Extending Andrews and Moshe Newman's Crank–Mex Refinement

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Background: Crank

Dyson defined the rank statistic that witnesses the modulo 5 and modul 7 Ramanujan congruences, but not the mod 11 congruences. He predicted (and named) the crank statistic for that case. George Andrews and Frank Garvan found it in 1988:

Let $\omega(\lambda)$ be the number of ones in λ and $\mu(\lambda)$ the number of parts of λ greater than $\omega(\lambda)$.

$$\text{crank}(\lambda) = \begin{cases} \lambda_1 & \text{if } \omega(\lambda) = 0, \\ \mu(\lambda) - \omega(\lambda) & \text{if } \omega(\lambda) > 0. \end{cases}$$

Note: The crank witnesses all three congruences.

Background: Crank

The following crank generating function follows from Garvan and Andrews–Garvan 1988:

$$(1-q) + \sum_{n\geq 1} \frac{q^n y^n}{(q^2;q)_{n-1}} + \sum_{n\geq 1} \frac{q^n y^{-n}}{(q^2;q)_{n-1}} \sum_{m\geq 0} \frac{q^{m(n+1)} y^m}{(q;q)_m}$$

where y keeps track of the crank.

While not apparent from the definition, there is crank symmetry in $P(n)$ for each $n \geq 2$. Berkovich–Garvan 2002 developed “pseudo-conjugation” that pairs partitions with crank k and crank $-k$.

Background: Mex

The minimal excludant (mex) of a partition is the smallest positive integer that is not a part. E.g.,

$$\mathrm{mex}(5) = \mathrm{mex}(32) = 1, \ \mathrm{mex}(311) = 2, \ \mathrm{mex}(221) = 3.$$

Sprague and Grundy 1930s analysis of combinatorial games.
Portmanteau coined by Berlekamp 1972.
In partitions, Grabner–Knopfmacher 2006 "least gap."
Andrews 2011 "smallest part that is *not* a summand,"
Andrews–D. Newman 2019 "mex."

Background: Mex

A partition with mex k

- must include at least one of each part from 1 to $k-1$,
- must exclude k as a part.

so the generating function for the number of mex k partitions is

$$\sum_{k\geq 1} q^{1+\cdots+(k-1)} \prod_{\substack{j\geq 1\\ j\neq k}} \frac{1}{1-q^j}$$

Background: Crank–Mex Theorem

Write $X^o(n)$ for the partitions of n with odd mex, $X^e(n)$ for even mex, $M_{\geq 0}(n)$ for the partitions of n with nonnegative crank, and lower case letters for set counts.

Andrews–D. Newman, H.–Sellers 2020

$$x^o(n) = m_{\geq 0}(n)$$

and, by complementarity and crank symmetry,

$$x^e(n) = m_{<0}(n) = m_{>0}(n).$$

Over 40 papers so far building on these ideas.

Background: Fixed Points

Blecher & Knopfmacher 2022 introduced the idea of fixed points to partitions, parts that satisfy $\lambda_i = i$.

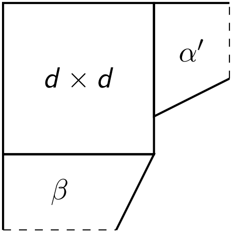
$$P(5) = \{5, 41, 3\underline{2}, 311, 2\underline{2}1, 2111, \underline{1}1111\}.$$

A fixed point is an "increasing" characteristic (Does $\lambda_1 = 1$? Does $\lambda_2 = 2$? etc.) so partitions have at most one fixed point.

Blecher–Knopfmacher

Write $F(n)$ for the partitions of n with a fixed point, $G(n)$ for those without. Conjecture: $g(n) > f(n)$ for $n \geq 3$.

Background: Fixed Points



If $\lambda_i = i$ then i is the dimension of the Durfee square leading to the generating function for partitions with a fixed point:

$$\sum_{d \geq 1} \frac{q^{d^2}}{(q;q)_{d-1}(q;q)_d}$$

Background: Fixed points

H.–Sellers 2024

By connecting fixed points to mex and crank: For $n \geq 3$,

$$g(n) - f(n) = m_{\geq 0}(n) - m_{>0}(n) = m_0(n),$$

i.e., $g(n)$ exceeds $f(n)$ by the number of crank zero partitions of n.

Extensions include generalizations to j-mex and generalized fixed points $\lambda_i = i + j$ which connect to the crank ranges $m_{>j}(n)$ using Durfee rectangles, analysis of number triangle from refining $f(n)$ by specific fixed point, etc.

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Combinatorial argument in Isaac Konan 2023 without explicitly mentioning fixed points.

Refining the Crank–Mex Theorem

For each set of partitions, add a parameter k for the number of parts greater than one.

Andrews–M. Newman 2025, arXiv:2508.17491

$$x^o(n,k) = m_{\geq 0}(n,k)$$

k	1	2	3	4
$X^o(8)$	$8, 21^6$	$62, 53, 44, 521, 4211, 221^4$	$422, 332, 2^3 11$	2^4
$M_{\geq 0}(8)$	$8, 71$	$62, 53, 521, 44, 431, 3311$	$422, 332, 3221$	2^4

Generating function proof, requested combinatorial one.

New: Extending that Crank–Mex Theorem Refinement

Andrews–H. 2025

For $k \geq 1$,

$$x^e(n,k) = f(n,k+1) = m_{<0}(n,k) = m_{>0}(n,k+1)$$

and $x^e(n,0) = f(n,0) = m_{<0}(n,0) = m_{>0}(n,1)$.

k	0	1	2	3
$X^e(7)$	1^7	$61, 511, 41^3, 31^4$	$331, 3211$	
$F(7)$	1^7		$52, 421, 3211, 221^3$	$322, 2^31$
$M_{<0}(7)$	1^7	$511, 41^3, 31^4, 21^5$	$3211, 221^3$	
$M_{>0}(7)$		7	$52, 43, 421, 331$	$322, 2221$

Generating Function Proof, Even Mex

The generating function for even mex partitions,

$$\sum_{n\geq 1} q^{1+\cdots+(2n-1)} \prod_{\substack{j\geq 1 \\ j\neq 2n}} \frac{1}{1-q^j},$$

with z tracking parts greater than one becomes

$$\sum_{n\geq 1} z^{2n-2} q^{1+\cdots+(2n-1)} \frac{1}{1-q} \prod_{\substack{j\geq 2 \\ j\neq 2n}} \frac{1}{1-zq^j}$$

which, with algebraic manipulation, is equal to

$$E(z,q) := \frac{1}{(1-q)(zq^2;q)_\infty} \sum_{n\geq 1} (-1)^{n-1} z^{n-1} q^{\binom{n+1}{2}}.$$

Generating Function Proof, Fixed Points

$$\sum_{n\geq 1} \frac{q^{n^2}}{(q;q)_{n-1}(q;q)_n}$$

with z tracking parts greater than one becomes, for $n \geq 2$,

$$\sum_{n\geq 2} \frac{z^n q^{n^2}}{(q;q)_{n-1}(1-q)(zq^2;q)_{n-1}}$$

which, using ${}_2\phi_1$ and the second Heine transform, becomes

$$zE(z,q) = \frac{z}{(1-q)(zq^2;q)_\infty} \sum_{n\geq 1} (-1)^{n-1} z^{n-1} q^{\binom{n+1}{2}}.$$

Generating Function Proof, Negative Crank

$$(1-q) + \sum_{n\geq 1} \frac{q^n y^n}{(q^2;q)_{n-1}} + \sum_{n\geq 1} \frac{q^n y^{-n}}{(q^2;q)_{n-1}} \sum_{m\geq 0} \frac{q^{m(n+1)} y^m}{(q;q)_m}$$

restricted to negative y exponents is

$$\sum_{n\geq 0} \frac{q^n y^{-n}}{(q^2;q)_{n-1}} \sum_{m=0}^{n-1} \frac{q^{m(n+1)} y^m}{(q;q)_m}.$$

Setting $y = 1$ and letting z track parts greater than one gives

$$\sum_{n=0}^{\infty} \frac{q^n}{(zq^2;q)_{n-1}} \sum_{m=0}^{n-1} \frac{z^m q^{m(n+1)}}{(q;q)_m}$$

which, using the second Heine transform, becomes $E(z,q)$.

Generating Function Proof, Positive Crank

The crank generating function restricted to positive y exponents is

$$\sum_{n\geq 1} \frac{q^n y^n}{(q^2;q)_{n-1}} + \sum_{n\geq 0} \frac{q^n y^{-n}}{(q^2;q)_{n-1}} \sum_{m\geq n+1} \frac{q^{m(n+1)} y^m}{(q;q)_m}.$$

Setting $y = 1$ and letting z track parts greater than one gives

$$zq + \sum_{n\geq 2} \frac{zq^n}{(zq^2;q)_{n-1}} + \sum_{n=1}^{\infty} \frac{q^n}{(zq^2;q)_{n-1}} \sum_{m=n+1}^{\infty} \frac{z^m q^{m(n+1)}}{(q;q)_m}$$

which, with an Andrews–M. Newman lemma and lot of algebraic manipulation, becomes $zE(z,q)$.

Combinatorial Proof, Even Mex and Fixed Points

Write $\beta(\lambda)$ for the number of parts of λ that are greater than one.

Konan 2023 outlines a bijection $X^e(n) \cong F(n)$.

- Adapted from his detailed bijection $X^o(n) \cong G(n)$.
- His maps are iterative (description & verification takes 8 *ElectronicJC* pages).
- Requires another bijection between $F(n)$ and partitions of n with no generalized fixed point $\lambda_i = i + 1$.
- We can show that the overall procedure increases $\beta(\lambda)$ by one except for $\lambda = (1^n)$ (the unique partitions with $\beta(\lambda) = 0$; it is fixed by the bijection).

See his 28 April 2022 video and slides from this seminar series.

Combinatorial Proof, Fixed Points and Negative Crank

Lemma 1

If $\lambda \in P(n)$ has a fixed point i and negative crank, then $\omega(\lambda) \geq i$.

If instead $\omega(\lambda) \leq i - 1$, since $\lambda_i = i$, we would have $\mu(\lambda) \geq i$ and

$$\text{crank}(\lambda) = \mu(\lambda) - \omega(\lambda) \geq i - (i - 1) = 1.$$

Lemma 2

If $\lambda \in P(n)$ has Durfee square size d, negative crank, and does not have a fixed point, then $\omega(\lambda) \geq d + 1$.

If instead $\omega(\lambda) \leq d$, since λ does not have a fixed point, we know $\lambda_d > d$ so that $\mu(\lambda) \geq d$ and

$$\text{crank}(\lambda) = \mu(\lambda) - \omega(\lambda) \geq d - d = 0.$$

Combinatorial Proof, Fixed Points and Negative Crank

Given $\lambda \in F(n, k+1)$ with $\lambda_i = i \geq 2$,

$$\boxed{\text{replace } \lambda_i \text{ with } i \text{ (more) parts 1, other parts unchanged}}$$

to make $\kappa \in P(n)$ with $\beta(\kappa) = \beta(\lambda) - 1$. Also, $\omega(\kappa) \geq i$ and $\mu(\kappa) < i - 1$ so $\text{crank}(\kappa) \leq i - 1 - i = -1$, i.e., $\kappa \in M_{<0}(n, k)$.

Note that if $\lambda_{i+1} = i$, then $\kappa_i = i$ and κ also has a fixed point. If instead $\lambda_{i+1} < i$, then κ does not have a fixed point.

$F(8)$	62	521	4211	321^3	221^4	422	3221	$2^3 11$	2^4
$M_{<0}(8)$	611	51^3	41^4	31^5	21^6	4211	321^3	221^4	$2^3 11$

Combinatorial Proof, Fixed Points and Negative Crank

Given $\kappa \in M_{<0}(n, k)$

- If $\kappa_i = i \geq 2$ then $\omega(\kappa) \geq i$ by Lemma 1.

$$\boxed{\text{Replace } i \text{ parts 1 with } i\text{, other parts unchanged}}$$

to make $\lambda \in P(n)$ with $\beta(\lambda) = \beta(\kappa) + 1$. Since $\lambda_i = \kappa_i = i$, we have $\lambda \in F(n, k+1)$. (Note that $\lambda_{i+1} = i$.)

Combinatorial Proof, Fixed Points and Negative Crank

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- If $\kappa_i = i \geq 2$ then $\omega(\kappa) \geq i$ by Lemma 1.

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  to make $\lambda \in P(n)$ with $\beta(\lambda) = \beta(\kappa) + 1$. Since $\lambda_i = \kappa_i = i$, we have $\lambda \in F(n, k+1)$. (Note that $\lambda_{i+1} = i$.)

- If κ has no fixed point and Durfee square size d, we know from Lemma 2 that $\omega(\kappa) \geq d + 1$.

  $$\boxed{\text{Replace } d+1 \text{ parts 1 with } d+1\text{, other parts unchanged}}$$

  to make $\lambda \in P(n)$ with $\beta(\lambda) = \beta(\kappa) + 1$. Now $\kappa \notin F(n)$ makes $\lambda_d = \kappa_d \geq d + 1$ and $\lambda_{d+1} = d + 1$ (the new part), so $\lambda \in F(n, k+1)$. (Note that $\lambda_{d+2} = \kappa_{d+1} \leq d$ since d is the size of the Durfee square of κ.)

Combinatorial Proof, Negative Crank and Positive Crank

Need a new map since, e.g., the pseudo-conjugate of 4211 is 431.

Given $\lambda = (\lambda_1, \ldots, \lambda_k, 1^w) \in M_{<0}(n,k)$, let $m = \mu(\lambda) \geq 0$ so that $\lambda_m > w \geq \lambda_{m+1}$.

$$\boxed{\lambda \mapsto \rho = (\lambda_1 - 1, \ldots, \lambda_m - 1, w, \lambda_{m+1}, \ldots, \lambda_k, 1^m)}$$

in $P(n)$ with $\beta(\rho) = \beta(\lambda) + 1$ since $\lambda_m > w \geq 2$.
Now $w > m$ since $\text{crank}(\lambda) < 0$
and $\mu(\rho) \geq m + 1$ since $\rho_{m+1} = w > m$,
so $\rho \in M_{>0}(n, k+1)$ because

$$\text{crank}(\rho) = \mu(\rho) - \omega(\rho) \geq m + 1 - m = 1.$$

Combinatorial Proof, Negative Crank and Positive Crank

Given $\rho = (\rho_1, \ldots, \rho_{k+1}, 1^v) \in M_{>0}(n, k+1)$ let $\ell = \rho_{v+1}$.
Now $\ell \geq v + 1$ since $\text{crank}(\rho) > 0$
(else $\ell \leq v$ and $\mu(\rho) - \omega(\rho) \leq v - v = 0$).

$$\boxed{\rho \mapsto \lambda = (\rho_1 + 1, \ldots, \rho_v + 1, \rho_{v+2}, \ldots, \rho_{k+1}, 1^\ell)}$$

in $P(n)$ with $\beta(\lambda) = \beta(\rho) - 1$ since $\ell = \rho_{v+1}$ became 1^ℓ.
Since $\ell = \rho_{v+1}$, we have $\mu(\lambda) = v$ and $\lambda \in M_{<0}(n, k)$ because

$$\text{crank}(\lambda) = \mu(\lambda) - \omega(\lambda) \leq v - (v + 1) = -1.$$

1^8	611	51^3	41^4	31^5	21^6	4211	321^3	221^4	2^311
8	521	431	44	53	62	3221	332	422	2^4

Crank Symmetry Too?

Although the new bijection does imply $m_{<0}(n) = m_{>0}(n)$, it does not satisfy crank symmetry on all pairs:

$$51^3 \longleftrightarrow 431 \qquad \text{crank}(51^3) = -2, \text{crank}(431) = 1.$$

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The crank -1 paritions of 7 are 511 ($\beta = 1$) and 3211 ($\beta = 2$).
The crank 1 partitions of 7 are 421 ($\beta = 2$) and 331 ($\beta = 2$).
No map can pair these while increasing β by 1.

Andrews–M. Newman Combinatorial Proof?

Konan's maps

$$X^o(n) \cong G(n) \overset{*}{\cong} M_{\leq 0}(n) \text{ (pseudo-conjugation) } M_{\geq 0}(n)$$

don't maintain the β relation (individually* or beginning to end).

- $X^o(5) = \{5, 32, 221, 2111\}$ and $G(5) = \{5, 41, 311, 21^3\}$ show the β relation is impossible.
- Crank 0 partitions of 12 have β values $1; 2, 2, 2; 3, 3; 4 \ldots$

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So combinatorial proof of $x^o(n) = m_{\geq 0}(n)$ is still open!