Combinatorial Perspectives on Dyson's Crank and the Mex of Partitions

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Specialty Seminar in Partitions and q-Series
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Overview

Features recent work of George Andrews and David Newman, JiSun Huh and Byungchan Kim. Collaborators James Sellers, Dennis Stanton, and Ae Ja Yee.

- Crank
- Mex
- Crank & Mex
- Crank, Mex, and Frobenius Symbols

Partitions and rank

Write $p(n)$ for the number of partitions of n.

Ramanujan 1919 proved (analytically) that

- $p(5n+4) \equiv 0 \bmod 5$,
- $p(7n+5) \equiv 0 \bmod 7$, and
- $p(11n+6) \equiv 0 \bmod 11$.

1944, "Some Guesses in the Theory of Partitions," *Eureka*
A young Freeman Dyson defined the rank of $\lambda = (\lambda_1, \ldots, \lambda_\ell)$ as $\lambda_1 - \ell$ and conjectured that this simple partition statistic combinatorially verifies the modulo 5 and 7 results by grouping the appropriate partitions into 5 or 7 equally numerous classes. Proven correct by Atkin–Swinnerton-Dyer, 1954.

Partitions and crank

But the rank does not show the modulo 11 identity. Dyson suggested that some "more recondite" partition statistic should. He gave it a name and a purpose, but no definition!

Definition (Andrews–Garvan 1988)

Given a partition λ, let $\omega(\lambda)$ be the number of ones in λ and let $\mu(\lambda)$ be the number of parts of λ greater than $\omega(\lambda)$. Then

$$\text{crank}(\lambda) = \begin{cases} \lambda_1 & \text{if } \omega(\lambda) = 0, \\ \omega(\lambda) - \mu(\lambda) & \text{if } \omega(\lambda) > 0. \end{cases}$$

They showed that this definition of the "elusive crank" does all that Dyson hoped for and gives a combinatorial verification of the modulo 5 and 7 identities, too (with different groupings).

Some crank results

For integers m and $n > 1$, let $M(n, m)$ be the number of partitions of n with crank m. We use standard q-series notation.

Theorem (Garvan 1988)

$$\sum_{n\geq 0} M(m,n)q^n = \frac{1}{(q;q)_\infty}\sum_{n\geq 1}(-1)^{n-1}q^{n(n-1)/2+n|m|}(1-q^n),$$

$$M(m,n) = M(-m,n).$$

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Compare the "not completely different" rank generating function

$$\sum_{n\geq 0} N(m,n)q^n = \frac{1}{(q;q)_\infty} \sum_{n\geq 1} (-1)^{n-1} q^{n(3n-1)/2+n|m|}(1-q^n).$$

Bounded crank

Given $j \geq 0$, we're interested in the number of partitions λ of n with $\text{crank}(\lambda) \geq j$. By Garvan's formula, we have

$$\sum_{m \geq j} \sum_{n \geq 0} M(m,n) q^n = \frac{1}{(q;q)_\infty} \sum_{n \geq 0} (-1)^n q^{n(n+1)/2 + j(n+1)}.$$

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Theorem (H., Sellers, Yee)

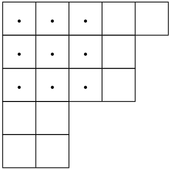
$$\sum_{m \geq j} \sum_{n \geq 0} M(m,n) q^n = \sum_{n \geq 0} \frac{q^{(n+1)(n+j)}}{(q;q)_n (q;q)_{n+j}}$$

Note that there is no alternating sum.

Proof ingredient: j-Durfee rectangles

Definition

The j-Durfee rectangle of a partition λ is the largest rectangle of size $d \times (d + j)$ that fits inside the Ferrers diagram of λ.



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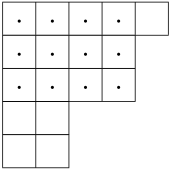

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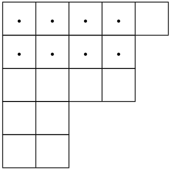

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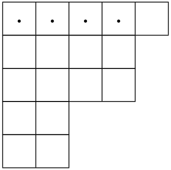

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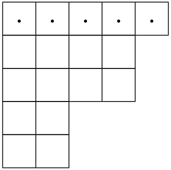

$(5, 4, 4, 2, 2)$ has 0-Durfee rectangle (Durfee square) size 3×3, 1-Durfee rectangle size 3×4, 2-Durfee rectangle size 2×4, 3-Durfee rectangle size 1×4, and 4-Durfee rectangle size 1×5.

Proof ingredient: symmetry insight

Use crank(λ) $\leq -j$ rather than crank(λ) $\geq j$.

Equal count since $M(m,n) = M(-m,n)$, but nonpositive cranks only come from the second part of the definition:

$$\text{crank}(\lambda) = \begin{cases} \lambda_1 & \text{if } \omega(\lambda) = 0, \quad \leftarrow \text{only positive crank} \\ \omega(\lambda) - \mu(\lambda) & \text{if } \omega(\lambda) > 0. \quad \leftarrow \text{any crank} \end{cases}$$

Combinatorial proof

Theorem (H., Sellers, Yee)

$$\sum_{m\geq j}\sum_{n\geq 0} M(m,n)q^n = \sum_{m\leq -j}\sum_{n\geq 0} M(m,n)q^n = \sum_{n\geq 0}\frac{q^{(n+1)(n+j)}}{(q;q)_n(q;q)_{n+j}}$$

Nonpositive crank implies $\omega(\lambda) > 0$. Consider the j-Durfee rectangle, size $d \times (d+j)$. Now, if $\omega(\lambda) < d+j$, then $\mu(\lambda) \geq d$ since $\lambda_d \geq d+j$ and

$$\text{crank}(\lambda) = \mu(\lambda) - \omega(\lambda) \ > \ d - (d+j) = -j.$$

So $\text{crank}(\lambda) \leq -j$ implies $\omega(\lambda) \geq d+j$. The generating function for such λ: the j-Durfee rectangle and the lower bound of $\omega(\lambda)$ give the exponent $d(d+j)+(d+j)$, boxes to the right of the j-Durfee rectangle account for $(q;q)_d$, boxes below give $(q;q)_{d+j}$.

Another combinatorial proof

We also show

$$(q;q)_\infty \sum_{n\geq 0} \frac{q^{n+j}}{(q;q)_n(q;q)_{n+j}} = \frac{1}{(q;q)_\infty} \sum_{n\geq 0} (-1)^n q^{n(n+1)/2+j(n+1)},$$

$$(q;q)_\infty \sum_{n\geq 0} \frac{q^{n+j}}{(q;q)_n(q;q)_{n+j}} = \sum_{n\geq 0} \frac{q^{(n+1)(n+j)}}{(q;q)_n(q;q)_{n+j}}$$

by considering two sign-reversing involutions on the triples of partitions (π, κ, ν) where π is a partition into distinct parts, κ is a nonempty partition with largest part at least j, and ν is a partition into parts that are at least j less than the largest part of κ.

Mex

The mex of a partition is the smallest missing (positive) part, e.g.,

$$\text{mex}((2,2,2)) = 1, \quad \text{mex}((3,1,1,1)) = 2, \quad \text{mex}((3,2,1)) = 4.$$

Terminology from combinatorial game theory (at least by 1973, Gundy values), combination of minimal excluded number.

References in partitions:

- Grabner–Knopfmacher 2006 "least gap"
- Andrews 2011 "smallest number that is *not* a summand"
- Andrews–Newman 2019 "minimal excludant"/mex

Mex count

Let $x(m,n)$ be the number of partitions of n with mex m.
Write $t_k = 1 + \cdots + k$ for the kth triangular number.

Proposition (H., Sellers, Stanton 2022)

$$x(m,n) = p(n - t_{m-1}) - p(n - t_m)$$

To have mex m, a partition must include parts $1, \ldots, m-1$.
Removing one of each of those leaves a partition of $n - t_{m-1}$.

And the partition must exclude m. The number of partitions of n with m also included is $p(n - t_m)$, so the number with m excluded is $p(n - t_{m-1}) - p(n - t_m)$.

Splitting the mexes

Definition

Let $m_{a,b}(n)$ to be the number of partitions of n with mex congruent to a modulo b.

Also, write superscript e for the number of partitions with an even number of parts, similarly for superscript o.

n	2	3	4	5	6	7	8	9	10	11	12
$m_{1,2}(n)$	1	2	3	4	6	8	12	16	23	30	42
$m_{1,4}(n)$	1	1	2	2	4	4	7	8	13	15	23
$m_{3,4}(n)$	0	1	1	2	2	4	5	8	10	15	19
$m^{o}_{1,2}(n)$	1	1	2	2	3	4	6	8	11	15	21
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Splitting the mexes

Proposition (H., Sellers, Yee)

$$m^o_{1,2}(n) = \begin{cases} m^e_{1,2}(n) + (-1)^{m+1} & \text{when } n = m(3m \pm 1), \\ m^e_{1,2}(n) & \text{otherwise.} \end{cases}$$

Combinatorial proof comes down to considering triples (π, μ, ν) where π is a partition into distinct even parts, μ is a partition into odd parts, and ν is a partition into distinct odd parts. A sign-reversing involutions leaves just $(\pi, \emptyset, \emptyset)$, then apply Franklin's bijection to $(\pi_1/2, \pi_2/2, \ldots)$.

Splitting the mexes

Theorem (Andrews, Newman 2019)

$m_{1,2}(n)$ is almost always even and is odd exactly when $n = m(3m \pm 1)$ for some m.

New proof:

$$m_{1,2}(n) = m^o_{1,2}(n) + m^e_{1,2}(n)$$
$$= \begin{cases} 2m^e_{1,2}(n) + (-1)^{m+1} & \text{when } n = m(3m \pm 1), \\ 2m^e_{1,2}(n) & \text{otherwise.} \end{cases}$$

Connecting crank and mex

Theorem (Andrews, Newman 2020; H., Sellers 2020)

The number of partitions of n with nonnegative crank equals the number of partitions of n with odd mex. I.e.,

$$M_{\geq 0}(n) = m_{1,2}(n).$$

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Generalize the mex: For j a part in λ, let $\mathrm{mex}_j(\lambda)$ to be the least integer greater than j that is not a part of λ. Not defined if j not a part of λ. (Thinking of 0 as a part of every partition, $\mathrm{mex}_0 = \mathrm{mex}$.)

Theorem (H., Sellers, Stanton 2022)

The number of partitions λ of n with $\mathrm{crank}(\lambda) \geq j$ equals the number of partitions of n with odd mex_j.

Connecting crank and mex

We have $m_{1,2}(n) = m_{1,4}(n) + m_{3,4}(n)$. What do those correspond to among the partitions with nonnegative crank?

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Theorem (Huh, Kim 2021)

$$M^e_{\geq 0}(n) = m_{1,4}(n), \quad M^o_{\geq 0}(n) = m_{3,4}(n).$$

Our proof: The generating function by number of parts is

$$\sum_{k,n\geq 0} M_{\leq 0}(k,n) z^k q^n = \sum_{n\geq 0} \frac{z^{2n} q^{n(n+1)}}{(zq;q)_n (q;q)_n}.$$

Substituting $z = -1$ gives

$$\sum_{n\geq 0} \left(M^e_{\leq 0}(n) - M^o_{\leq 0}(n)\right) q^n = \sum_{n\geq 0} \frac{q^{n(n+1)}}{(-q;q)_n (q;q)_n} = \sum_{n\geq 0} \frac{q^{n(n+1)}}{(q^2;q^2)_n}.$$

This connects to formulas derived from our first theorem.

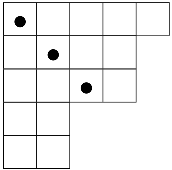
Another crank result

Proposition (H., Sellers, Yee)

$$M^e_{\leq 0}(n) = \begin{cases} M^o_{\leq 0}(n) & \text{if } n \text{ is odd,} \\ M^o_{\leq 0}(n) + q(n/2) & \text{if } n \text{ is even.} \end{cases}$$

The proof adjusts the partitions to the right of and below the Durfee square to make a sign-reversing involution. The fixed points are in bijection to partitions of n into distinct even parts.

$(5, 4, 4, 2, 2)$ Ferrers diagram and Frobenius symbol



$$\begin{pmatrix} 4 & 2 & 1 \\ 4 & 3 & 0 \end{pmatrix}$$

Theorem (Andrews 2011)

The number of partitions of n with no 0 in the top row of their Frobenius symbols equals the number of partitions of n with odd mex.

Crank and Frobenius symbols

Theorem (H., Sellers, Stanton 2022)

The number of partitions of $n-j$ with no j in the top row of their Frobenius symbols equals the number of partitions λ of n with $\text{crank}(\lambda) \geq j$.

New proof: By our first theorem, partitions with this bounded crank are in bijection with partitions of n with at least $d+j$ parts 1 where the j-Durfee rectangle is $d \times (d+j)$. Delete $d+j$ parts 1 and increase each of the d largest parts by 1. This makes a partition λ' of $n-j$ with

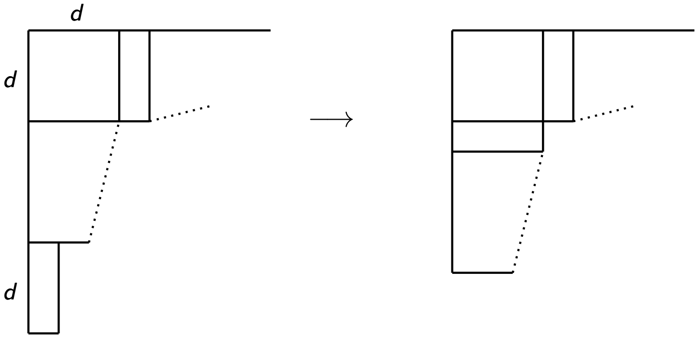
$$\lambda'_d \geq d+j+1 \quad \text{and} \quad \lambda'_{d+1} \leq d+j = (d+1)+(j-1).$$

Thus the top entries in columns d and $d+1$ of the Frobenius symbol are at least $j+1$ and at most $j-1$, respectively: no j.

Crank and Frobenius symbols

Theorem (H., Sellers, Yee)

The number of partitions of n with crank 0 equals the number of partitions of n whose Frobenius symbol has no 0 and the first two entries of the bottom row differ by 1.



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- B. Hopkins, J. A. Sellers, A. J. Yee, Combinatorial perspectives on the crank and mex partition statistics, submitted, arXiv:2108.09414