Overview


- Crank
- Mex
- Crank & Mex
- Crank, Mex, and Frobenius Symbols
Write $p(n)$ for the number of partitions of $n$.

Ramanujan 1919 proved (analytically) that

- $p(5n + 4) \equiv 0 \text{ mod } 5$,
- $p(7n + 5) \equiv 0 \text{ mod } 7$, and
- $p(11n + 6) \equiv 0 \text{ mod } 11$.

1944, “Some Guesses in the Theory of Partitions,” *Eureka*

A young Freeman Dyson defined the rank of $\lambda = (\lambda_1, \ldots, \lambda_\ell)$ as $\lambda_1 - \ell$ and conjectured that this simple partition statistic combinatorially verifies the modulo 5 and 7 results by grouping the appropriate partitions into 5 or 7 equally numerous classes. Proven correct by Atkin–Swinnerton-Dyer, 1954.
Partitions and crank

But the rank does not show the modulo 11 identity. Dyson suggested that some “more recondite” partition statistic should. He gave it a name and a purpose, but no definition!

Definition (Andrews–Garvan 1988)

Given a partition $\lambda$, let $\omega(\lambda)$ be the number of ones in $\lambda$ and let $\mu(\lambda)$ be the number of parts of $\lambda$ greater than $\omega(\lambda)$. Then

$$\text{crank}(\lambda) = \begin{cases} 
\lambda_1 & \text{if } \omega(\lambda) = 0, \\
\omega(\lambda) - \mu(\lambda) & \text{if } \omega(\lambda) > 0.
\end{cases}$$

They showed that this definition of the “elusive crank” does all that Dyson hoped for and gives a combinatorial verification of the modulo 5 and 7 identities, too (with different groupings).
Some crank results

For integers $m$ and $n > 1$, let $M(n, m)$ be the number of partitions of $n$ with crank $m$. We use standard $q$-series notation.

**Theorem (Garvan 1988)**

$$\sum_{n \geq 0} M(m, n) q^n = \frac{1}{(q; q)_{\infty}} \sum_{n \geq 1} (-1)^{n-1} q^{n(n-1)/2+n|m|} (1 - q^n),$$

and $M(m, n) = M(-m, n)$. 

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$$

$$
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$$

Compare the “not completely different” rank generating function

$$
\sum_{n \geq 0} N(m, n)q^n = \frac{1}{(q; q)_{\infty}} \sum_{n \geq 1} (-1)^{n-1} q^{n(3n-1)/2+n|m|}(1 - q^n).
$$
Given $j \geq 0$, we’re interested in the number of partitions $\lambda$ of $n$ with $\text{crank}(\lambda) \geq j$. By Garvan’s formula, we have

$$\sum_{m \geq j} \sum_{n \geq 0} M(m, n)q^n = \frac{1}{(q; q)_{\infty}} \sum_{n \geq 0} (-1)^n q^{n(n+1)/2+j(n+1)}.$$
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**Theorem (H., Sellers, Yee)**

$$\sum_{m \geq j} \sum_{n \geq 0} M(m, n) q^n = \sum_{n \geq 0} \frac{q^{(n+1)(n+j)}}{(q; q)_n(q; q)_{n+j}}.$$

Note that there is no alternating sum.
Proof ingredient: \(j\)-Durfee rectangles

**Definition**

The \(j\)-Durfee rectangle of a partition \(\lambda\) is the largest rectangle of size \(d \times (d + j)\) that fits inside the Ferrers diagram of \(\lambda\).

\((5, 4, 4, 2, 2)\) has 0-Durfee rectangle (Durfee square) size \(3 \times 3\),
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(5, 4, 4, 2, 2) has 0-Durfee rectangle (Durfee square) size $3 \times 3$, 1-Durfee rectangle size $3 \times 4$, 2-Durfee rectangle size $2 \times 4$, 3-Durfee rectangle size $1 \times 4$, and 4-Durfee rectangle size $1 \times 5$. 

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Partition Crank and Mex
Use \( \text{crank}(\lambda) \leq -j \) rather than \( \text{crank}(\lambda) \geq j \).

Equal count since \( M(m, n) = M(-m, n) \), but nonpositive cranks only come from the second part of the definition:

\[
\text{crank}(\lambda) = \begin{cases} 
\lambda_1 & \text{if } \omega(\lambda) = 0, \\
\omega(\lambda) - \mu(\lambda) & \text{if } \omega(\lambda) > 0.
\end{cases}
\]

← only positive crank

← any crank
Theorem (H., Sellers, Yee)

\[
\sum_{m \geq j} \sum_{n \geq 0} M(m, n)q^n = \sum_{m \leq -j} \sum_{n \geq 0} M(m, n)q^n = \sum_{n \geq 0} \frac{q^{(n+1)(n+j)}}{(q; q)_n(q; q)_{n+j}}
\]

Nonpositive crank implies \(\omega(\lambda) > 0\). Consider the \(j\)-Durfee rectangle, size \(d \times (d + j)\). Now, if \(\omega(\lambda) < d + j\), then \(\mu(\lambda) \geq d\) since \(\lambda_d \geq d + j\) and

\[
\text{crank}(\lambda) = \mu(\lambda) - \omega(\lambda) > d - (d + j) = -j.
\]

So \(\text{crank}(\lambda) \leq -j\) implies \(\omega(\lambda) \geq d + j\). The generating function for such \(\lambda\): the \(j\)-Durfee rectangle and the lower bound of \(\omega(\lambda)\) give the exponent \(d(d + j) + (d + j)\), boxes to the right of the \(j\)-Durfee rectangle account for \((q; q)_d\), boxes below give \((q; q)_{d+j}\).
Another combinatorial proof

We also show

\[(q; q)_{\infty} \sum_{n \geq 0} \frac{q^{n+j}}{(q; q)_n(q; q)_{n+j}} = \frac{1}{(q; q)_{\infty}} \sum_{n \geq 0} (-1)^n q^{n(n+1)/2 + j(n+1)},\]

\[(q; q)_{\infty} \sum_{n \geq 0} \frac{q^{n+j}}{(q; q)_n(q; q)_{n+j}} = \sum_{n \geq 0} \frac{q^{(n+1)(n+j)}}{(q; q)_n(q; q)_{n+j}}\]

by considering two sign-reversing involutions on the triples of partitions \((\pi, \kappa, \nu)\) where \(\pi\) is a partition into distinct parts, \(\kappa\) is a nonempty partition with largest part at least \(j\), and \(\nu\) is a partition into parts that are at least \(j\) less than the largest part of \(\kappa\).
The mex of a partition is the smallest missing (positive) part, e.g.,

\[
\text{mex}((2, 2, 2)) = 1, \quad \text{mex}((3, 1, 1, 1)) = 2, \quad \text{mex}((3, 2, 1)) = 4.
\]

Terminology from combinatorial game theory (at least by 1973, Gundy values), combination of minimal excluded number.

References in partitions:
- Grabner–Knopfmacher 2006 “least gap”
- Andrews 2011 “smallest number that is not a summand”
- Andrews–Newman 2019 “minimal excludant” / mex
Let $x(m, n)$ be the number of partitions of $n$ with mex $m$.
Write $t_k = 1 + \cdots + k$ for the $k$th triangular number.

**Proposition (H., Sellers, Stanton 2022)**

$$x(m, n) = p(n - t_{m-1}) - p(n - t_m)$$

To have mex $m$, a partition must include parts $1, \ldots, m - 1$. Removing one of each of those leaves a partition of $n - t_{m-1}$.
And the partition must exclude $m$. The number of partitions of $n$ with $m$ also included is $p(n - t_m)$, so the number with $m$ excluded is $p(n - t_{m-1}) - p(n - t_m)$. 
Definition

Let $m_{a,b}(n)$ to be the number of partitions of $n$ with mex congruent to $a$ modulo $b$.

Also, write superscript $e$ for the number of partitions with an even number of parts, similarly for superscript $o$.

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Splitting the mexes

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Proposition (H., Sellers, Yee)

\[
m_{1,2}^o(n) = \begin{cases} 
m_{1,2}^e(n) + (-1)^{m+1} & \text{when } n = m(3m \pm 1), \\
m_{1,2}^o(n) & \text{otherwise}.
\end{cases}
\]

Combinatorial proof comes down to considering triples \((\pi, \mu, \nu)\) where \(\pi\) is a partition into distinct even parts, \(\mu\) is a partition into odd parts, and \(\nu\) is a partition into distinct odd parts. A sign-reversing involutions leaves just \((\pi, \emptyset, \emptyset)\), then apply Franklin’s bijection to \((\pi_1/2, \pi_2/2, \ldots)\).
Splitting the mexes

Theorem (Andrews, Newman 2019)

$m_{1,2}(n)$ is almost always even and is odd exactly when $n = m(3m \pm 1)$ for some $m$.

New proof:

$$m_{1,2}(n) = m_{1,2}^o(n) + m_{1,2}^e(n)$$

$$= \begin{cases} 
2m_{1,2}^e(n) + (-1)^{m+1} & \text{when } n = m(3m \pm 1), \\
2m_{1,2}^e(n) & \text{otherwise.}
\end{cases}$$
Theorem (Andrews, Newman 2020; H., Sellers 2020)

The number of partitions of $n$ with nonnegative crank equals the number of partitions of $n$ with odd mex. I.e.,

$$M_{\geq 0}(n) = m_{1,2}(n).$$
Connecting crank and mex

Theorem (Andrews, Newman 2020; H., Sellers 2020)

The number of partitions of \( n \) with nonnegative crank equals the number of partitions of \( n \) with odd mex. I.e.,

\[
M_{\geq 0}(n) = m_{1,2}(n).
\]

Generalize the mex: For \( j \) a part in \( \lambda \), let \( \text{mex}_j(\lambda) \) to be the least integer greater than \( j \) that is not a part of \( \lambda \). Not defined if \( j \) not a part of \( \lambda \). (Thinking of 0 as a part of every partition, \( \text{mex}_0 = \text{mex} \).)

Theorem (H., Sellers, Stanton 2022)

The number of partitions \( \lambda \) of \( n \) with \( \text{crank}(\lambda) \geq j \) equals the number of partitions of \( n \) with odd \( \text{mex}_j \).
We have $m_{1,2}(n) = m_{1,4}(n) + m_{3,4}(n)$. What do those correspond to among the partitions with nonnegative crank?
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\textbf{Theorem (Huh, Kim 2021)}

\[ M_{\geq 0}^e(n) = m_{1,4}(n), \quad M_{\geq 0}^o(n) = m_{3,4}(n). \]

Our proof: The generating function by number of parts is

\[ \sum_{k,n \geq 0} M_{\leq 0}(k,n)z^k q^n = \sum_{n \geq 0} \frac{z^{2n} q^{n(n+1)}}{(zq; q)_n(q; q)_n}. \]

Substituting \( z = -1 \) gives

\[ \sum_{n \geq 0} \left( M_{\leq 0}^e(n) - M_{\leq 0}^o(n) \right) q^n = \sum_{n \geq 0} \frac{q^{n(n+1)}}{(-q; q)_n(q; q)_n} = \sum_{n \geq 0} \frac{q^{n(n+1)}}{(q^2; q^2)_n}. \]

This connects to formulas derived from our first theorem.
Another crank result

Proposition (H., Sellers, Yee)

\[ M_{\leq 0}^e(n) = \begin{cases} M_{\leq 0}^o(n) & \text{if } n \text{ is odd}, \\ M_{\leq 0}^o(n) + q(n/2) & \text{if } n \text{ is even}. \end{cases} \]

The proof adjusts the partitions to the right of and below the Durfee square to make a sign-reversing involution. The fixed points are in bijection to partitions of \( n \) into distinct even parts.
The number of partitions of $n$ with no 0 in the top row of their Frobenius symbols equals the number of partitions of $n$ with odd mex.
Theorem (H., Sellers, Stanton 2022)

The number of partitions of $n - j$ with no $j$ in the top row of their Frobenius symbols equals the number of partitions $\lambda$ of $n$ with crank($\lambda$) $\geq j$.

New proof: By our first theorem, partitions with this bounded crank are in bijection with partitions of $n$ with at least $d + j$ parts 1 where the $j$-Durfee rectangle is $d \times (d + j)$. Delete $d + j$ parts 1 and increase each of the $d$ largest parts by 1. This makes a partition $\lambda'$ of $n - j$ with

$$\lambda'_d \geq d + j + 1 \quad \text{and} \quad \lambda'_{d+1} \leq d + j = (d + 1) + (j - 1).$$

Thus the top entries in columns $d$ and $d + 1$ of the Frobenius symbol are at least $j + 1$ and at most $j - 1$, respectively: no $j$. 

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Theorem (H., Sellers, Yee)

The number of partitions of $n$ with crank 0 equals the number of partitions of $n$ whose Frobenius symbol has no 0 and the first two entries of the bottom row differ by 1.
