j) = \begin{cases} 5\binom{k+2}{2} + \binom{k+1}{2} - 9\binom{k+2-j}{2} - 9\binom{k+1-j}{2} + 4\binom{k+2-2j}{2} \\ +13\binom{k+1-2j}{2} + \binom{k-2j}{2} - 5\binom{k+1-3j}{2} - \binom{k-3j}{2} \end{cases}$$

36 Constituents of the quasipolynomial for $p(n,3,N)$

$$p\left(n,3,6j+1\right)$$

$$p(6k,3,6j+1) = \begin{cases} \binom{k+2}{2} + 4\binom{k+1}{2} + \binom{k}{2} - 9\binom{k+1-j}{2} - 9\binom{k-j}{2} \\ +2\binom{k+1-2j}{2} + 14\binom{k-2j}{2} + 2\binom{k-1-2j}{2} \\ -3\binom{k-3j}{2} - 3\binom{k-1-3j}{2} \end{cases}$$

$$p(6k+1,3,6j+1) = \begin{cases} \binom{k+2}{2} + 5\binom{k+1}{2} - 12\binom{k+1-j}{2} - 6\binom{k-j}{2} + 4\binom{k+1-2j}{2} \\ +13\binom{k-2j}{2} + \binom{k-1-2j}{2} - 4\binom{k-3j}{2} - 2\binom{k-1-3j}{2} \end{cases}$$

$$p(6k+2,3,6j+1) = \begin{cases} 2\binom{k+2}{2} + 4\binom{k+1}{2} - \binom{k+2-j}{2} - 13\binom{k+1-j}{2} - 4\binom{k-j}{2} \\ +6\binom{k+1-2j}{2} + 12\binom{k-2j}{2} - 5\binom{k-3j}{2} - \binom{k-1-3j}{2} \end{cases}$$

$$p(6k+3,3,6j+1) = \begin{cases} 3\binom{k+2}{2} + 3\binom{k+1}{2} - 2\binom{k+2-j}{2} - 14\binom{k+1-j}{2} \\ -2\binom{k-j}{2} + 9\binom{k+1-2j}{2} + 9\binom{k-2j}{2} - \binom{k+1-3j}{2} \\ -4\binom{k-3j}{2} - \binom{k-1-3j}{2} \end{cases}$$

$$p(6k+4,3,6j+1) = \begin{cases} 4\binom{k+2}{2} + 2\binom{k+1}{2} - 4\binom{k+2-j}{2} - 13\binom{k+1-j}{2} - \binom{k-j}{2} \\ +12\binom{k+1-2j}{2} + 6\binom{k-2j}{2} - \binom{k+1-3j}{2} - 5\binom{k-3j}{2} \end{cases}$$

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$$p\left(n,3,6j+2\right)$$

$$p(6k,3,6j+2) = \begin{cases} \binom{k+2}{2}+4\binom{k+1}{2}+\binom{k}{2}-6\binom{k+1-j}{2}-12\binom{k-j}{2} \\ +12\binom{k-2j}{2}+6\binom{k-1-2j}{2}-\binom{k-1-3j}{2} \\ -4\binom{k-2-3j}{2}-\binom{k-3-3j}{2} \end{cases}$$

$$p(6k+1,3,6j+2) = \begin{cases} \binom{k+2}{2}+5\binom{k+1}{2}-9\binom{k+1-j}{2}-9\binom{k-j}{2}+\binom{k+1-2j}{2} \\ +13\binom{k-2j}{2}+4\binom{k-1-2j}{2}-\binom{k-3j}{2}-5\binom{k-1-3j}{2} \end{cases}$$

$$p(6k+2,3,6j+2) = \begin{cases} 2\binom{k+2}{2}+4\binom{k+1}{2}-12\binom{k+1-j}{2}-6\binom{k-j}{2}+2\binom{k+1-2j}{2} \\ +14\binom{k-2j}{2}+2\binom{k-1-2j}{2}-2\binom{k-3j}{2}-4\binom{k-1-3j}{2} \end{cases}$$

$$p(6k+3,3,6j+2) = \begin{cases} 3\binom{k+2}{2}+3\binom{k+1}{2}-\binom{k+2-j}{2}-13\binom{k+1-j}{2}-4\binom{k-j}{2} \\ +4\binom{k+1-2j}{2}+13\binom{k-2j}{2}+\binom{k-1-2j}{2} \\ -3\binom{k-3j}{2}-3\binom{k-1-3j}{2} \end{cases}$$

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36 Constituents of the quasipolynomial for $p(n, 3, N)$

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$$p(6k,3,6j+3) = \begin{cases} \binom{k+2}{2} + 4\binom{k+1}{2} + \binom{k}{2} - 4\binom{k+1-j}{2} - 13\binom{k-j}{2} \\ -\binom{k-1-j}{2} + 6\binom{k-2j}{2} + 12\binom{k-1-2j}{2} \\ -3\binom{k-1-3j}{2} - 3\binom{k-2-3j}{2} \end{cases}$$

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36 Constituents of the quasipolynomial for $p(n, 3, N)$

$$p\left(n, 3, 6j+4\right)$$

$$p(6k,3,6j+4) = \begin{cases} \binom{k+2}{2} + 4\binom{k+1}{2} + \binom{k}{2} - 2\binom{k+1-j}{2} - 14\binom{k-j}{2} \\ -2\binom{k-1-j}{2} + 2\binom{k-2j}{2} + 14\binom{k-1-2j}{2} + 2\binom{k-2-2j}{2} \\ -\binom{k-1-3j}{2} - 4\binom{k-2-3j}{2} - \binom{k-3-3j}{2} \end{cases}$$

$$p(6k+1,3,6j+4) = \begin{cases} \binom{k+2}{2} + 5\binom{k+1}{2} - 4\binom{k+1-j}{2} - 13\binom{k-j}{2} - \binom{k-1-j}{2} \\ +4\binom{k-2j}{2} + 13\binom{k-1-2j}{2} + \binom{k-2-2j}{2} \\ -\binom{k-1-3j}{2} - 5\binom{k-2-3j}{2} \end{cases}$$

$$p(6k+2,3,6j+4) = \begin{cases} 2\binom{k+2}{2} + 4\binom{k+1}{2} - 6\binom{k+1-j}{2} - 12\binom{k-j}{2} \\ +6\binom{k-2j}{2} + 6\binom{k-1-2j}{2} - 2\binom{k-1-3j}{2} - 4\binom{k-2-3j}{2} \end{cases}$$

$$p(6k+3,3,6j+4) = \begin{cases} 3\binom{k+2}{2} + 3\binom{k+1}{2} - 9\binom{k+1-j}{2} - 9\binom{k-j}{2} \\ +9\binom{k-2j}{2} + 9\binom{k-1-2j}{2} - 3\binom{k-1-3j}{2} - 3\binom{k-2-3j}{2} \end{cases}$$

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$$p(6k+3,3,6j+5) = \begin{cases} 3\binom{k+2}{2} + 3\binom{k+1}{2} - 6\binom{k+1-j}{2} - 12\binom{k-j}{2} \\ +4\binom{k-2j}{2} + 13\binom{k-1-2j}{2} + \binom{k-2-2j}{2} \\ -\binom{k-1-3j}{2} - 4\binom{k-2-3j}{2} - \binom{k-3-3j}{2} \end{cases}$$

$$p(6k+4,3,6j+5) = \begin{cases} 4\binom{k+2}{2} + 2\binom{k+1}{2} - 9\binom{k+1-j}{2} - 9\binom{k-j}{2} \\ +6\binom{k-2j}{2} + 12\binom{k-1-2j}{2} - \binom{k-1-3j}{2} - 5\binom{k-2-3j}{2} \end{cases}$$

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Definition 3

For $m, N \geq 0$ the expression

$$\begin{bmatrix} N+m \\ m \end{bmatrix} = \begin{cases} \dfrac{(q;q)_{N+m}}{(q;q)_m (q;q)_N} = \dfrac{\left(q^{N+1};q\right)_m}{(q;q)_m} & \text{for } m, N \geq 1 \\ 1 \text{ for } m = 0 \end{cases} \tag{7}$$

is known as a Gaussian polynomial.

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Example 4

$$\begin{bmatrix} 4+3 \\ 3 \end{bmatrix} = \frac{(q;q)_7}{(q;q)_3 (q;q)_4} = \frac{\left(q^5;q\right)_3}{(q;q)_3} = \frac{\left(1-q^5\right)\left(1-q^6\right)\left(1-q^7\right)}{(1-q)\left(1-q^2\right)\left(1-q^3\right)}$$

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$$= 1 + q + 2q^2 + 3q^3 + 4q^4 + 4q^5 + 5q^6 + 4q^7 + 4q^8 + 3q^9 + 2q^{10} + q^{11} + q^{12}.$$

Gaussian polynomials have a well known partition theoretic interpretation.

Proposition 5

$$\begin{bmatrix} N+m \\ m \end{bmatrix} = \sum_{n=0}^{mN} p(n,m,N)q^n \tag{8}$$

where $p(n,m,N)$ enumerates the number of partitions of n into at most m parts with no part larger than N.

Introduction

$$\begin{bmatrix} 4+3 \\ 3 \end{bmatrix} = \frac{(1-q^5)(1-q^6)(1-q^7)}{(1-q)(1-q^2)(1-q^3)}$$

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3. *They are unimodal: for* $n \leq \frac{mN}{2}$, *the coefficient of* q^{n-1} *is less than or equal to the coefficient of* q^n.

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David Bressoud, Discrete Mathematics, 1992

The first proof of unimodality is due to Sylvester (1878). I. J. Schur's later proof employs the theory of invariants. White (1980), and Proctor (1982) have also proved unimodality. The title of Proctor's paper is telling: *Solution of two difficult combinatorial problems with linear algebra*. Of course, there is the celebrated proof of unimodality by O'Hara (1990). So celebrated is this proof that Zeilberger wrote several follow-up papers about O'Hara's proof. Bressoud's paper is itself a follow-up to Zeilberger's paper.

Introduction

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A polynomial $P(q) = a_0 + a_1 q + \cdots + a_d q^d$ is called unimodal if there exists x such that

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We can show unimodality for $\begin{bmatrix} N+2 \\ 2 \end{bmatrix}$ too...

The Gaussian polynomials possess three characteristics which are not immediately obvious from the definition:

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$$C_{m,A}(N) = p\left(\left\lfloor \frac{mN}{2} \right\rfloor - A, m, N\right).$$

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and for mN odd,

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$\begin{bmatrix}0+3\\3\end{bmatrix}$	$\begin{bmatrix}1+3\\3\end{bmatrix}$	$\begin{bmatrix}2+3\\3\end{bmatrix}$	$\begin{bmatrix}3+3\\3\end{bmatrix}$	$\begin{bmatrix}4+3\\3\end{bmatrix}$	$\begin{bmatrix}5+3\\3\end{bmatrix}$	$\begin{bmatrix}6+3\\3\end{bmatrix}$
$p(0,3,0)$	$p(1,3,1)$	$p(3,3,2)$	$p(4,3,3)$	$p(6,3,4)$	$p(7,3,5)$	$p(9,3,6)$
	$p(0,3,1)$	$p(2,3,2)$	$p(3,3,3)$	$p(5,3,4)$	$p(6,3,5)$	$p(8,3,6)$
		$p(1,3,2)$	$p(2,3,3)$	$p(4,3,4)$	$p(5,3,5)$	$p(7,3,6)$
		$p(0,3,2)$	$p(1,3,3)$	$p(3,3,4)$	$p(4,3,5)$	$p(6,3,6)$
			$p(0,3,3)$	$p(2,3,4)$	$p(3,3,5)$	$p(5,3,6)$
				$p(1,3,4)$	$p(2,3,5)$	$p(4,3,6)$
				$p(0,3,4)$	$p(1,3,5)$	$p(3,3,6)$
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$C_{3,0}(N)$	$p(0,3,0)$	$p(1,3,1)$	$p(3,3,2)$	$p(4,3,3)$	$p(6,3,4)$	$p(7,3,5)$	$p(9,3,6)$
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$C_{3,1}(N)$		$p(0,3,1)$	$p(2,3,2)$	$p(3,3,3)$	$p(5,3,4)$	$p(6,3,5)$	$p(8,3,6)$
			$p(1,3,2)$	$p(2,3,3)$	$p(4,3,4)$	$p(5,3,5)$	$p(7,3,6)$
			$p(0,3,2)$	$p(1,3,3)$	$p(3,3,4)$	$p(4,3,5)$	$p(6,3,6)$
				$p(0,3,3)$	$p(2,3,4)$	$p(3,3,5)$	$p(5,3,6)$
					$p(1,3,4)$	$p(2,3,5)$	$p(4,3,6)$
					$p(0,3,4)$	$p(1,3,5)$	$p(3,3,6)$
						$p(0,3,5)$	$p(2,3,6)$
							$p(1,3,6)$
							$p(0,3,6)$

	$\begin{bmatrix}0+3\\3\end{bmatrix}$	$\begin{bmatrix}1+3\\3\end{bmatrix}$	$\begin{bmatrix}2+3\\3\end{bmatrix}$	$\begin{bmatrix}3+3\\3\end{bmatrix}$	$\begin{bmatrix}4+3\\3\end{bmatrix}$	$\begin{bmatrix}5+3\\3\end{bmatrix}$	$\begin{bmatrix}6+3\\3\end{bmatrix}$
$C_{3,0}(N)$	$p(0,3,0)$	$p(1,3,1)$	$p(3,3,2)$	$p(4,3,3)$	$p(6,3,4)$	$p(7,3,5)$	$p(9,3,6)$
$C_{3,1}(N)$		$p(0,3,1)$	$p(2,3,2)$	$p(3,3,3)$	$p(5,3,4)$	$p(6,3,5)$	$p(8,3,6)$
$C_{3,2}(N)$			$p(1,3,2)$	$p(2,3,3)$	$p(4,3,4)$	$p(5,3,5)$	$p(7,3,6)$
			$p(0,3,2)$	$p(1,3,3)$	$p(3,3,4)$	$p(4,3,5)$	$p(6,3,6)$
				$p(0,3,3)$	$p(2,3,4)$	$p(3,3,5)$	$p(5,3,6)$
					$p(1,3,4)$	$p(2,3,5)$	$p(4,3,6)$
					$p(0,3,4)$	$p(1,3,5)$	$p(3,3,6)$
						$p(0,3,5)$	$p(2,3,6)$
							$p(1,3,6)$
							$p(0,3,6)$

Background

	$\begin{bmatrix}0+3\\3\end{bmatrix}$	$\begin{bmatrix}1+3\\3\end{bmatrix}$	$\begin{bmatrix}2+3\\3\end{bmatrix}$	$\begin{bmatrix}3+3\\3\end{bmatrix}$	$\begin{bmatrix}4+3\\3\end{bmatrix}$	$\begin{bmatrix}5+3\\3\end{bmatrix}$	$\begin{bmatrix}6+3\\3\end{bmatrix}$
$C_{3,0}(N)$	$p(0,3,0)$	$p(1,3,1)$	$p(3,3,2)$	$p(4,3,3)$	$p(6,3,4)$	$p(7,3,5)$	$p(9,3,6)$
$C_{3,1}(N)$		$p(0,3,1)$	$p(2,3,2)$	$p(3,3,3)$	$p(5,3,4)$	$p(6,3,5)$	$p(8,3,6)$
$C_{3,2}(N)$			$p(1,3,2)$	$p(2,3,3)$	$p(4,3,4)$	$p(5,3,5)$	$p(7,3,6)$
$C_{3,3}(N)$			$p(0,3,2)$	$p(1,3,3)$	$p(3,3,4)$	$p(4,3,5)$	$p(6,3,6)$
				$p(0,3,3)$	$p(2,3,4)$	$p(3,3,5)$	$p(5,3,6)$
					$p(1,3,4)$	$p(2,3,5)$	$p(4,3,6)$
					$p(0,3,4)$	$p(1,3,5)$	$p(3,3,6)$
						$p(0,3,5)$	$p(2,3,6)$
							$p(1,3,6)$
							$p(0,3,6)$

	$\begin{bmatrix}0+3\\3\end{bmatrix}$	$\begin{bmatrix}1+3\\3\end{bmatrix}$	$\begin{bmatrix}2+3\\3\end{bmatrix}$	$\begin{bmatrix}3+3\\3\end{bmatrix}$	$\begin{bmatrix}4+3\\3\end{bmatrix}$	$\begin{bmatrix}5+3\\3\end{bmatrix}$	$\begin{bmatrix}6+3\\3\end{bmatrix}$
$C_{3,0}(N)$	$p(0,3,0)$	$p(1,3,1)$	$p(3,3,2)$	$p(4,3,3)$	$p(6,3,4)$	$p(7,3,5)$	$p(9,3,6)$
$C_{3,1}(N)$		$p(0,3,1)$	$p(2,3,2)$	$p(3,3,3)$	$p(5,3,4)$	$p(6,3,5)$	$p(8,3,6)$
$C_{3,2}(N)$			$p(1,3,2)$	$p(2,3,3)$	$p(4,3,4)$	$p(5,3,5)$	$p(7,3,6)$
$C_{3,3}(N)$			$p(0,3,2)$	$p(1,3,3)$	$p(3,3,4)$	$p(4,3,5)$	$p(6,3,6)$
$C_{3,4}(N)$				$p(0,3,3)$	$p(2,3,4)$	$p(3,3,5)$	$p(5,3,6)$
$C_{3,5}(N)$					$p(1,3,4)$	$p(2,3,5)$	$p(4,3,6)$
$C_{3,6}(N)$					$p(0,3,4)$	$p(1,3,5)$	$p(3,3,6)$
$C_{3,7}(N)$						$p(0,3,5)$	$p(2,3,6)$
$C_{3,8}(N)$							$p(1,3,6)$
$C_{3,9}(N)$							$p(0,3,6)$
$\vdots$							

Background

$$\begin{aligned}
\{C_{3,0}(N)\}_{N\geq 0} &= p(0,3,0), p(1,3,1), p(3,3,2), p(4,3,3), p(6,3,4), p(7,3,5), p(9,3,6), \cdots \\
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Background

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Sequences!

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Sequences!

$$\{C_{3,3a}(N)\}_{a,N=0} = \Big\{p\big(3k, 3, 2(k+a)\big),\ p\big(3k+1, 3, 2(k+a)+1\big)\Big\}_{a,k=0}$$

$$\{C_{3,3a+1}(N)\}_{a,N=0} = \Big\{p\big(3k, 3, 2(k+a)+1\big),\ p\big(3k+2, 3, 2(k+1+a)\big)\Big\}_{a,k=0}$$

$$\begin{aligned}
&\{C_{3,3a+2}(N)\}_{a,N=0} \\
&\quad = \Big\{p\big(3k+2, 3, 2(k+1+a)\big),\ p\big(3k+2, 3, 2(k+1+a)+1\big)\Big\}_{a,k=0}
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We will turn these sequences into generating functions.

To accommodate the expressions in the quasipolynomial for $p(n,3,N)$, we replace k with $6k$ and a with $3a$ and write

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$$
\begin{aligned}
\{C_{3,3a}(N)\}_{a,N=0} &= \{p\big(3k,3,2(k+a)\big), p\big(3k+1,3,2(k+a)+1\big)\}_{k,a=0} \\
&= \{p\big(6(3k),3,6(2k+a)\big), p\big(6(3k)+1,3,6(2k+a)+1\big), \\
&\quad p\big(6(3k)+3,3,6(2k+a)+2\big),\ \ p\big(6(3k)+4,3,6(2k+a))+3\big), \\
&\quad p\big(6(3k+1),3,6(2k+a)+4\big),\ \ p\big(6(3k+1)+1,3,6(2k+a)+5\big), \\
&\quad p\big(6(3k+1)+3,3,6(2k+1+a)\big),\ \ p\big(6(3k+1)+4,3,6(2k+1+a)+1\big), \\
&\quad p\big(6(3k+2),3,6(2k+1+a)+2\big),\ \ p\big(6(3k+2)+1,3,6(2k+1+a)+3\big), \\
&\quad p\big(6(3k+2)+3,3,6(2k+1+a)+4\big),\ \ p\big(6(3k+2)+4,3,6(2k+1+a)+5\big)\}_{k,a=0}.
\end{aligned}
$$

We note that there are two further cases to consider; replace a by $3a+1$ and again by $3a+2$. We omit these computations.

Now we turn the sequence $\{C_{3,A}(N)\}_{A,N=0}$ into a generating function.

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$$= \sum_{k=0}^{\infty} p\big(6(3k),3,6(2k+a)\big) z^{12k} + \sum_{k=0}^{\infty} p\big(6(3k)+1,3,6(2k+a)+1\big) z^{12k+1}$$

$$+ \sum_{k=0}^{\infty} p\big(6(3k)+3,3,6(2k+a)+2\big) z^{12k+2} + \sum_{k=0}^{\infty} p\big(6(3k)+4,3,6(2k+a)+3\big) z^{12k+3}$$

$$+ \sum_{k=0}^{\infty} p\big(6(3k+1),3,6(2k+a)+4\big) z^{12k+4} + \sum_{k=0}^{\infty} p\big(6(3k+1)+1,3,6(2k+a)+5\big) z^{12k+5}$$

$$+ \sum_{k=0}^{\infty} p\big(6(3k+1)+3,3,6(2k+1+a)\big) z^{12k+6} + \sum_{k=0}^{\infty} p\big(6(3k+1)+4,3,6(2k+1+a)+1\big) z^{12k+7}$$

$$+ \sum_{k=0}^{\infty} p\big(6(3k+2),3,6(2k+1+a)+2\big) z^{12k+8} + \sum_{k=0}^{\infty} p\big(6(3k+2)+1,3,6(2k+1+a)+3\big) z^{12k+9}$$

$$+ \sum_{k=0}^{\infty} p\big(6(3k+2)+3,3,6(2k+1+a)+4\big) z^{12k+10} + \sum_{k=0}^{\infty} p\big(6(3k+2)+4,3,6(2k+1+a)+5\big) z^{12k+11}$$

Stuff

The formulas for $p(n, 3, N)$ are lengthy:

The formulas for $p(n,3,N)$ are lengthy:

$$p(6k+4,3,6j+1) = 4\binom{k+2}{2}+2\binom{k+1}{2}-4\binom{k+2-j}{2}-13\binom{k+1-j}{2}$$
$$-\binom{k-j}{2}+12\binom{k+1-2j}{2}+6\binom{k-2j}{2}-\binom{k+1-3j}{2}-5\binom{k-3j}{2}$$

Stuff

$$\sum_{N=0}^{\infty} C_{3,3(3a)}(N) z^{N-2a} = \sum_{k=0}^{\infty} \left(p(3k, 3, 2(k+a)) z^{2k} + p(3k+1, 3, 2(k+a)+1) z^{2k+1} \right)$$

Stuff

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$$= \sum_{k=0}^{\infty} \left(\binom{3k+2}{2} + 4\binom{3k+1}{2} + \binom{3k}{2} \right) z^{12k} - 12 \sum_{k=a-1}^{\infty} \binom{k+1-a}{2} z^{12k} - 6\sum_{k=a}^{\infty} \binom{k-a}{2} z^{12k} + \sum_{k=0}^{\infty} \left(\binom{3k+2}{2} + 5\binom{3k+1}{2} \right) z^{12k+1}$$

$$- 12 \sum_{k=a-1}^{\infty} \binom{k+1-a}{2} z^{12k+1} - 6\sum_{k=a}^{\infty} \binom{k-a}{2} z^{12k+1} + \sum_{k=0}^{\infty} \left(3\binom{3k+2}{2} + 3\binom{3k+1}{2} \right) z^{12k+2} - \sum_{k=a-2}^{\infty} \binom{k+2-a}{2} z^{12k+2}$$

$$- 13 \sum_{k=a-1}^{\infty} \binom{k+1-a}{2} z^{12k+2} - 4\sum_{k=a}^{\infty} \binom{k-a}{2} z^{12k+2} + \sum_{k=0}^{\infty} \left(3\binom{3k+2}{2} + 3\binom{3k+1}{2} \right) z^{12k+3} - \sum_{k=a-2}^{\infty} \binom{k+2-a}{2} z^{12k+3}$$

$$- 13 \sum_{k=a-1}^{\infty} \binom{k+1-a}{2} z^{12k+3} - 4\sum_{k=a}^{\infty} \binom{k-a}{2} z^{12k+3} + \sum_{k=0}^{\infty} \left(\binom{3k+3}{2} + 4\binom{3k+2}{2} + \binom{3k+1}{2} \right) z^{12k+4} - 2 \sum_{k=a-2}^{\infty} \binom{k+2-a}{2} z^{12k+4}$$

$$-14 \sum_{k=a-1}^{\infty} \binom{k+1-a}{2} - 2\sum_{k=a}^{\infty} \binom{k-a}{2} z^{12k+4} + \sum_{k=0}^{\infty} \left(\binom{3k+3}{2} + 5\binom{3k+2}{2} \right) z^{12k+5} - 2 \sum_{k=a-2}^{\infty} \binom{k+2-a}{2} z^{12k+5} - 14 \sum_{k=a-1}^{\infty} \binom{k+1-a}{2} z^{12k+5}$$

$$-2\sum_{k=a}^{\infty} \binom{k-a}{2} z^{12k+5} + \sum_{k=0}^{\infty} \left(3\binom{3k+3}{2} + 3\binom{3k+2}{2} \right) z^{12k+6} - 4 \sum_{k=a-2}^{\infty} \binom{k+2-a}{2} z^{12k+6} - 13 \sum_{k=a-1}^{\infty} \binom{k+1-a}{2} z^{12k+6} - \sum_{k=a}^{\infty} \binom{k-a}{2} z^{12k+6}$$

$$+ \sum_{k=0}^{\infty} \left(4\binom{3k+2}{2} + 2\binom{3k+1}{2} \right) z^{12k+7} - 4 \sum_{k=a-2}^{\infty} \binom{k+2-a}{2} z^{12k+7} - 13 \sum_{k=a-1}^{\infty} \binom{k+1-a}{2} z^{12k+7} - \sum_{k=a}^{\infty} \binom{k-a}{2} z^{12k+7}$$

$$+ \sum_{k=0}^{\infty} \left(\binom{3k+4}{2} + 4\binom{3k+3}{2} + \binom{3k+2}{2} \right) z^{12k+8} - 6 \sum_{k=a-2}^{\infty} \binom{k+2-a}{2} z^{12k+8} - 12 \sum_{k=a-1}^{\infty} \binom{k+1-a}{2} z^{12k+8} + \sum_{k=0}^{\infty} \left(\binom{3k+4}{2} + 5\binom{3k+3}{2} \right) z^{12k+9}$$

$$- 6 \sum_{k=a-2}^{\infty} \binom{k+2-a}{2} z^{12k+9} - 12 \sum_{k=a-1}^{\infty} \binom{k+1-a}{2} z^{12k+9} + \sum_{k=0}^{\infty} \left(3\binom{3k+4}{2} + 3\binom{3k+3}{2} z^{12k+10} \right) - 9 \sum_{k=a-2}^{\infty} \binom{k+2-a}{2} z^{12k+10}$$

$$-9 \sum_{k=a-1}^{\infty} \binom{k+1-a}{2} z^{12k+10} + \sum_{k=0}^{\infty} \left(4\binom{3k+4}{2} + 2\binom{3k+3}{2} \right) z^{12k+11} - 9 \sum_{k=a-2}^{\infty} \binom{k+2-a}{2} z^{12k+11} - 9 \sum_{k=a-1}^{\infty} \binom{k+1-a}{2} z^{12k+11}.$$

Turn this into rational functions

$$\sum_{N=0}^{\infty} C_{3,3(3a)}(N) z^{N-2a} =$$

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$$\sum_{N=0}^{\infty} C_{3,3(3a)}(N) z^{N-2a} = \frac{1+34z^{12}+19z^{24}-12z^{12(a+1)}-6z^{12(a+2)}}{(1-z^{12})^3}$$
$$+\frac{z\left(1+37z^{12}+16z^{24}-12z^{12(a+1)}-6z^{12(a+2)}\right)}{(1-z^{12})^3}+\frac{z^2\left(1+13z^{12}+4z^{24}\right)\left(3-z^{12a}\right)}{(1-z^{12})^3}$$
$$+\frac{z^3\left(4+40z^{12}+10z^{24}-z^{12a}-13z^{12(a+1)}-4z^{12(a+2)}\right)}{(1-z^{12})^3}$$
$$+\frac{z^4\left(7+40z^{12}+7z^{24}-2z^{12a}-14z^{12(a+1)}-2z^{12(a+2)}\right)}{(1-z^{12})^3}$$
$$+\frac{z^5\left(8+41z^{12}+5z^{24}-2z^{12a}-14z^{12(a+1)}-2z^{12(a+2)}\right)}{(1-z^{12})^3}$$
$$+\frac{z^6\left(4+13z^{12}+z^{24}\right)\left(3-z^{12a}\right)}{(1-z^{12})^3}+\frac{z^7\left(14+38z^{12}+2z^{24}-4z^{12a}-13z^{12(a+1)}-z^{12(a+2)}\right)}{(1-z^{12})^3}$$
$$+\frac{z^8\left(19+34z^{12}+z^{24}-6z^{12a}-12z^{12(a+1)}\right)}{(1-z^{12})^3}+\frac{3z^9\left(7+11z^{12}-2z^{12a}-4z^{12(a+1)}\right)}{(1-z^{12})^3}$$
$$+\frac{9z^{10}\left(1+z^{12}\right)\left(3-z^{12a}\right)}{(1-z^{12})^3}+\frac{3z^{11}\left(10+8z^{12}-3z^{12a}-3z^{12(a+1)}\right)}{(1-z^{12})^3}$$

It all simplifies to $\rightarrow$

Turn this into rational functions

$$\sum_{N=0}^{\infty} C_{3,3(3a)}(N) z^{N-2a} = \frac{1+34z^{12}+19z^{24}-12z^{12(a+1)}-6z^{12(a+2)}}{(1-z^{12})^3}$$

$$+\frac{z\left(1+37z^{12}+16z^{24}-12z^{12(a+1)}-6z^{12(a+2)}\right)}{(1-z^{12})^3}+\frac{z^2\left(1+13z^{12}+4z^{24}\right)\left(3-z^{12a}\right)}{(1-z^{12})^3}$$

$$+\frac{z^3\left(4+40z^{12}+10z^{24}-z^{12a}-13z^{12(a+1)}-4z^{12(a+2)}\right)}{(1-z^{12})^3}$$

$$+\frac{z^4\left(7+40z^{12}+7z^{24}-2z^{12a}-14z^{12(a+1)}-2z^{12(a+2)}\right)}{(1-z^{12})^3}$$

$$+\frac{z^5\left(8+41z^{12}+5z^{24}-2z^{12a}-14z^{12(a+1)}-2z^{12(a+2)}\right)}{(1-z^{12})^3}$$

$$+\frac{z^6\left(4+13z^{12}+z^{24}\right)\left(3-z^{12a}\right)}{(1-z^{12})^3}+\frac{z^7\left(14+38z^{12}+2z^{24}-4z^{12a}-13z^{12(a+1)}-z^{12(a+2)}\right)}{(1-z^{12})^3}$$

$$+\frac{z^8\left(19+34z^{12}+z^{24}-6z^{12a}-12z^{12(a+1)}\right)}{(1-z^{12})^3}+\frac{3z^9\left(7+11z^{12}-2z^{12a}-4z^{12(a+1)}\right)}{(1-z^{12})^3}$$

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It all simplifies to $\rightarrow$ $= \dfrac{1+z^2+z^3-z^{12a+2}}{(1-z)\left(1-z^2\right)\left(1-z^4\right)}.$

Stuff

In the end, we have computed the three rational functions.

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$$\sum_{N=0}^{\infty} C_{3,3(3a)}(N) z^{N-2a} = \frac{1+z^2+z^3-z^{12a+2}}{(1-z)(1-z^2)(1-z^4)} = \frac{1+z^2+z^3-z^{4(3a)+2}}{(1-z)(1-z^2)(1-z^4)} \quad (9)$$

$$\sum_{N=0}^{\infty} C_{3,3(3a+1)}(N) z^{N-2a} = \frac{1+z^2+z^3-z^{12a+6}}{(1-z)(1-z^2)(1-z^4)} = \frac{1+z^2+z^3-z^{4(3a+1)+2}}{(1-z)(1-z^2)(1-z^4)} \quad (10)$$

$$\sum_{N=0}^{\infty} C_{3,3(3a+2)}(N) z^{N-2a} = \frac{1+z^2+z^3-z^{12a+10}}{(1-z)(1-z^2)(1-z^4)} = \frac{1+z^2+z^3-z^{4(3a+2)+2}}{(1-z)(1-z^2)(1-z^4)} \quad (11)$$

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Combining the rational functions in (9), (10) and (11) and reincorporating the factor z^{2a}, we arrive at:

Stuff

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Combining the rational functions in (9), (10) and (11) and reincorporating the factor z^{2a}, we arrive at:

$$\sum_{N=0}^{\infty} C_{3,3a}(N) z^{N} = \frac{z^{2a}\left(1+z^2+z^3-z^{4a+2}\right)}{(1-z)(1-z^2)(1-z^4)}.$$

Treating the remaining cases $C_{3,3a+1}(N)$ and $C_{3,3a+2}(N)$, we have

Proposition 10

For $a \geq 0$, the coefficients of $\left[{N+3 \atop 3}\right]$ are produced by the following generating function which comes in three cases.

$$\sum_{N=0}^{\infty} C_{3,A}(N)z^N = \begin{cases} \sum_{N=0}^{\infty} C_{3,3a}(N)z^N & = \dfrac{z^{2a}\left(1+z^2+z^3-z^{4a+2}\right)}{(1-z)(1-z^2)(1-z^4)} \\ \sum_{N=0}^{\infty} C_{3,3a+1}(N)z^N & = \dfrac{z^{2a+1}\left(1+z+z^3-z^{4a+3}\right)}{(1-z)(1-z^2)(1-z^4)} \\ \sum_{N=0}^{\infty} C_{3,3a+2}(N)z^N & = \dfrac{z^{2a+2}\left(1+z+z^2-z^{4a+4}\right)}{(1-z)(1-z^2)(1-z^4)}. \end{cases}$$

Proposition 11

For all $N, A, a \geq 0$, *one has*

$$\sum_{N=0}^{\infty} C_{2,A}(N) z^N = \begin{cases} \displaystyle\sum_{N=0}^{\infty} C_{2,2a}(N) z^N & = \dfrac{z^{2a}(1+z)}{(1-z^2)^2} \\ \displaystyle\sum_{N=0}^{\infty} C_{2,2a+1}(N) z^N & = \dfrac{z^{2a+1}(1+z)}{(1-z^2)^2}. \end{cases}$$

Stuff

Proposition 11

For all $N, A, a \geq 0$, *one has*

$$\sum_{N=0}^{\infty} C_{2,A}(N)z^N = \begin{cases} \sum_{N=0}^{\infty} C_{2,2a}(N)z^N &= \frac{z^{2a}(1+z)}{(1-z^2)^2} \\ \sum_{N=0}^{\infty} C_{2,2a+1}(N)z^N &= \frac{z^{2a+1}(1+z)}{(1-z^2)^2}. \end{cases}$$

Proposition 12

For all $N, A, a \geq 0$ *and* $0 \leq b \leq 5$, *one has*

$$\sum_{N=0}^{\infty} C_{4,A}(N)z^N = \begin{cases} \sum_{N=0}^{\infty} C_{4,12a+2b}(N)z^N = \frac{z^{12a+b}\left(1+z+z^2+z^3-z^{12a+1+b}-z^{12a+2+b}\right)}{(1-z)(1-z^2)^2(1-z^3)} \\ \sum_{N=0}^{\infty} C_{4,12a+2b+1}(N)z^N = \frac{z^{12a+1+b}\left(1+2z+z^2-z^{12a+1+b}-z^{12a+2+b}\right)}{(1-z)(1-z^2)^2(1-z^3)}. \end{cases}$$

The coefficients of $\begin{bmatrix} N+4 \\ 4 \end{bmatrix}$ are unimodular.

Proof:

The coefficients of $\begin{bmatrix} N+4 \\ 4 \end{bmatrix}$ are unimodular.

Proof:
We consider the differences of generating functions. For any $a \geq 0$ and $0 \leq b \leq 5$ we have

$$\sum_{N=0}^{\infty} C_{4,12a+2b}(N)z^N - \sum_{N=0}^{\infty} C_{4,12a+2b+1}(N)z^N = \frac{z^{12a+b}(1-z^{12a+b+1})}{(z;z)_3} \quad (15)$$

and

$$\sum_{N=0}^{\infty} C_{4,12a+2b+1}(N)z^N - \sum_{N=0}^{\infty} C_{4,12a+2(b+1)}(N)z^N = \frac{z^{12a+b+2}(1-z^{12a+b})}{(z;z)_3} \quad (16)$$

The coefficients of $\begin{bmatrix} N+4 \\ 4 \end{bmatrix}$ are unimodular.

Proof:
We consider the differences of generating functions. For any $a \geq 0$ and $0 \leq b \leq 5$ we have

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and

$$\sum_{N=0}^{\infty} C_{4,12a+2b+1}(N)z^N - \sum_{N=0}^{\infty} C_{4,12a+2(b+1)}(N)z^N = \frac{z^{12a+b+2}(1 - z^{12a+b})}{(z;z)_3} \tag{16}$$

Note:

$$\frac{1}{(z;z)_3} = \sum_{n=0}^{\infty} p(n,3)z^n.$$

An inspection of the six constituents of the quasipolynomial in (18) reveals that for any n,

$$p(n,3) \geq p(n-1,3). \tag{17}$$

$$p(n,3) = \begin{cases} p(6k,3) & = 1\binom{k+2}{2} + 4\binom{k+1}{2} + 1\binom{k}{2} = 3k^2 + 3k + 1 \\ p(6k+1,3) & = 1\binom{k+2}{2} + 5\binom{k+1}{2} = 3k^2 + 4k + 1 \\ p(6k+2,3) & = 2\binom{k+2}{2} + 4\binom{k+1}{2} = 3k^2 + 5k + 2 \\ p(6k+3,3) & = 3\binom{k+2}{2} + 3\binom{k+1}{2} = 3k^2 + 6k + 3 \\ p(6k+4,3) & = 4\binom{k+2}{2} + 2\binom{k+1}{2} = 3k^2 + 7k + 4 \\ p(6k+5,3) & = 5\binom{k+2}{2} + 1\binom{k+1}{2} = 3k^2 + 8k + 5. \end{cases} \tag{18}$$

The coefficients of $\begin{bmatrix} N+4 \\ 4 \end{bmatrix}$ are unimodular.

We will show $C_{4,12a+2b+1}(N) \geq C_{4,12a+2b+2}(N)$. Working from (16) we see that

$$\sum_{N=0}^{\infty} \left(C_{4,12a+2b+1}(N) - C_{4,12a+2(b+1)}(N) \right) z^{N-12a+b+2}$$
$$= \frac{1 - z^{12a+b}}{(z;z)_3} = \sum_{n=0}^{\infty} \left(p(n,3) - p(n-12a-b,3) \right) z^n. \quad (19)$$

We must show

$$p(n,3) \geq p(n-12a-b,3) \quad (20)$$

for all n, a, b. Setting $a = b = 0$ in (20) so that $p(n-12a-b,3)$ is as large as possible, we have $p(n,3) \geq p(n,3)$ which is true. Hence, by (17), for any $a, b \geq 0$ (20), is true.

The coefficients of $\begin{bmatrix} N+4 \\ 4 \end{bmatrix}$ are unimodular.

Replicating the above argument for

$$\begin{aligned} &\sum_{N=0}^{\infty} \left(C_{4,12a+2b}(N) z^N - C_{4,12a+2b+1}(N) \right) z^{N+12a+b} \\ &\quad = \frac{1 - z^{12a+b+1}}{(z;z)_3} = \sum_{n=0}^{\infty} \Big(p\big(n,3\big) - p\big(n - 12a - b - 1, 3\big) \Big) z^n \end{aligned} \tag{21}$$

will show

$$p\big(n,3\big) \geq p\big(n - 12a - b - 1, 3\big) \tag{22}$$

is likewise true for all $a, b \geq 0$.

Replicating the above argument for

$$\sum_{N=0}^{\infty} \left(C_{4,12a+2b}(N) z^N - C_{4,12a+2b+1}(N) \right) z^{N+12a+b}$$
$$= \frac{1 - z^{12a+b+1}}{(z;z)_3} = \sum_{n=0}^{\infty} \left(p(n,3) - p(n-12a-b-1,3) \right) z^n \quad (21)$$

will show

$$p(n,3) \geq p(n-12a-b-1,3) \quad (22)$$

is likewise true for all $a, b \geq 0$. Hence, it follows that for all a, b and any fixed N, one has

$$C_{4,12a+2b}(N) \geq C_{4,12a+2b+1}(N) \geq C_{4,12a+2(b+1)}(N).$$

The coefficients of $\begin{bmatrix} N+4 \\ 4 \end{bmatrix}$ are unimodular.

Recall: For mN even,

$$C_{m,A}(N) = p\left(\left\lfloor \frac{mN}{2} \right\rfloor - A, m, N\right) = p\left(\left\lfloor \frac{mN}{2} \right\rfloor + A, m, N\right) \qquad (23)$$

The coefficients of $\begin{bmatrix} N+4 \\ 4 \end{bmatrix}$ are unimodular.

Recall: For mN even,

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Now, with the central coefficient of $\begin{bmatrix} N+4 \\ 4 \end{bmatrix}$ being $C_{4,0}(N)$, we have

$$\begin{aligned} C_{4,2N}(N) \le \cdots \le C_{4,2}(N) \le C_{4,1}(N) \le C_{4,0}(N) \\ \ge C_{4,-1}(N) \ge C_{4,-2}(N) \ge \cdots C_{4,-2N}(N). \end{aligned} \tag{24}$$

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$$C_{m,A}(N) = p\left(\left\lfloor \frac{mN}{2} \right\rfloor - A, m, N\right) = p\left(\left\lfloor \frac{mN}{2} \right\rfloor + A, m, N\right) \quad (23)$$

Now, with the central coefficient of $\begin{bmatrix} N+4 \\ 4 \end{bmatrix}$ being $C_{4,0}(N)$, we have

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Equivalently,

$$\begin{aligned} p(0,4,N) \cdots \leq p(2N-2,4,N) \leq p(2N-1,4,N) \leq p(2N,4,N) \\ \geq p(2N-1,4,N) \geq p(2N-2,4,N) \geq \cdots p(-4N,4,N). \quad (25) \end{aligned}$$

Thus, the coefficients of $\begin{bmatrix} N+4 \\ 4 \end{bmatrix}$ are unimodular.

Background

The generating function for $C_{4,A}(N)$ comes from the 144 constituents for $p(n, 4, N)$.

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$$
\begin{aligned}
&p(12k+0,4,12j+2)\\
&\quad = 1\binom{k+3}{3} + 30\binom{k+2}{3} + 39\binom{k+1}{3} + 2\binom{k}{3} - 53\binom{k+2-j}{3}\\
&\quad - 194\binom{k+1-j}{3} - 41\binom{k-j}{3} + 20\binom{k+2-2j}{3} + 250\binom{k+1-2j}{3}\\
&+160\binom{k-2j}{3}+2\binom{k-1-2j}{3}-1\binom{k+2-3j}{3}-98\binom{k+1-3j}{3}-173\binom{k-3j}{3}\\
&\quad - 16\binom{k-1-3j}{3} + 9\binom{k+1-4j}{3} + 48\binom{k-4j}{3} + 15\binom{k-1-4j}{3}.
\end{aligned}
$$