Method of Weighted Words on Cylindric Partitions

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Ordinary Integer Partitions

- An ordinary partition of a positive integer n is a finite non-increasing sequence of positive integers $b_1, b_2, \ldots b_k$ such that $n = b_1 + b_2 + \ldots + b_k$ where the b_i are called parts of the partition for $i = 1, 2, \ldots, k$.

- If we take $n = 6$, the corresponding partitions are

$$6,\ 5{+}1,\ 4{+}2,\ 3{+}3,\ 4{+}1{+}1,\ 3{+}2{+}1,\ 2{+}2{+}2,\ 3{+}1{+}1{+}1,$$

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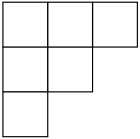
Ordinary Integer Partitions

- The Ferrers board of the partition $3 + 2 + 1$ is



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- We have the following infinite product representation of the partition function:

$$P(q) = \prod_{k \geq 1} \frac{1}{1 - q^k}.$$

The last series and the infinite product converge absolutely for $|q| < 1$.

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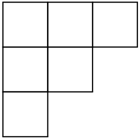

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Plane Partitions

- If we call ordinary partitions as one-dimensional partitions, we also have two-dimensional partitions, in other words, plane partitions.
- Plane partitions of n are two-dimensional arrays of non-negative integers such that rows from left to right and columns from top to bottom are non-increasing.
- One of the plane partitions of $n = 30$ is

$$\begin{matrix} 4 & 4 & 3 & 2 & 1 \\ 4 & 3 & 1 & 1 & \\ 3 & 2 & 1 & & \\ 1 & & & & \end{matrix}$$

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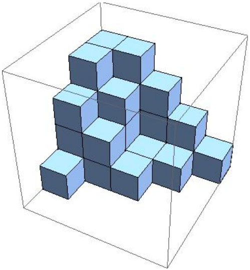
3-D Ferrers board of Plane Partitions

3-D Ferrers board of the running example is



(figure credit: https://en.wikipedia.org/wiki/Plane_partition)

The Generating Function for Plane Partitions

The generating function for plane partitions is given by the following MacMahon formula [16]:

$$PP(q) = \sum_{n \geq 0} pp(n) \cdot q^n = \prod_{n \geq 1} \frac{1}{(1-q^n)^n},$$

where $pp(n)$ gives the number of plane partitions of n.

Cylindric Partitions (Gessel and Krattenthaler [12])

- Let r and ℓ be positive integers. Let $c = (c_1, c_2, \ldots, c_r)$ be a composition where $c_1 + c_2 + \ldots + c_r = \ell$.
- The integer r is called the rank of the cylindric partition, while the integer ℓ is called the level of the cylindric partition.
- A cylindric partition with profile c is a vector partition $\Lambda = (\lambda^{(1)}, \lambda^{(2)}, \ldots, \lambda^{(r)})$ where each $\lambda^{(i)}$ is an ordinary partition such that $\lambda^{(i)} = \lambda_1^{(i)} + \lambda_2^{(i)} + \ldots + \lambda_{s_i}^{(i)}$ and for all i and j,

$$\lambda_j^{(i)} \geq \lambda_{j+c_{i+1}}^{(i+1)}, \; \lambda_j^{(r)} \geq \lambda_{j+c_1}^{(1)}.$$

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An Example of a Cylindric Partition with a profile $c = (c_1, c_2, \ldots, c_r)$

- Consider the cylindric partition $\Lambda = ((5,4),(8,2),(7,5,1))$ with profile $c = (0,1,2)$ where $\lambda^{(1)} = (5,4)$, $\lambda^{(2)} = (8,2)$, and $\lambda^{(3)} = (7,5,1)$ are ordinary partitions
- $c_1 = 0$, $c_2 = 1$, $c_3 = 2$ and $\lambda^{(2)}$ is shifted to the left by c_2, $\lambda^{(3)}$ is shifted to the left by c_3, and as a hidden condition $\lambda^{(1)}$ is shifted to the left by c_1, so that horizontally and vertically we have numbers in the non-increasing form.

$$
\begin{array}{cccccc}
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 & & 8 & 2 & 0 & \\
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3-D Ferrers board of the running example

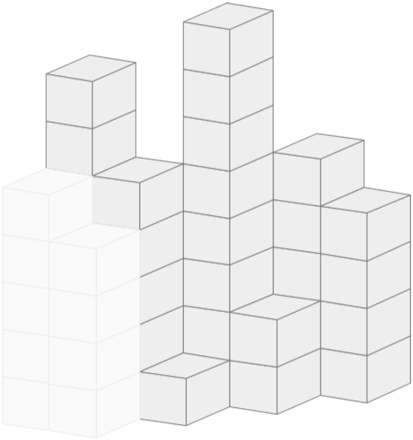

Figure: 3-D Ferrers board of cylindric partition $\Lambda = ((5,4),(8,2),(7,5,1))$ with profile $c = (0,1,2)$

Slices of the running example

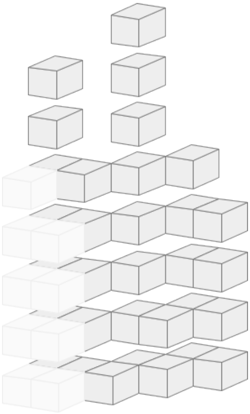

Figure: Slices of cylindric partition $\Lambda = ((5,4),(8,2),(7,5,1))$ with profile $c = (0,1,2)$

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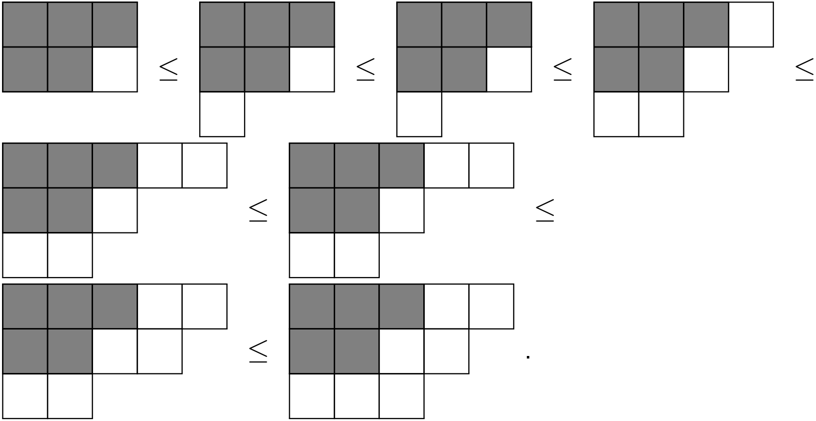

Shapes of Slices [15]

- Each slice having a profile $c = (c_1, c_2, \ldots, c_r)$ has a shape. We define the shape of a slice as follows:
- Without distinguishing between the white and gray squares in the representation of a slice, we remove k squares from each row, where k is the length of the r^{th} row.
- Since in a slice with a profile $c = (c_1, c_2, \ldots, c_r)$, there are r rows, after the operation above, we will be left with $(r-1)$ rows (counting zeroes), and starting from the top row, we write the number of squares left in the rows, respectively. We call this the shape of a slice.

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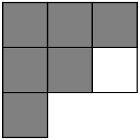
Examples: Shapes of slices

Shape of the slice with profile $c = (1,1,1)$ below is $(2,2)$:



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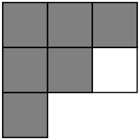

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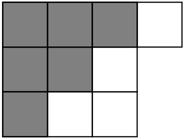

Generating Functions of Cylindric Partitions with a given profile $c = (c_1, c_2, \ldots, c_r)$

- The size of a cylindric partition $\Lambda = (\lambda^{(1)}, \lambda^{(2)}, \ldots, \lambda^{(r)})$ with a profile $c = (c_1, c_2, \ldots, c_r)$ is denoted by $|\Lambda|$.
- The largest part of a cylindric partition is denoted by $\max(\Lambda)$
- For cylindric partition $\Lambda = ((5,4),(8,2),(7,5,1))$ with profile $c = (0,1,2)$, $|\Lambda| = 32$ and $\max(\Lambda) = 8$.
- The following generating function

$$F_c(z,q) := \sum_{\Lambda \in P_c} z^{\max(\Lambda)} q^{|\Lambda|}$$

is the generating function for cylindric partitions, where P_c denotes the set of all cylindric partitions with profile c.

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Borodin's formula [5] for Cylindric Partitions with a given profile $c = (c_1, c_2, \ldots, c_r)$

Let r and ℓ be positive integers. Let $c = (c_1, c_2, \ldots, c_r)$ be a composition where $\ell = c_1 + c_2 + \ldots + c_r$. Define $t := r + \ell$ and $s(i,j) := c_i + c_{i+1} + \ldots + c_j$. Then,

$$F_c(1,q) = \frac{1}{(q^t;q^t)_\infty} \prod_{i=1}^{r}\prod_{j=i}^{r}\prod_{m=1}^{c_i} \frac{1}{(q^{m+j-i+s(i+1,j)};q^t)_\infty} \prod_{i=2}^{r}\prod_{j=2}^{i}\prod_{m=1}^{c_i} \frac{1}{(q^{t-m+j-i-s(j,i-1)};q^t)_\infty}.$$

Figure: Borodin formula

where the definition for q-Pochhammer symbols:

$$(a;q)_n := \prod_{j=1}^{n}(1 - aq^{j-1}),\ (a;q)_\infty := \lim_{n\to\infty}(a;q)_n.$$

Some of the Works on Cylindric Partitions

- In his proof for the above formula [5], Borodin used a probabilistic approach.
- Foda and Welsh gave a proof for Borodin's formula in the context of affine and $\mathcal{W}_r$ algebras [10].
- Corteel and Welsh used cylindric partitions to reproduce the four A_2 Rogers - Ramanujan type identities and to prove a similar fifth identity [6].
- Corteel, Dousse, and Uncu worked on the cylindric partitions with a given profile for rank 3 and level 5, thus obtaining new A_2 Rogers - Ramanujan type identities [7].
- Warnaar found results for $F_c(z,q)$ for rank 3 and level $\not\equiv 0 \pmod 3$ [19].
- Alternative generating functions for cylindric partitions of various profiles in the form of series or infinite product times series have been considered in the works of Kanade and Russell [13], Uncu [18], and Warnaar [20] using other approaches.

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What is our aim?

- We try to give a more combinatorial explanation of generating functions of cylindric partitions with a given profile and we are trying to find expressions alternative to Borodin's formula.
- For this purpose, we tried to use method of weighted words [1, 8, 9].
- The method of weighted words was first introduced by Alladi and Gordon [1] to prove Schur's identity [17].
- They computed the generating functions for the minimal partitions satisfying some minimal distance conditions, and used q-series identities.
- Later, this method was used by Dousse to prove other partition identities, but she created a new version of the method in which she used recurrences and q-difference equations instead of using minimal partitions and q-series identities [8, 9].

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What is our strategy?

- We fix a profile $c = (c_1, c_2, \ldots, c_r)$.

  Let us consider cylindric partitions having profile $c = (2, 1)$ with rank $r = 2$ and level $\ell = 3$.

- We determine the shapes of the slices for the cylindric partitions with the fixed profile.

  For the slices of the cylindric partitions with the given profile, there are exactly $\binom{\ell+r-1}{r-1} = \binom{4}{1}$ shapes.

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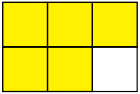
The corresponding shapes are $(0), (1), (2), (3)$ and let us denote these shapes by $\mathbf{a}, \mathbf{c}, \mathbf{b}$, and $\mathbf{d}$, respectively.
Slices that have these shapes with minimum positive weight are shown below.



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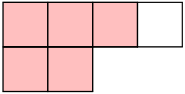
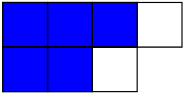
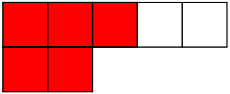


(bq^1),



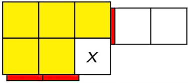
(cq^2),



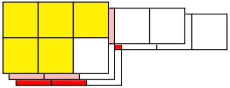
(dq^2).

What is our strategy?

- We find the minimum distance conditions between slices with certain shapes.
  For example, as shown below, after the slice aq^1 we cannot put the slice dq^2, as the crossed white square in the slice aq^1 is not contained in dq^2.



But after aq^1, we can put dq^4. Moreover, we can have the following placement with the slices aq^1, bq^3 and dq^4, respectively.



The above placement of the slices creates cylindric partition $\Lambda = ((2,2,1),(3))$ with profile $c = (2,1)$.

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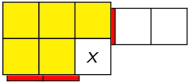

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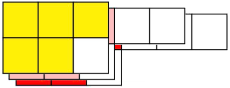
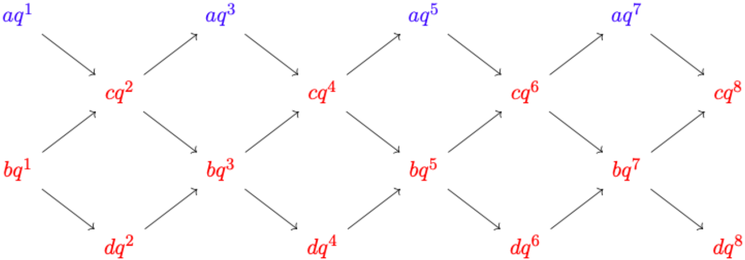


The above placement of the slices creates cylindric partition $\Lambda = ((2,2,1),(3))$ with profile $c = (2,1)$.

What is our strategy?

- Creating slice flows for the given profiles, we compute the beginning terms of the generating functions. And we search for a pattern!

Slice flow for profile $c = (2, 1)$ is given below:



Sketch of the Proof for Profile $c = (2, 1)$

The beginning terms of the generating function for profile $c = (2, 1)$ when repetition of slices is not allowed is as follows:

$$(1+bq^1)(1+cq^2+dq^2)(1+bq^3)(1+cq^4+dq^4)(1+bq^5)(1+cq^6+dq^6)$$

$$+(\mathbf{aq^1})(\mathbf{1+cq^2})(1+bq^3)(1+cq^4+dq^4)(1+bq^5)(1+cq^6+dq^6)$$

$$+(1+bq^1)(\mathbf{1+cq^2})(\mathbf{aq^3})(\mathbf{1+cq^4})(1+bq^5)(1+cq^6+dq^6)$$

$$+(1+bq^1)(1+cq^2+dq^2)(1+bq^3)(\mathbf{1+cq^4})(\mathbf{aq^5})(\mathbf{1+cq^6})$$

$$+(\mathbf{aq^1})(\mathbf{1+cq^2})(\mathbf{aq^3})(\mathbf{1+cq^4})(1+bq^5)(1+cq^6+dq^6)$$

$$+(\mathbf{aq^1})(\mathbf{1+cq^2})(1+bq^3)(\mathbf{1+cq^4})(\mathbf{aq^5})(\mathbf{1+cq^6})$$

$$+(1+bq^1)(\mathbf{1+cq^2})(\mathbf{aq^3})(\mathbf{1+cq^4})(\mathbf{aq^5})(\mathbf{1+cq^6})$$

$$+(\mathbf{aq^1})(\mathbf{1+cq^2})(\mathbf{aq^3})(\mathbf{1+cq^4})(\mathbf{aq^5})(\mathbf{1+cq^6}).$$

Sketch of the Proof for Profile $c = (2, 1)$

If we allow slices to repeat, the expression becomes:

$$(1+\frac{bq^1}{1-bq^1})(1+\frac{cq^2}{1-cq^2}+\frac{dq^2}{1-dq^2})(1+\frac{bq^3}{1-bq^3})(1+\frac{cq^4}{1-cq^4}+\frac{dq^4}{1-dq^4})(1+\frac{bq^5}{1-bq^5})(1+\frac{cq^6}{1-cq^6}+\frac{dq^6}{1-dq^6})$$

$$+(\frac{\mathbf{aq^1}}{\mathbf{1-aq^1}})(\mathbf{1}+\frac{\mathbf{cq^2}}{\mathbf{1-cq^2}})(1+\frac{bq^3}{1-bq^3})(1+\frac{cq^4}{1-cq^4}+\frac{dq^4}{1-dq^4})(1+\frac{bq^5}{1-bq^5})(1+\frac{cq^6}{1-cq^6}+\frac{dq^6}{1-dq^6})$$

$$+(1+\frac{bq^1}{1-bq^1})(\mathbf{1}+\frac{\mathbf{cq^2}}{\mathbf{1-cq^2}})(\frac{\mathbf{aq^3}}{\mathbf{1-aq^3}})(\mathbf{1}+\frac{\mathbf{cq^4}}{\mathbf{1-cq^4}})(1+\frac{bq^5}{1-bq^5})(1+\frac{cq^6}{1-cq^6}+\frac{dq^6}{1-dq^6})$$

$$+(1+\frac{bq^1}{1-bq^1})(1+\frac{cq^2}{1-cq^2}+\frac{dq^2}{1-dq^2})(1+\frac{bq^3}{1-bq^3})(\mathbf{1}+\frac{\mathbf{cq^4}}{\mathbf{1-cq^4}})(\frac{\mathbf{aq^5}}{\mathbf{1-aq^5}})(\mathbf{1}+\frac{\mathbf{cq^6}}{\mathbf{1-cq^6}})$$

$$+(\frac{\mathbf{aq^1}}{\mathbf{1-aq^1}})(\mathbf{1}+\frac{\mathbf{cq^2}}{\mathbf{1-cq^2}})(\frac{\mathbf{aq^3}}{\mathbf{1-aq^3}})(\mathbf{1}+\frac{\mathbf{cq^4}}{\mathbf{1-cq^4}})(1+\frac{bq^5}{1-bq^5})(1+\frac{cq^6}{1-cq^6}+\frac{dq^6}{1-dq^6})$$

$$+(\frac{\mathbf{aq^1}}{\mathbf{1-aq^1}})(\mathbf{1}+\frac{\mathbf{cq^2}}{\mathbf{1-cq^2}})(1+\frac{bq^3}{1-bq^3})(\mathbf{1}+\frac{\mathbf{cq^4}}{\mathbf{1-cq^4}})(\frac{\mathbf{aq^5}}{\mathbf{1-aq^5}})(\mathbf{1}+\frac{\mathbf{cq^6}}{\mathbf{1-cq^6}})$$

$$+(1+\frac{bq^1}{1-bq^1})(\mathbf{1}+\frac{\mathbf{cq^2}}{\mathbf{1-cq^2}})(\frac{\mathbf{aq^3}}{\mathbf{1-aq^3}})(\mathbf{1}+\frac{\mathbf{cq^4}}{\mathbf{1-cq^4}})(\frac{\mathbf{aq^5}}{\mathbf{1-aq^5}})(\mathbf{1}+\frac{\mathbf{cq^6}}{\mathbf{1-cq^6}})$$

$$+(\frac{\mathbf{aq^1}}{\mathbf{1-aq^1}})(\mathbf{1}+\frac{\mathbf{cq^2}}{\mathbf{1-cq^2}})(\frac{\mathbf{aq^3}}{\mathbf{1-aq^3}})(\mathbf{1}+\frac{\mathbf{cq^4}}{\mathbf{1-cq^4}})(\frac{\mathbf{aq^5}}{\mathbf{1-aq^5}})(\mathbf{1}+\frac{\mathbf{cq^6}}{\mathbf{1-cq^6}}).$$

Sketch of the Proof for Profile $c = (2, 1)$

Putting $a = b = c = d = 1$, we obtain the following sum:

$$
\begin{aligned}
&(\tfrac{1}{1-q^1})(\tfrac{1+q^2}{1-q^2})(\tfrac{1}{1-q^3})(\tfrac{1+q^4}{1-q^4})(\tfrac{1}{1-q^5})(\tfrac{1+q^6}{1-q^6})\\
&+(\tfrac{\mathbf{q^1}}{\mathbf{1-q^1}})(\tfrac{\mathbf{1}}{\mathbf{1-q^2}})(\tfrac{1}{1-q^3})(\tfrac{1+q^4}{1-q^4})(\tfrac{1}{1-q^5})(\tfrac{1+q^6}{1-q^6})\\
&+(\tfrac{1}{1-q^1})(\tfrac{\mathbf{1}}{\mathbf{1-q^2}})(\tfrac{\mathbf{q^3}}{\mathbf{1-q^3}})(\tfrac{\mathbf{1}}{\mathbf{1-q^4}})(\tfrac{1}{1-q^5})(\tfrac{1+q^6}{1-q^6})\\
&+(\tfrac{1}{1-q^1})(\tfrac{1+q^2}{1-q^2})(\tfrac{1}{1-q^3})\tfrac{\mathbf{1}}{\mathbf{1-q^4}}(\tfrac{\mathbf{q^5}}{\mathbf{1-q^5}})(\tfrac{\mathbf{1}}{\mathbf{1-q^6}})\\
&+(\tfrac{\mathbf{q^1}}{\mathbf{1-q^1}})(\tfrac{\mathbf{1}}{\mathbf{1-q^2}})(\tfrac{\mathbf{q^3}}{\mathbf{1-q^3}})(\tfrac{\mathbf{1}}{\mathbf{1-q^4}})(\tfrac{1}{1-q^5})(\tfrac{1+q^6}{1-q^6})\\
&+(\tfrac{\mathbf{q^1}}{\mathbf{1-q^1}})(\tfrac{\mathbf{1}}{\mathbf{1-q^2}})(\tfrac{1}{1-q^3})(\tfrac{\mathbf{1}}{\mathbf{1-q^4}})(\tfrac{\mathbf{q^5}}{\mathbf{1-q^5}})(\tfrac{\mathbf{1}}{\mathbf{1-q^6}})\\
&+(\tfrac{1}{1-q^1})(\tfrac{\mathbf{1}}{\mathbf{1-q^2}})(\tfrac{\mathbf{q^3}}{\mathbf{1-q^3}})(\tfrac{\mathbf{1}}{\mathbf{1-q^4}})(\tfrac{\mathbf{q^5}}{\mathbf{1-q^5}})(\tfrac{\mathbf{1}}{\mathbf{1-q^6}})\\
&+(\tfrac{\mathbf{q^1}}{\mathbf{1-q^1}})(\tfrac{\mathbf{1}}{\mathbf{1-q^2}})(\tfrac{\mathbf{q^3}}{\mathbf{1-q^3}})(\tfrac{\mathbf{1}}{\mathbf{1-q^4}})(\tfrac{\mathbf{q^5}}{\mathbf{1-q^5}})(\tfrac{\mathbf{1}}{\mathbf{1-q^6}}).
\end{aligned}
$$

Sketch of the Proof for Profile $c = (2,1)$

Let us denote the first part of this new sum above by $P(q)$, that is,

$$P(q) := (\frac{1}{1-q^1})(\frac{1+q^2}{1-q^2})(\frac{1}{1-q^3})(\frac{1+q^4}{1-q^4})(\frac{1}{1-q^5})(\frac{1+q^6}{1-q^6}).$$

Then, how we get the other parts of the sum is as follows:

$$(\frac{\mathbf{q^1}}{\mathbf{1-q^1}})(\frac{\mathbf{1}}{\mathbf{1-q^2}})(\frac{1}{1-q^3})(\frac{1+q^4}{1-q^4})(\frac{1}{1-q^5})(\frac{1+q^6}{1-q^6}) = P(q)\cdot\frac{q^1}{1+q^2}$$

$$(\frac{1}{1-q^1})(\frac{\mathbf{1}}{\mathbf{1-q^2}})(\frac{\mathbf{q^3}}{\mathbf{1-q^3}})(\frac{\mathbf{1}}{\mathbf{1-q^4}})(\frac{1}{1-q^5})(\frac{1+q^6}{1-q^6}) = P(q)\cdot\frac{q^3}{(1+q^2)(1+q^4)}$$

$$(\frac{1}{1-q^1})(\frac{1+q^2}{1-q^2})(\frac{1}{1-q^3})\frac{\mathbf{1}}{\mathbf{1-q^4}}(\frac{\mathbf{q^5}}{\mathbf{1-q^5}})(\frac{\mathbf{1}}{\mathbf{1-q^6}}) = P(q)\cdot\frac{q^5}{(1+q^4)(1+q^6)}$$

$$(\frac{\mathbf{q^1}}{\mathbf{1-q^1}})(\frac{\mathbf{1}}{\mathbf{1-q^2}})(\frac{\mathbf{q^3}}{\mathbf{1-q^3}})(\frac{\mathbf{1}}{\mathbf{1-q^4}})(\frac{1}{1-q^5})(\frac{1+q^6}{1-q^6}) = P(q)\cdot\frac{q^1\cdot q^3}{(1+q^2)(1+q^4)}$$

$$(\frac{\mathbf{q^1}}{\mathbf{1-q^1}})(\frac{\mathbf{1}}{\mathbf{1-q^2}})(\frac{1}{1-q^3})(\frac{\mathbf{1}}{\mathbf{1-q^4}})(\frac{\mathbf{q^5}}{\mathbf{1-q^5}})(\frac{\mathbf{1}}{\mathbf{1-q^6}}) = P(q)\cdot\frac{q^1\cdot q^5}{(1+q^2)(1+q^4)(1+q^6)}$$

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$$(\frac{\mathbf{q^1}}{\mathbf{1-q^1}})(\frac{\mathbf{1}}{\mathbf{1-q^2}})(\frac{\mathbf{q^3}}{\mathbf{1-q^3}})(\frac{\mathbf{1}}{\mathbf{1-q^4}})(\frac{\mathbf{q^5}}{\mathbf{1-q^5}})(\frac{\mathbf{1}}{\mathbf{1-q^6}}) = P(q)\cdot\frac{q^1\cdot q^3\cdot q^5}{(1+q^2)(1+q^4)(1+q^6)}.$$

Sketch of the Proof for Profile $c = (2, 1)$

Let n be the number of distinct sizes of slices with shape a.

When $n = 0$, we have the following generating function:

$$S(q) := \frac{\prod_{k\geq 1}(1+q^{2k})}{\prod_{k\geq 1}(1-q^{k})} = \frac{(-q^2;q^2)_\infty}{(q;q)_\infty}.$$

When $n = 1$, we have the following generating function:

$$\left(\frac{q^1}{1+q^2} + \sum_{k\geq 1} \frac{q^{2k+1}}{(1+q^{2k})(1+q^{2k+2})}\right) \cdot S(q).$$

Sketch of the Proof for Profile $c = (2, 1)$

When $n = 2$, we have the following generating function:

$$\Bigg(\frac{q^1 \cdot q^3}{(1+q^2)(1+q^4)} + \sum_{k\geq 1} \frac{q^1 \cdot q^{2k+3}}{(1+q^2)(1+q^{2k+2})(1+q^{2k+4})}$$

$$+ \sum_{k\geq 1} \frac{q^{2k+1} \cdot q^{2k+3}}{(1+q^{2k})(1+q^{2k+2})(1+q^{2k+4})}$$

$$+ \sum_{n\geq 1} \sum_{k\geq 1} \frac{q^{2k+1} \cdot q^{2k+2n+3}}{(1+q^{2k})(1+q^{2k+2})(1+q^{2k+2n+2})(1+q^{2k+2n+4})} \Bigg) \cdot S(q).$$

Sketch of the Proof for Profile $c = (2, 1)$

Lemma

For any integer $k \geq 0$, we have the following identity:

$$\frac{q^{2k+1}}{(1+q^{2k})(1+q^{2k+2})} = \frac{q}{1-q^2} \cdot \left(\frac{-1}{1+q^{2k}} + \frac{1}{1+q^{2k+2}} \right).$$

For any integer $k \geq 0, n \geq 2$, we have the following identity:

$$\frac{q^{2k+1} \cdot q^{2k+3} \dots q^{2k+2n-1}}{(1+q^{2k})(1+q^{2k+2}) \dots (1+q^{2k+2n})} =$$
$$\frac{q}{1-q^{2n}} \left(\frac{-q^{2k+3} \cdot q^{2k+5} \dots q^{2k+2n-1}}{(1+q^{2k})(1+q^{2k+2}) \dots (1+q^{2k+2n-2})} + \frac{q^{2k+3} \cdot q^{2k+5} \dots q^{2k+2n-1}}{(1+q^{2k+2})(1+q^{2k+4}) \dots (1+q^{2k+2n})} \right).$$

Sketch of the Proof for Profile $c = (2,1)$

Lemma

For any integer $m \geq 1$, the following holds.

$$\sum_{k\geq 1} \frac{q^{2k+1}\cdot q^{2k+3}\dots q^{2k+2m-1}}{(1+q^{2k})(1+q^{2k+2})\dots(1+q^{2k+2m})}$$
$$= \frac{q^{2m}}{1-q^{2m}} \cdot \frac{q^1 q^3 \dots q^{2m-1}}{(1+q^2)(1+q^4)\dots(1+q^{2m})}.$$

Sketch of the Proof for Profile $c = (2, 1)$

Lemma

For any integer $m_i \geq 1$ where $i = 1, 2, \ldots, n$, the following identity holds:

$$\sum_{k_n \geq 1} \cdots \sum_{k_2 \geq 1} \sum_{k_1 \geq 1} \left(\frac{q^{2k_1+1} q^{2k_1+3} \ldots q^{2k_1+2m_1-1}}{(1+q^{2k_1})(1+q^{2k_1+2}) \ldots (1+q^{2k_1+2m_1})} \right)$$

$$\left(\frac{q^{2k_1+2k_2+(2m_1+1)}}{(1+q^{2k_1+2k_2+2m_1})} \frac{q^{2k_1+2k_2+(2m_1+3)} \ldots q^{2k_1+2k_2+(2m_1+2m_2-1)}}{(1+q^{2k_1+2k_2+2m_1+2}) \ldots (1+q^{2k_1+2k_2+2m_1+2m_2})} \right)$$

$$\left(\frac{q^{2k_1+2k_2+2k_3+(2m_1+2m_2+1)}}{(1+q^{2k_1+2k_2+2k_3+2m_1+2m_2})} \frac{q^{2k_1+2k_2+2k_3+(2m_1+2m_2+3)} \ldots q^{2k_1+2k_2+2k_3+(2m_1+2m_2+2m_3-1)}}{(1+q^{2k_1+2k_2+2k_3+2m_1+2m_2+2}) \ldots (1+q^{2k_1+2k_2+2k_3+2m_1+2m_2+2m_3})} \right)$$

$$\ldots \left(\frac{q^{2k_1+2k_2+\ldots+2k_n+(2m_1+2m_2+\ldots+2m_{n-1}+1)} q^{2k_1+2k_2+\ldots+2k_n+(2m_1+2m_2+\ldots+2m_{n-1}+3)}}{(1+q^{2k_1+2k_2+\ldots+2k_n+(2m_1+2m_2+\ldots+2m_{n-1})})(1+q^{2k_1+2k_2+\ldots+2k_n+(2m_1+2m_2+\ldots+2m_{n-1}+2)})} \right.$$

$$\left. \frac{\ldots q^{2k_1+2k_2+\ldots+2k_n+(2m_1+2m_2+\ldots+2m_n-1)}}{\ldots (1+q^{2k_1+2k_2+\ldots+2k_n+(2m_1+2m_2+\ldots+2m_{n-1}+2m_n)})} \right)$$

$$= \frac{q^1 q^3 \ldots q^{2m_1+2m_2+\ldots+2m_n-1}}{(1+q^2)(1+q^4) \ldots (1+q^{2m_1+2m_2+\ldots+2m_n})}.$$

$$\frac{q^{2(m_1+m_2+\ldots+m_n)}}{(1-q^{2(m_1+m_2+\ldots+m_n)})} \frac{q^{2(m_2+m_3+\ldots+m_n)}}{(1-q^{2(m_2+m_3+\ldots+m_n)})} \cdots \frac{q^{2(m_{n-1}+m_n)}}{(1-q^{2(m_{n-1}+m_n)})} \frac{q^{2m_n}}{(1-q^{2m_n})}.$$

Sketch of the Proof for Profile $c = (2, 1)$

- We compute the sums for cases $n = 1$ and $n = 2$.
- We reach a formula for general n.
- We proceed by induction on n, and we conclude the proof.

Our Results for $r = 2$, $\ell = 2$

The results we have found are given below, where the left hand sides are the alternative expressions we find while the right hand sides are the infinite products obtained by the Borodin's formula.

Profile $\mathbf{c} = (\mathbf{1}, \mathbf{1})$:

$$\frac{(-q;q^2)_\infty}{(q;q)_\infty} = \frac{1}{(q^4;q^4)_\infty (q;q^4)_\infty^2 (q^3;q^4)_\infty^2}.$$

Profile $\mathbf{c} = (\mathbf{2}, \mathbf{0})$:

$$\frac{(-q^2;q^2)_\infty}{(q;q)_\infty} = \frac{1}{(q;q)_\infty (q^2;q^4)_\infty}.$$

Our Results for $r = 2$, $\ell = 3$ and $r = 3$, $\ell = 2$

Profiles $\mathbf{c} = (\mathbf{2}, \mathbf{1})$ and $\mathbf{c} = (\mathbf{1}, \mathbf{1}, \mathbf{0})$:

$$\left(\sum_{n \geq 0} \frac{q^{n^2}}{(q^4; q^4)_n} \right) \cdot \frac{(-q^2; q^2)_\infty}{(q; q)_\infty} = \frac{1}{(q; q)_\infty (q; q^5)_\infty (q^4; q^5)_\infty}.$$

Profiles $\mathbf{c} = (\mathbf{3}, \mathbf{0})$ and $\mathbf{c} = (\mathbf{2}, \mathbf{0}, \mathbf{0})$:

$$\left(\sum_{n \geq 0} \frac{q^{n^2 + 2 \cdot n}}{(q^4; q^4)_n} \right) \cdot \frac{(-q^2; q^2)_\infty}{(q; q)_\infty} = \frac{1}{(q; q)_\infty (q^2; q^5)_\infty (q^3; q^5)_\infty}.$$

Our Results for $r = 2$, $\ell = 4$ and $r = 4$, $\ell = 2$

Profiles $\mathbf{c} = (4, 0)$ and $\mathbf{c} = (2, 0, 0, 0)$:

$$\left(\sum_{n\geq 0} \frac{q^{n(n+1)} \cdot (-q^2;q^2)_n}{(-q^3;q^2)_n \cdot (q^2;q^2)_n}\right) \cdot \frac{(-q^3;q^2)_\infty}{(q;q)_\infty} = \frac{1}{(q;q)_\infty (q^2;q^6)_\infty (q^3;q^6)_\infty (q^4;q^6)_\infty}.$$

Profiles $\mathbf{c} = (2, 2)$ and $\mathbf{c} = (1, 0, 1, 0)$:

$$\left(1 + \sum_{n\geq 1} \frac{2 \cdot q^{n(n+1)} \cdot (-q^2;q^2)_{n-1}}{(q^2;q^2)_n \cdot (-q;q^2)_n}\right) \cdot \frac{(-q;q^2)_\infty}{(q;q)_\infty} = \frac{1}{(q^6;q^6)_\infty (q;q^6)_\infty^2 (q^2;q^6)_\infty^2 (q^4;q^6)_\infty^2 (q^5;q^6)_\infty^2}.$$

Profiles $\mathbf{c} = (3, 1)$ and $\mathbf{c} = (1, 1, 0, 0)$:

$$\left(\sum_{n\geq 0} \frac{q^{n^2}}{(q^2;q^2)_n}\right) \cdot \frac{(-q^2;q^2)_\infty}{(q;q)_\infty} = \frac{1}{(q;q)_\infty (q;q^6)_\infty (q^3;q^6)_\infty (q^5;q^6)_\infty}.$$

Future Work

- Our plan is to continue to work on cylindric partitions with rank 2 and levels ≥ 5.
- We will try to generalize our results.
- We will also simultaneously work on cylindric partitions with profile $c = (1, 1, 1)$.

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THANK YOU!