Partition Fixed Points: Connections, Generalizations, and Refinements

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Seminar in Partition Theory, q-Series and Related Topics
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Partition Fixed Points

Blecher & Knopfmacher (*Ramanujan J.* 2022) introduced the idea of fixed points to partitions, parts that satisfy $\lambda_i = i$.

Here order matters. Most of their work concerns fixed points in partitions written in non-decreasing order, e.g.,

$$P(5) = \{\underline{1}1111, \underline{1}112, \underline{1}1\underline{3}, \underline{12}2, \underline{1}4, 23, 5\}.$$

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They do discuss, and we will focus on, fixed points in partitions written in non-increasing order, e.g.,

$$P(5) = \{5, 41, 3\underline{2}, 311, 2\underline{2}1, 2111, \underline{1}1111\}.$$

Partition Fixed Points Conjecture

A fixed point is an "increasing" characteristic (Does $\lambda_1 = 1$? Does $\lambda_2 = 2$? etc.) so partitions (in non-increasing order) have at most one fixed point.

$$P(5) = \{5, 41, 3\underline{2}, 311, 2\underline{2}1, 2111, \underline{1}1111\}.$$

Note that 3 partitions of 5 have a fixed point while 4 do not.

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Blecher–Knopfmacher conjecture

For $n \geq 2$, there are more partitions of n without a fixed point than with a fixed point.

But $P(2) = \{2, \underline{1}1\}$ has one of each, so ...

Partition Fixed Points Conjecture

Having a fixed point is "increasing" (Does $\lambda_1 = 1$? Does $\lambda_2 = 2$? etc.) so non-increasing partitions have at most one fixed point.

$$P(5) = \{5, 41, 3\underline{2}, 311, 2\underline{2}1, 2111, \underline{1}1111\}.$$

Note that 3 partitions of 5 have a fixed point while 4 do not.

Adjusted Blecher–Knopfmacher conjecture

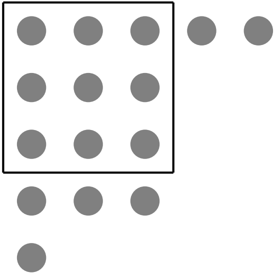
For $n > 2$, there are more partitions of n without a fixed point than with a fixed point.

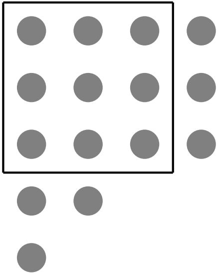
n	1	2	3	4	5	6	7	8	9	10
no fixed point	0	1	2	3	4	6	8	12	16	23
1 fixed point	1	1	1	2	3	5	7	10	14	19

Overview

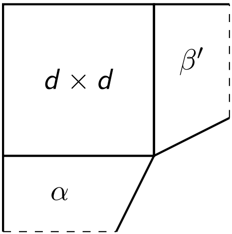
1. Proving their conjecture connects to several partition concepts:
   - Frobenius symbol
   - Dyson's crank
   - minimal excluded part, mex
2. Generalization to k-fixed points uses mex_k and k-Durfee rectangle, gives results about intervals of crank values
3. Refinement of fixed points into an integer triangle has many patterns, connects to excedances of partitions and unimodal compositions

Ferrers diagrams of $(5,3,3,3,1)$ and $(4,4,4,2,1)$ which both have 3×3 Durfee squares.





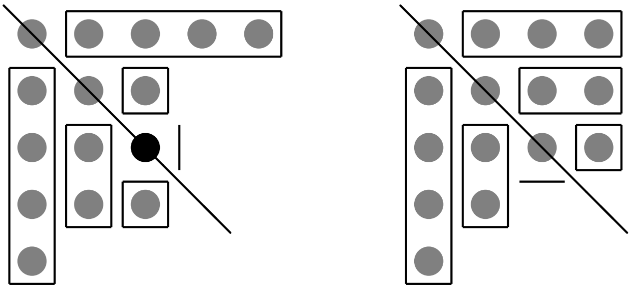
1. A Jacobi formula



A partition consists of its Durfee square, a subpartition α below and a subpartition β' to the right, each with first part at most d.

$$\sum_{n \geq 0} p(n) q^n = \sum_{d \geq 0} \frac{q^{d^2}}{(q;q)_d^2}.$$

1. Fixed points to Frobenius symbols



$$53331 \sim \begin{pmatrix} 4 & 1 & 0 \\ 4 & 2 & 1 \end{pmatrix} \text{ and } 44421 \sim \begin{pmatrix} 3 & 2 & 1 \\ 4 & 2 & 0 \end{pmatrix}$$

The black dot shows the fixed point,
corresponds to a 0 in the top row of the Frobenius symbol,
equivalently, at the bottom right edge of the Durfee square.

1. Connections

$$\begin{array}{ccc} \text{no fixed point} & ? & \text{fixed point} \\ \updownarrow & & \updownarrow \\ \text{top Frob. no 0} & & \text{top Frob. has 0} \end{array}$$

The minimal excludant (mex) of a partition is the smallest positive integer that is not a part. E.g.,

$$\operatorname{mex}(5) = \operatorname{mex}(32) = 1, \ \operatorname{mex}(311) = 2, \ \operatorname{mex}(221) = 3.$$

Sprague and Grundy 1930s analysis of combinatorial games.
In partitions, Grabner–Knopfmacher 2006 "least gap." Andrews 2011 "smallest part that is *not* a summand," Andrews–Newman 2019 mex.

Write $\operatorname{mex}_{a,b}(n)$ for the number of partitions of n with mex congruent to a mod b.

1. Connections

no fixed point	?	fixed point
$\updownarrow$		$\updownarrow$
top Frob. no 0		top Frob. 0
$\updownarrow$		$\updownarrow$
odd mex		even mex

Andrews 2011

by generating function arguments.

(Liu 2012 gives some related combinatorial arguments.)

1. Connections

$$\begin{array}{rcll} \text{no fixed point} & ? & \text{fixed point} & \\ \updownarrow & & \updownarrow & \\ \text{top Frob. no 0} & & \text{top Frob. 0} & \\ \updownarrow & & \updownarrow & \text{Andrews 2011} \\ \text{odd mex} & > & \text{even mex} & \text{Shen 2019} \end{array}$$

Yufei Shen was an undergraduate student of George Andrews, actual result

$$\mathrm{mex}_{1,2}(n) \geq \mathrm{mex}_{0,2}(n) \text{ for } n \geq 2$$

via extensive q-series manipulations.

Done, but . . . is there a better answer?

Dyson predicted (and named) the crank statistic in 1944, George Andrews and Frank Garvan found it in 1988.

Let $\omega(\lambda)$ be the number of ones in λ and $\mu(\lambda)$ the number of parts of λ greater than $\omega(\lambda)$. The crank c of $\lambda = (\lambda_1, \ldots, \lambda_k)$ is

$$c(\lambda) = \begin{cases} \lambda_1 & \text{if } \omega(\lambda) = 0, \\ \mu(\lambda) - \omega(\lambda) & \text{if } \omega(\lambda) > 0. \end{cases}$$

Write $M(m, n)$ for the number of partitions of n with crank m.

1. Connections

$$\begin{array}{rcll}
\text{no fixed point} & ? & \text{fixed point} & \\
\updownarrow & & \updownarrow & \\
\text{top Frob. no 0} & & \text{top Frob. 0} & \\
\updownarrow & & \updownarrow & \text{Andrews 2011} \\
\text{odd mex} & > & \text{even mex} & \text{Shen 2019} \\
\updownarrow & & \updownarrow & \text{H.–S., A.–N. 2020} \\
\text{nonneg. crank} & & \text{negative crank} &
\end{array}$$

by generating function arguments.

(Konan 2023 gives a bijective proof, presented spring 2022 here.)

1. Connections

no fixed point	?	fixed point	
$\updownarrow$		$\updownarrow$	
top Frob. no 0		top Frob. 0	
$\updownarrow$		$\updownarrow$	Andrews 2011
odd mex	$>$	even mex	Shen 2019
$\updownarrow$		$\updownarrow$	H.–S., A.–N. 2020
nonneg. crank		negative crank	
$\updownarrow$		$\updownarrow$	Garvan 1988
nonneg. crank	$>$	positive crank	

clear from the crank generating function.

(Berkovich–Garvan 2002 define "pseudo-conjugation" which proves crank symmetry combinatorially.)

1. Connections

$$\begin{array}{rcll}
\text{no fixed point} & ? & \text{fixed point} & \\
\updownarrow & & \updownarrow & \\
\text{top Frob. no } 0 & & \text{top Frob. } 0 & \\
\updownarrow & & \updownarrow & \text{Andrews 2011} \\
\text{odd mex} & > & \text{even mex} & \text{Shen 2019} \\
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\end{array}$$

H.-Sellers 2020

In $P(n)$, the number with odd mex exceeds the number with even mex by the number of crank 0 partitions.

$$\text{mex}_{1,2}(n) - \text{mex}_{0,2}(n) = M(0, n) \text{ for } n \geq 3.$$

1. Connections

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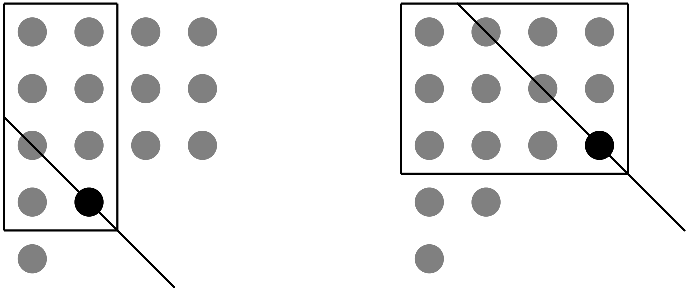
H.-Sellers 2023?

In $P(n)$, the number without a fixed point exceeds the number with a fixed point by the number of crank 0 partitions.

$$\boxed{g(n) - f(n) =} \operatorname{mex}_{1,2}(n) - \operatorname{mex}_{0,2}(n) = M(0, n) \text{ for } n \geq 3.$$

Given an integer k, say a partition λ has a k-fixed point if there is an i such that $\lambda_i = i + k$.



44421 has a -2-fixed point and a 1-fixed point.

2. k-Durfee rectangle

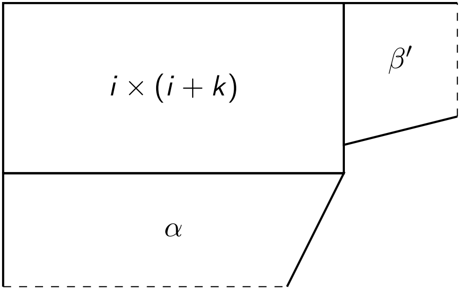
From H., Sellers, Yee 2022:

Given an integer k, the k-Durfee rectangle of $\lambda \in P(n)$ is the largest $d \times (d+k)$ rectangle that fits in the Ferrers diagram of λ.

Leads to a nice expression for the number of partitions with crank bounded from below:

$$\sum_{m \geq j} \sum_{n \geq 0} M(m,n) q^n = \sum_{i \geq 0} \frac{q^{(i+1)(i+j)}}{(q;q)_i (q;q)_{i+j}}.$$

2. k-fixed point generating function



The number of partitions of n with a k-fixed point is given by

$$\sum_{n \geq 0} f_k(n) q^n = \sum_{i \geq 1} \frac{q^{i(i+k)}}{(q;q)_{i+k}(q;q)_{i-1}}.$$

For $k \geq 0$, the difference between partitions of n without and with k-fixed points is the number of partitions of n with crank in the centered interval from $-k$ to k:

$$g_k(n) - f_k(n) = \sum_{\ell=-k}^{k} M(\ell, n)$$
$$= p(n) + 2\sum_{j\geq 1}(-1)^j\, p\left(n - \frac{j(j+2k-1)}{2}\right).$$

The $k = 0$ case is $g(n) - f(n) = M(0, n)$; this generalizes the resolution of the Blecher–Knopfmacher conjecture.

2. Generalized mex

from H., Sellers, Stanton 2022:

Given a partition λ and a nonnegative integer j, let $\text{mex}_j(\lambda)$ be the smallest integer greater than j that is not a part of λ.

For example, $\alpha = (5, 3, 3, 3, 1)$ has $\text{mex}_1(\alpha) = 2$ and $\text{mex}_2(\alpha) = 4$. The unindexed mex corresponds to the $j = 0$ case.

Given $n \geq 2$ and $j \geq 0$,

$$\sum_{m \geq j} M(m, n) = \#\{\mu \vdash n \mid \text{mex}_j - j \equiv 1 \bmod 2 \text{ and } j \text{ is a part of } \mu\}$$

$$= \#\{\nu \vdash n - j \mid \text{mex}_j - j \equiv 1 \bmod 2\}$$

2. Equivalences for k-fixed points, k positive

For $k \geq 0$, the following are equal:

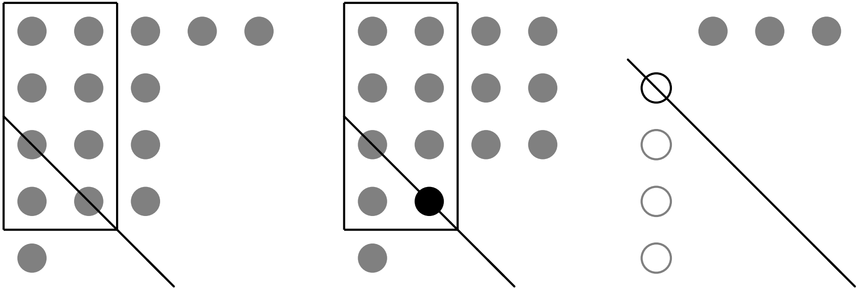
- partitions of n without a k-fixed point (i.e., $g_k(n)$),
- partitions of n whose Frobenius symbol has no k in the top row,
- partitions of n with $mex_k(\lambda) - k$ odd,
- partitions of n with crank at least $-k$,
- partitions of $n + k$ with crank at least k.

Generating function proofs. The equality of the last two,

$$\sum_{m \geq -k} M(m, n) = \sum_{m \geq k} M(m, n + k),$$

calls for a "move these dots around" combinatorial proof.

2. Issue for $-k$-fixed points (k positive)



53331 does not have a -2-fixed point, 44421 does. What about 3?

Best to use $f_{-k}(n)$, $p(n, k-1)$ partitions with up to $k-1$ parts,
$g'_{-k}(n)$ partitions with at least k parts and no $-k$-fixed points.

2. Results for $-k$-fixed points

$$f_{-k}(n) + p(n, k-1) = \sum_{m \geq -k+1} M(m, n), \quad g'_{-k}(n) = \sum_{m \geq k} M(m, n);$$

$$\begin{aligned} f_{-k}(n) + p(n, k-1) - g'_{-k}(n) &= \sum_{m=-k+1}^{k-1} M(m, n) \\ &= p(n) + 2 \sum_{j \geq 1} (-1)^j \, p\left(n - \frac{j(j+2k-3)}{2}\right). \end{aligned}$$

No analogues for generalized mex or Frobenius symbol entries.

H., Sellers, On Blecher and Knopfmacher's fixed points for integer partitions, submitted, arXiv:2305.05096.

Separate fixed point partitions by which part is fixed. E.g.,

$$P(5) = \{5, 41, 3\underline{2}, 311, 2\underline{2}1, 2111, \underline{1}1111\}.$$

5 has 1 fixed partition with $\lambda_1 = 1$ and 2 with $\lambda_2 = 2$.

3?

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3? The smallest fixed point partition with $\lambda_3 = 3$ is 333 in $P(9)$.

1?

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3? The smallest fixed point partition with $\lambda_3 = 3$ is 333 in $P(9)$.

1? Clear that, for each n, the only fixed point partition with $\lambda_1 = 1$ is the all-1 partition 1^n.

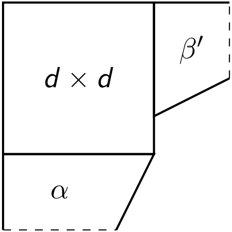
3. Triangle of fixed point partitions $f(n, d)$

n	1	2	3	4	Σ
1	1				1
2	1				1
3	1				1
4	1	1			2
5	1	2			3
6	1	4			5
7	1	6			7
8	1	9			10
9	1	12	1		14
10	1	16	2		19
11	1	20	5		26
12	1	25	9		35
13	1	30	16		47
14	1	36	25		62
15	1	42	39		82
16	1	49	56	1	107

$(d, \ldots, d) \in P(d^2)$ is first partition with fixed point d, so column d begins in row d^2.

Row sums are the number of partitions of n with positive crank/ even mex ...

3. $f(n,d)$ triangle column formulas



In order to have a fixed point $\lambda_d = d$, need the first part of β to be at most $d-1$.

$$\sum_{n \geq d^2} f(n,d) q^n = \frac{q^{d^2}}{(q;q)_d (q;q)_{d-1}}.$$

3. $f(n,d)$ triangle entries

By the generating function, know $f(n,d)$ satisfies a degree d^2 linear recurrence relation.

Can do better using the previous column of the triangle:

For $d \geq 2$ and $n \geq d^2$,

$$\begin{aligned} f(n,d) =& f(n-d+1,d) + f(n-d,d) \\ & - f(n-2d+1,d) + f(n-2d+1,d-1). \end{aligned}$$

A moderately involved combinatorial proof shows

$$\begin{aligned} &F(n,d) \cup F(n-2d+1,d) \\ &\qquad \cong F(n-d+1,d) \cup \underline{F(n-d,d)} \cup F(n-2d+1,d-1). \end{aligned}$$

3. $f(n,d)$ triangle examples of inter-column recurrence

n	1	2	3	4	Σ
1	1				1
2	1				1
3	1				1
4	1	1			2
5	1	2			3
6	1	4			5
7	1	6			7
8	1	9			10
9	1	12	1		14
10	1	16	2		19
11	$\boxed{1}$	$\boxed{-20}$	5		26
12	1	$\boxed{+25}$	9		35
13	1	$\boxed{+30}$	16		47
14	1	$\boxed{=36}$	25		62

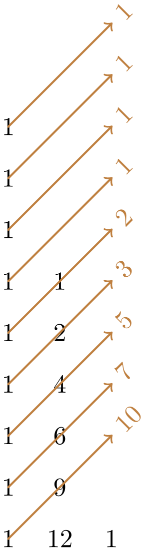
$f(14,2) =$
$30 + 25 - 20 + 1 = 36$

3. $f(n,d)$ triangle examples of inter-column recurrence

n	1	2	3	4	Σ
1	1				1
2	1				1
3	1				1
4	1	1			2
5	1	2			3
6	1	4			5
7	1	6			7
8	1	9			10
9	1	$\boxed{12}$	$\boxed{-1}$		14
10	1	16	2		19
11	1	20	$\boxed{+5}$		26
12	1	25	$\boxed{+9}$		35
13	1	30	16		47
14	1	36	$\boxed{=25}$		62

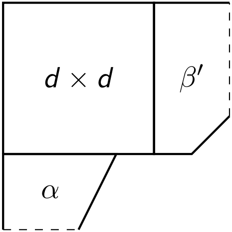
$f(14,2) =$
$30 + 25 - 20 + 1 = 36$

$f(14,3) =$
$9 + 5 - 1 + 12 = 25$



1, 1, 1, 1, 2, 3, 5, 7, 10, 13, 18, 23, 31, 40, 53, 68 is OEIS A118199, “partitions of n having no part equal to the size of their Durfee square.”

Partitions of n having no part equal to the size of their Durfee square are conjugate to



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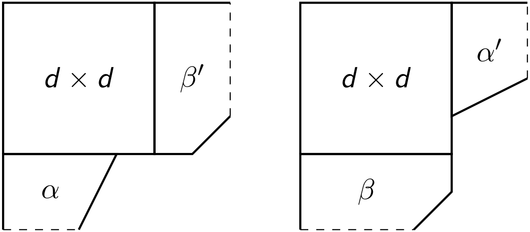


partitions of n with a fixed point λ_d and also $\lambda_{d+1} = d$.

3. $f(n,d)$ triangle diagonal sums

Let $A(n)$ be the partitions of n with $\lambda_d = \lambda_{d+1} = d$ for any d.

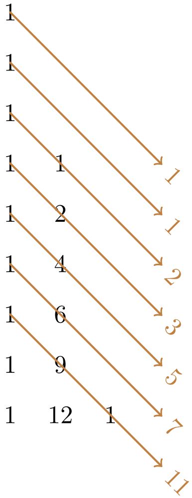
$$FP(n-1,1) \cup FP(n-2,2) \cup \cdots \cong A(n)$$

by simply adding a copy of the fixed point. E.g.,

$FP(8,1) \cup FP(7,2)$	$A(9)$
$\underline{1}1111111$	1^9
$5\underline{2}$	522
$4\underline{2}1$	4221
$3\underline{2}2$	3222
$3\underline{2}11$	32211
$2\underline{2}21$	22221
$2\underline{2}111$	222111

(Note that 333 is not in $A(9)$: $\lambda_3 = 3$ but there is no λ_4.)

3. $f(n,d)$ triangle anti-diagonal sums



1, 1, 2, 3, 5, 7, 11, 15, 22, 30, 42, 56, 77, 101, . . . , the partition numbers $p(n)$.

3. $f(n,d)$ triangle anti-diagonal sums

$$FP(n+1,1) \cup FP(n+2,2) \cup \cdots \cong P(n)$$

by simply removing the fixed point. E.g.,

$FP(7,1) \cup FP(8,2) \cup FP(9,3)$	$P(6)$
1111111	1^6
62	6
521	51
422	42
4211	411
3221	321
32111	3111
2222	222
22211	2211
221111	21111
333	33

3. Partition excedance

Co-opting another permutation statistic, the excedance of a partition is the number of parts for which $\lambda_i > i$.

$$P(5) = \{\underset{\bullet}{5}, \underset{\bullet}{4}1, \underset{\bullet}{2}2, \underset{\bullet}{3}11, \underset{\bullet}{2}21, \underset{\bullet}{2}111, 11111\}.$$

The smallest partition with two excedances is $\underset{\bullet\bullet}{33} \in P(6)$.

With non-increasing order, if λ has k excedances, those occur in the first k parts.

3. Triangle of partition excedances $e(n, k)$

n	1	2	3	4	Σ
1	1				1
2	1	1			2
3	1	2			3
4	1	4			5
5	1	6			7
6	1	9	1		11
7	1	12	2		15
8	1	16	5		22
9	1	20	9		30
10	1	25	16		42
11	1	30	25		56
12	1	36	39	1	77
13	1	42	56	2	101

$e(n, k)$ with $k \geq 1$ counts partitions of n with $k - 1$ excedances.

OEIS A353318, column k begins in row $(k - 1)k$.

Same columns as $f(n, d)$: $e(n, k) = f(n+k, k)$ [easy] which gives a version of Pascal's lemma.

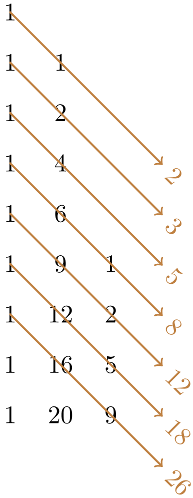
Clearly row sums are $p(n)$, match antidiagonal sums in $f(n, d)$ triangle.

3. $e(n,k)$ triangle example of inter-column recurrence

n	1	2	3	4	Σ
1	1				1
2	1	1			2
3	1	2			3
4	1	4			5
5	1	6			7
6	1	9	$\boxed{-1}$		11
7	1	$\boxed{12}$	2		15
8	1	16	$\boxed{+5}$		22
9	1	20	$\boxed{+9}$		30
10	1	25	16		42
11	1	30	$\boxed{=25}$		56
12	1	36	39	1	77
13	1	42	56	2	101

$e(11,3) =$
$9+5-1+12 = 25$

3. $e(n,k)$ triangle anti-diagonal sums



2, 3, 5, 8, 12, 18, 26, 37, 52, 98, . . . , is A084376, $p(n)+p(n+1)$ [moderate].

A unimodal composition of n is an ordered collection of positive integer parts c_i with $\sum_i c_i = n$ and

$$c_1 \leq \cdots \leq c_{p-1} \leq c_p \geq c_{p+1} \geq \cdots \geq c_s$$

for some index p; call c_p the peak.

Among the 16 compositions of 5, only 212 is not unimodal.
Unimodal with peak 1: 11111.
Unimodal with peak 2: 221, 2111, 122, 1211, 1121, 1112.
Unimodal with peak 3: 32, 311, 23, 131, 113.
Unimodal with peak 4: 41, 14.
Unimodal with peak 5: 5.

1	2	3	4	5	6	7	8
1							
1	1						
1	2	1					
1	4	2	1				
1	6	5	2	1			
1	9	9	5	2	1		
1	12	16	10	5	2	1	
1	16	25	19	10	5	2	1

$u(n, p)$ with $p \geq 1$ counts unimodal compositions of n with peak p.

OEIS A229706, column k begins in row k

Same columns as $f(n, d)$: $u(n, p) = f(n+p(p-1), p)$ [moderate] which gives a version of Pascal's lemma.

3. $u(n, p)$ triangle example of inter-column recurrence

1	2	3	4	5	6	7	8
1							
1	1						
1	2	$\boxed{-1}$					
1	4	2	1				
1	6	$\boxed{+5}$	2	1			
1	9	$\boxed{+9}$	5	2	1		
1	$\boxed{12}$	16	10	5	2	1	
1	16	$\boxed{=25}$	19	10	5	2	1

$u(8, 3) = 25$

Could pursue analagous combinatorial analysis of k-fixed points for any integer k ...