A Combinatorial Proof of a Restricted Overpartition Count Modulo Four

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Partitions and Overpartitions

A partition of n is an unordered collection of positive integers whose sum is n. Write $\lambda \in P(n)$ as $\lambda = (\lambda_1, \ldots, \lambda_t)$ where the λ_i are the parts, listed in nonincreasing order. The length of λ is the number of parts t.

$$P(3) = \{3, 21, 111\}; \qquad p(3) = 3.$$

In 2004, Sylvie Corteel and Jeremy Lovejoy unified disparate studies of a partition generalization they named overpartitions: at most one of each part size can be overlined. E.g.,

$$P(3) \subset O(3) = \{3, \bar{3}, 21, \bar{2}1, 2\bar{1}, \bar{2}\bar{1}, 111, \bar{1}11\}; \qquad o(3) = 8.$$

Some parity results

n	1	2	3	4	5	6	7	8	9	10	11	12
$p(n)$	1	2	3	5	7	11	15	22	30	42	56	77
$o(n)$	2	4	8	14	24	40	64	100	154	232	344	504

Easy to see that $o(n)$ is always even: Toggle overlining the largest part. E.g., $o(4) = 14$:

$$\{4, 31, 22, 211, 1111, 3\bar{1}, 2\bar{1}1\} \longleftrightarrow \{\bar{4}, \bar{3}1, \bar{2}2, \bar{2}11, \bar{1}111, \bar{3}\bar{1}, \bar{2}\bar{1}1\}.$$

Augustine Munagi and James Sellers 2014 considered $D_2(n)$ where only even parts can be overlined, e.g.,

$$D_2(3) = \{3, 21, \bar{2}1, 111\},\ D_2(4) = \{4, \bar{4}, 31, 22, \bar{2}2, 211, \bar{2}11, 1111\},$$

and showed that $d_2(n)$ is almost always even.

A stronger result

n	1	2	3	4	5	6	7	8	9	10	11	12
$p(n)$	1	2	3	5	7	11	15	22	30	42	56	77
$d_2(n)$	1	3	4	8	11	20	27	44	60	92	124	183
$o(n)$	2	4	8	14	24	40	64	100	154	232	344	504

Aidan Carlson, H., Sellers

$d_2(n)$ is a multiple of 4 unless $n = k(3k \pm 1)/2$ for some $k \geq 0$.

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Glaisher's bijection

Euler (1748)

For each positive n, the number of odd part partitions of n equals the number of distinct part partitions of n.

Glaisher's 1883 proof: merge duplicates / split evens.

$$6 \longleftrightarrow 33$$
$$51$$
$$42 \longleftrightarrow \{411, 222, 2211, 21^4\} \longleftrightarrow 1^6$$
$$321 \longleftrightarrow 3111$$

Franklin's near-involution

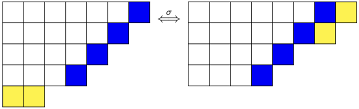
Euler (1740–1750)

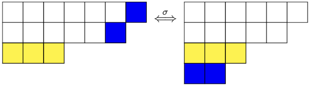
$$p(n) = p(n-1) + p(n-2) - p(n-5) - p(n-7) + \cdots$$

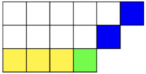
is equivalent to, for n not a generalized pentagonal number,

$\#$ distinct part partitions of n with an even number of parts
$= \#$ distinct part partitions of n with an odd number of parts.

Franklin's 1881 map moves the smallest part or a diagonal strip, changes the number of parts by one when it works.







van Leeuwen's map

Let $Q_o(n)$ be the distinct part partitions of n with only odd parts and $R(n) = P(n) \setminus Q_o(n)$. E.g., $R(8)$ is $P(8)$ except for 71 and 53:

$$\{62, 5111, 44, 4211, 3311, 3221, 31^5, 2^4, 221^4, 1^8,$$
$$8, 611, 521, 431, 422, 41^4, 332, 32111, 22211, 21^6\}$$

Write $R^e(n)$ for the partitions in $R(n)$ with even length, $R^o(n)$ for odd length. Marc van Leeuwen 2011 MSE showed $r^e(n) = r^o(n)$ by a more refined merging/ splitting procedure.

van Leeuwen's map

Given $\lambda \in R(n)$ for $n \geq 2$:

- Let m be the least positive odd integer such that the set of parts of the form $2^k m$ for $k \geq 0$ is nonempty and not a single part m.
- Choose the maximal k such that some part $\lambda_i = 2^k m$.
- If the part $2^k m$ is unique in λ, then we know $k \neq 0$ by the choice of m; split λ_i into two new parts each $2^{k-1} m$.
- If $2^k m$ appears two or more times in λ, then merge two of those parts to make a new part $2^{k+1} m$.

In either case, the resulting partition is in $R(n)$ and the length has changed by one. For example, $(6, 6, 5, 5, 1)$ has $m = 3$ and the two parts 6 are merged, therefore $(6, 6, 5, 5, 1) \xleftrightarrow{vL} (12, 5, 5, 1)$.

Every partition of $R(8)$ has $m = 1$, i.e., parts in $\{1, 2, 4, 8\}$ not just a single 1.

Van Leeuwen's pairing of the 20 partitions in $R(8)$:

$$\begin{array}{cccccccccc} 8 & 611 & 521 & 431 & 422 & 41^4 & 332 & 32111 & 22211 & 21^6 \\ \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow \\ 44 & 62 & 5111 & 3221 & 2^4 & 221^4 & 3311 & 31^5 & 4211 & 1^8 \end{array}$$

In the Q&A after this presentation, Dennis Eichhorn noted that this map dates back to Sylvester. Specifically, see section 19 of his 1882 "Constructive theory of partitions" where he talks about *batches* such as $\{1, 2, 4, 8, \ldots\}$ and $\{3, 6, 12, 24, \ldots\}$.

Combinatorial proof overview

A combinatorial proof showing that a quantity is even often uses an involution (which might fail for certain values). It's uncommon to show that a count is a multiple of four combinatorially.

Overview

Partition $D_2(n)$ into $D_2^o(n) \cup D_2^e(n)$ by whether a partition has an odd or even number of parts. For $n \neq k(3k \pm 1)/2$, we show

1. $d_2^o(n)$ and $d_2^e(n)$ are both even,
2. $d_2^o(n) = d_2^e(n)$.

Therefore, the possibilities for $d_2(n) = d_2^o(n) + d_2^e(n)$ are only $0 + 0$ mod 4 and $2 + 2$ mod 4.

1. Both subsets have even size

For $n \neq k(3k \pm 1)/2$, we can pair up the partitions in $D_2^o(n)$. Similarly for $D_2^e(n)$.

If a partition includes an even part, toggle overlining the largest even part (as in the argument that $o(n)$ is even).

If the partition has only odd parts, apply Glaisher's map, then Franklin's map, and then Glaisher's map again. E.g., in $D_2^e(10)$,

$$91 \xleftrightarrow{G} 91 \xleftrightarrow{F} 10 \xleftrightarrow{G} 55$$

$$73 \xleftrightarrow{G} 73 \xleftrightarrow{F} 631 \xleftrightarrow{G} 3331$$

$$7111 \xleftrightarrow{G} 721 \xleftrightarrow{F} 82 \xleftrightarrow{G} 1^{10}$$

$$5311 \xleftrightarrow{G} 532 \xleftrightarrow{F} 4321 \xleftrightarrow{G} 31^7$$

$$51^5 \xleftrightarrow{G} 541 \xleftrightarrow{F} 64 \xleftrightarrow{G} 331^4$$

For $n \neq k(3k \pm 1)/2$, we can match up $D_2^o(n)$ and $D_2^e(n)$. Write a partition in $D_2(n)$ as (μ, ν) where μ consists of any unoverlined parts and ν consists of any overlined parts which must be distinct and all even (so overpartitions are a special kind of bipartition).

If μ is nonempty and not a distinct odd part partition, connect (μ, ν) and $(\mathrm{vL}(\mu), \nu)$ (which changes the number of parts by one). For $D_2(8)$, the $\nu = \varnothing$ case is the 10 pairs from the van Leeuwen slide. The other cases with these kind of μ are

$$\begin{array}{c|ccc|ccccc} \bar{6}11 & \bar{4}22 & \bar{4}1^4 & \bar{4}\bar{2}2 & 4\bar{2}2 & 33\bar{2} & 3\bar{2}111 & \bar{2}2211 & \bar{2}1^6 \\ \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow \\ \bar{6}2 & \bar{4}4 & \bar{4}211 & \bar{4}\bar{2}11 & \bar{2}222 & 6\bar{2} & 3\bar{2}21 & 4\bar{2}11 & \bar{2}21^4 \end{array}$$

2. The subsets have equal size

The remaining $D_2(n)$ elements correspond to (μ, ν) with μ empty or consisting of only odd parts that are distinct. Remember that ν, the overlined parts, are only even parts and distinct.

Insight: Ignore the overlines! Such (μ, ν) are just the partitions of n into distinct parts (separated into odd and even parts).

Apply Franklin's map to the (combined) distinct part partition and then overline any even parts. The examples in $D_2(8)$ are

$$\begin{array}{ccc} \bar{8} & 5\bar{2}1 & \bar{4}31 \\ \updownarrow & \updownarrow & \updownarrow \\ 71 & \bar{6}\bar{2} & 53 \end{array}$$

Link and addendum

Manuscript submitted and available on the arXiv, includes all details including what happens when $n = k(3k \pm 1)/2$.

Also has a combinatorial proof of the 2014 mod 2 result using

Half-conjugation

The half-conjugate of an overpartition fixes any overlined parts and conjugates the subpartition consisting of any nonoverlined parts.

E.g., the half-conjugate of $(3, \bar{2}, 1)$ is $(\bar{2}, 2, 1, 1)$. (NB: Different from Corteel and Lovejoy's overpartition conjugation, which is not closed on $D_2(n)$.)

We show that $H(n)$, the set of self-half-conjugate overpartitions of $D_2(n)$, is in bijection with $Q(n)$, using a map due to Sylvester.