Rooted partitions and number-theoretic functions

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Introduction

An identity with ϕ

An identity with μ

Outline

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Ex. $\mu(70) = \mu(2 \cdot 5 \cdot 7) = (-1)^3 = -1$ but $\mu(50) = \mu(2 \cdot 5^2) = 0$.

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Theorem (Merca-Schmidt)

1. $S_1(n) = \sum_{k=2}^{n+1} \phi(k)\ S_k^{\geq 2}(n+1).$
2. $p(n) = \sum_{k=3}^{n+3} \frac{\phi(k)}{2}\ S_k^{\geq 3}(n+3).$
3. $p(n) = \sum_{k=1}^{n+1} \mu(k)\ S_k(n+1).$
4. $p(n) = -\sum_{k=2}^{n+2} \mu(k)\ S_k^{\geq 2}(n+2).$

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Ex. If $\lambda = (5, 2, 2, 2, 1, 1)$ then the ways to root λ at 2 are

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Note that this operation is not commutative as

$$(4, 4, 2, \hat{2}, 2, 1, 1) \oplus (5, 2, 2, 1) = (5, 4, 4, 2, \hat{2}, 2, 2, 2, 1, 1, 1).$$

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Proof. (Sagan) We give a bijection $\mathcal{S}_1(n) \to \mathcal{S}'(n+1)$ where

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Ex. Suppose that

$$\lambda = (4, 4, 2, 1, 1, \hat{1}, 1, 1) \in \mathcal{S}_1(15).$$

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One can show that the map $\lambda \mapsto (\lambda', r)$ is a bijection by constructing its inverse. $\square$

Outline

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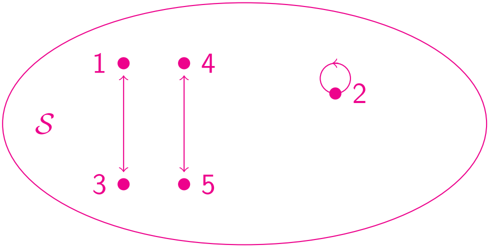

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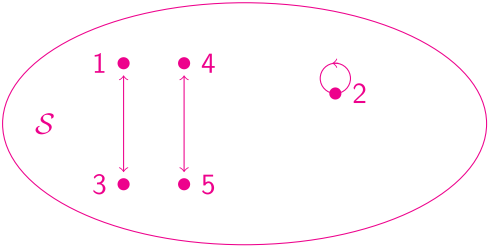
$\iota =$ 

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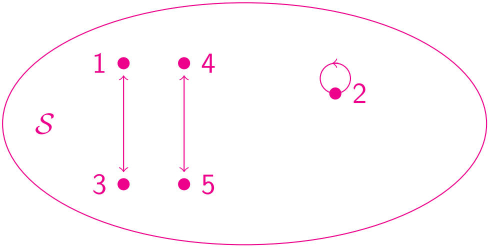
$\iota =$ 

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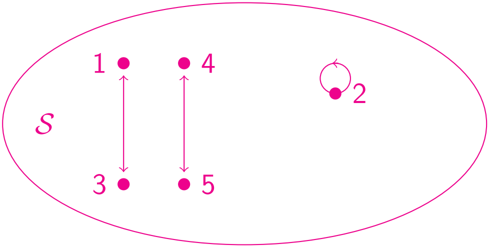
$\iota =$ 

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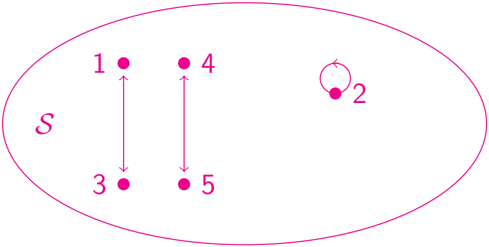
$\iota =$ 

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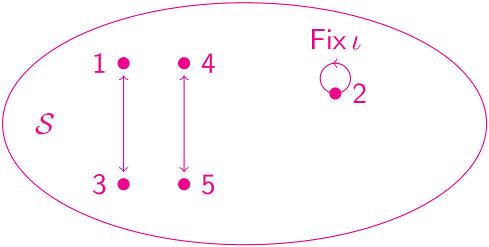
$\iota =$ 

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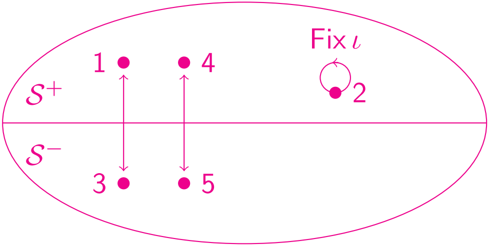
$\iota =$ 

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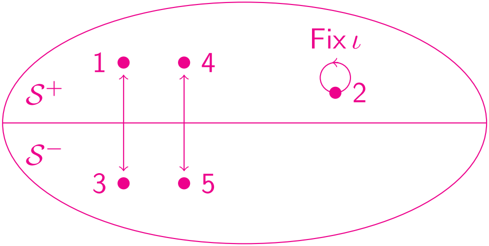
$\iota =$ 

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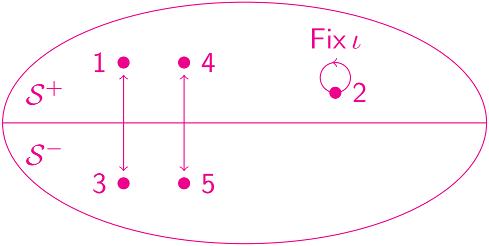
$\iota =$ 

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So it suffices to produce a sign-reversing involution ι on $\mathcal{S}(n+1)$ with the rooted partitions ending in $\hat{1}$ as fixed points.

To construct the sign-reversion involution, we will need

$$\pi(n) = \begin{cases} \text{smallest prime dividing } n & \text{if } n \geq 2, \\ \infty & \text{if } n = 1, \end{cases}$$

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One can check that Cases 1 and 2 are sign-reversing inverses. ☐

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