d-Fold Partition Diamonds: Generating Functions and Partition Analysis

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Seminar in Partitions, q-Series, and Related Topics
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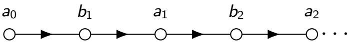
Integer partitions

A partition of a natural number n is a non-increasing sequence of integers

$$a_0 \geq b_1 \geq a_1 \geq b_2 \geq a_2 \geq \cdots \geq 0$$

whose sum is n.



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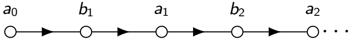
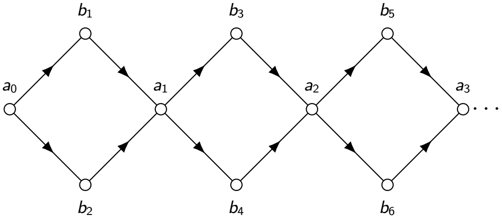


The counting function for partitions of n is $p(n)$, which has generating function

$$\sum_{n=0}^{\infty} p(n)q^n = \prod_{n=1}^{\infty} \frac{1}{1-q^n}.$$

Plane partition diamonds

Andrews, Paule, and Riese (2001) introduced plane partition diamonds, partitions whose parts satisfy



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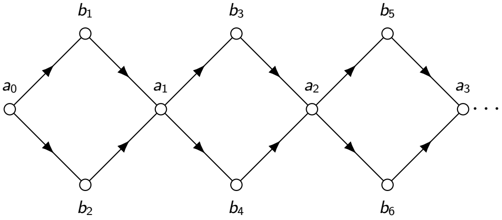

They found that

$$\sum_{n=0}^{\infty} d(n)q^n = \prod_{n=1}^{\infty} \frac{1+q^{3n-1}}{1-q^n},$$

where $d(n)$ counts the number of plane partition diamonds of n.

Schmidt type partitions

A (classical) Schmidt type partition of a natural number n is a non-increasing sequence of integers

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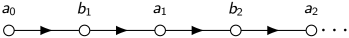

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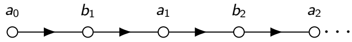

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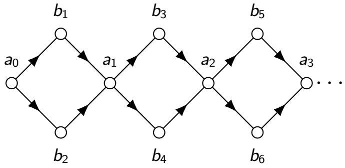

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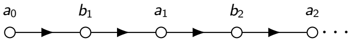

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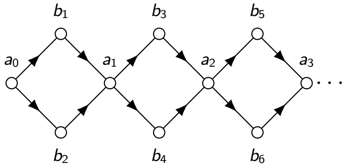

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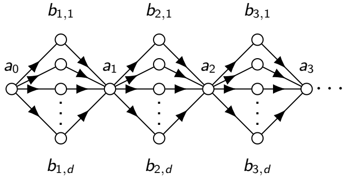

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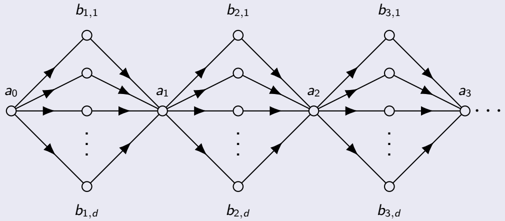

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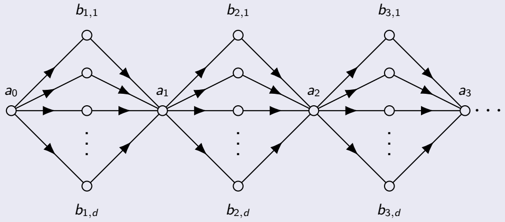

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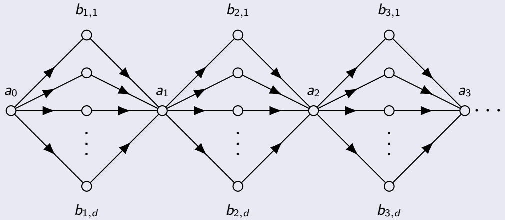

Schmidt type d-fold partition diamonds are defined in the natural way. We let $r_d(n)$ and $s_d(n)$ be the counting functions for d-fold partition diamonds of n and their Schmidt type counterpart, respectively.

Results

Proposition (D.-Jameson-Sellers-Wilson)

We have

$$\sum_{n=0}^{\infty} r_d(n)q^n = \prod_{n=1}^{\infty} \frac{F_d(q^{(n-1)(d+1)+1}, q)}{1-q^n}$$

and

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Here $F_d(q_0, w)$ are polynomials defined recursively by $F_1(q_0, w) := 1$ and

$$F_d(q_0, w) = \frac{(1-q_0 w^d)F_{d-1}(q_0, w) - w(1-q_0)F_{d-1}(q_0 w, w)}{1-w}.$$

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The proof relies on MacMahon's partition analysis.

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$$\underset{\geq}{\Omega} \sum_{s_1=-\infty}^{\infty} \cdots \sum_{s_r=-\infty}^{\infty} A_{s_1,\cdots,s_r} \lambda_1^{s_1} \cdots \lambda_r^{s_r} := \sum_{s_1=0}^{\infty} \cdots \sum_{s_r=0}^{\infty} A_{s_1,\ldots,s_r},$$

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Elimination formulae make $\Omega_{\geq}$ a very powerful tool in computing generating functions.

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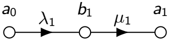

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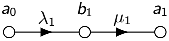

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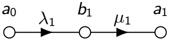

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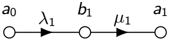

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Letting $q_0 = q_1 = w = q$, the generating function is

$$\frac{1}{(1-q)(1-q^2)(1-q^3)}.$$

Extending MacMahon's identities

We will need the following generalization of the previous identities.

Lemma (D.-Jameson-Sellers-Wilson)

For $d \geq 1$ and $j \in \mathbb{Z}$, we have

$$\underset{\geq}{\Omega} \frac{\lambda^j}{(1-\lambda x_1)\cdots(1-\lambda x_d)(1-\lambda^{-1}y)} = \frac{1}{(1-y)}\left[\frac{1}{(1-x_1)\cdots(1-x_d)} - \frac{y^{j+1}}{(1-x_1 y)\cdots(1-x_d y)}\right].$$

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Proof (sketch). We have

$$\underset{\geq}{\Omega} \frac{\lambda^j}{(1-\lambda x_1)\cdots(1-\lambda x_d)(1-\lambda^{-1}y)} = \underset{\geq}{\Omega} \sum_{a_1,\ldots,a_{d+1}\geq 0} \lambda^{j+a_1+\cdots+a_d-a_{d+1}} x_1^{a_1}\cdots x_d^{a_d} y^{a_{d+1}}$$

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$$\underset{\geq}{\Omega} \frac{\lambda^j}{(1-\lambda x_1)\cdots(1-\lambda x_d)(1-\lambda^{-1}y)} = \frac{1}{(1-y)}\left[\frac{1}{(1-x_1)\cdots(1-x_d)} - \frac{y^{j+1}}{(1-x_1 y)\cdots(1-x_d y)}\right].$$

Proof (sketch). We have

$$\begin{aligned}\underset{\geq}{\Omega} \frac{\lambda^j}{(1-\lambda x_1)\cdots(1-\lambda x_d)(1-\lambda^{-1}y)} &= \underset{\geq}{\Omega} \sum_{a_1,\ldots,a_{d+1}\geq 0} \lambda^{j+a_1+\cdots+a_d-a_{d+1}} x_1^{a_1}\cdots x_d^{a_d} y^{a_{d+1}} \\ &= \underset{\geq}{\Omega} \sum_{\substack{a_1,\ldots,a_d\geq 0 \\ 0\leq k\leq a_1+\cdots+a_d+j}} \lambda^k x_1^{a_1}\cdots x_d^{a_d} y^{a_1+\cdots+a_d+j-k}\end{aligned}$$

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\end{aligned}$$

and the result follows from finite geometric series. $\square$

Crude forms for d-fold partition diamonds

Set $D_{d,n} := D_{d,n}(q_0, q_1, \ldots, q_n; w)$ to be the generating function for d-fold partition diamonds of fixed length n.

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$$g_n := g_{n,d} := \frac{1 - \frac{\lambda_{n+1,1} \cdots \lambda_{n+1,d}}{\mu_{n,1} \cdots \mu_{n,d}} q_{n+1}}{1 - \frac{q_{n+1}}{\mu_{n,1} \cdots \mu_{n,d}}}$$

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Fact:

$$D_{d,n} = \underset{\geq}{\Omega}\, h \cdot f_1 \cdots f_n \cdot g_n.$$

The crude forms (continued)

For $\rho \geq 0$ we will need $D_{d,n}^{(\rho)} := D_{d,n}^{(\rho)}(q_0, q_1, \cdots, q_n; w)$, which is defined to be the generating function for d-fold partition diamonds of fixed length n, with $a_n \geq \rho$.

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This can be shown algebraically, or combinatorially via a bijection between the sets of partitions.

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Theorem

Theorem (D.-Jameson-Sellers-Wilson)

For $d \geq 1$ and $n \geq 1$,

$$D_{d,n}(q_0, \ldots, q_n; w) = \left(\prod_{k=0}^{n-1} \frac{F_d(Q_k w^{dk}, w)}{(1 - Q_k w^{dk}) \cdots (1 - Q_k w^{dk+d})} \right) \frac{1}{1 - Q_n w^{dn}},$$

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where F_d is as previously defined and $Q_k := q_0 q_1 \cdots q_k$.

Note that

$$\sum_{n=0}^{\infty} r_d(n) q^n = \lim_{n\to\infty} D_{d,n}(q, \cdots, q; q) = \prod_{n=1}^{\infty} \frac{F_d(q^{(n-1)(d+1)+1}, q)}{1 - q^n}$$

$$\sum_{n=0}^{\infty} s_d(n) q^n = \lim_{n\to\infty} D_{d,n}(q, \cdots, q, 1) = \prod_{n=1}^{\infty} \frac{F_d(q^n), 1}{(1 - q^n)^{d+1}}.$$

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Here we sketch the proof, by:

1. treating the case $d = 1, n = 1$ (i.e., $D_{1,1}$)
2. for $n = 1$ fixed, inductively proving the result for $D_{d,1}$
3. for arbitrary d, inducting on n starting from $D_{d,1}$

The base case $D_{1,1}$

- We must show

$$D_{1,1}(q_0, q_1; w) = \frac{1}{(1 - q_0)(1 - q_0 w)(1 - q_0 q_1 w)}.$$

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- But a 1-fold partition diamond of length 1 is classical partition with at most three parts.
- This was exactly the conclusion of the previous example.

Induction on d

- Now we want to show for $d \geq 1$ that

$$D_{d,1}(q_0, q_1; w) = \frac{F_d(q_0, w)}{(1-q_0)(1-q_0 w)\cdots(1-q_0 w^d)\cdot(1-q_0 q_1 w^d)}$$

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- Suppose the result holds for $d - 1$. Then

$$D_{d,1}(q_0, q_1; w) = \underset{\geq}{\Omega} \frac{1}{(1 - \lambda_1 \cdots \lambda_d \, q_0) \cdot (1 - \lambda_1^{-1} \mu_1 w) \cdots (1 - \lambda_d^{-1} \mu_d w)} \cdot \frac{1}{(1 - \mu_1^{-1} \cdots \mu_d^{-1} q_1)}$$

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&= \underset{\geq}{\Omega} \frac{F_{d-1}(\lambda_d q_0, w)}{(1-\lambda_d q_0)\cdots(1-\lambda_d q_0 w^{d-1})(1-\frac{\lambda_d}{\mu_d} q_0 q_1 w^{d-1})(1-\frac{\mu_d}{\lambda_d} w)} \\
&= \frac{1}{1-q_0 q_1 w^d}\, \underset{\geq}{\Omega} \frac{F_{d-1}(\lambda_d q_0, w)}{(1-\lambda_d q_0)\cdots(1-\lambda_d q_0 w^{d-1})(1-\lambda_d^{-1} w)}
\end{aligned}$$

by eliminating μ_d.

Induction on d (continued)

- Set

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$$= \frac{\sum_{i=0}^{n} a_i(w) q_0^i}{(1 - q_0) \cdots (1 - q_0 w^d)(1 - q_0 q_1 w^d)} \left[\frac{1 - q_0 w^d}{1 - w} - \frac{w^{i+1}(1 - q_0)}{1 - w} \right],$$

by applying our lemma.

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by applying our lemma. So we have shown the numerator is

$$\sum_{i=0}^{n} a_i(w) q_0^i \left[\frac{1-q_0 w^d}{1-w} - \frac{w^{i+1}(1-q_0)}{1-w} \right],$$

which simplifies to $F_d(q_0, w)$.

Induction on n (sketch)

- We want to show

$$D_{d,n+1}(q_0,\ldots,q_n;w) = \frac{F_d(Q_n w^{dn},w)D_n(q_0,\cdots,q_n;w)}{(1-Q_n w^{dn+1})\cdots(1-Q_n w^{dn+d})(1-Q_{n+1}w^{dn+d})}.$$

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- Use the case $D_{d,1}$ on the last $d+2$ terms (in red), obtaining an expression involving $D_{d,n}^{(\rho)}$.
- Apply the relationship between $D_{d,n}$ and $D_{d,n}^{(\rho)}$ and simplify.

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Eulerian polynomials

The Eulerian polynomials A_d are defined for non-negative integers d by $A_0(q) = 1$ and for $d \geq 1$,

$$A_d(q) = (1 + (d-1)q)A_{d-1}(q) + q(1-q)A'_{d-1}(q).$$

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Thus

$$\sum_{n=0}^{\infty} s_d(n) = \prod_{n=1}^{\infty} \frac{F_d(q^n, 1)}{(1-q^n)^{d+1}} = \prod_{n=1}^{\infty} \frac{A_d(q^n)}{(1-q^n)^{d+1}}$$

Eulerian polynomials

The Eulerian polynomials A_d are defined for non-negative integers d by $A_0(q) = 1$ and for $d \geq 1$,

$$A_d(q) = (1 + (d-1)q)A_{d-1}(q) + q(1-q)A'_{d-1}(q).$$

One quickly verifies that $F_d(q, 1) = A_d(q)$ for all $d \geq 1$.

Euler (1768) showed

$$\frac{A_d(q)}{(1-q)^{d+1}} = \sum_{j=0}^{\infty} (j+1)^d q^j.$$

Thus

$$\sum_{n=0}^{\infty} s_d(n) = \prod_{n=1}^{\infty} \frac{F_d(q^n, 1)}{(1-q^n)^{d+1}} = \prod_{n=1}^{\infty} \frac{A_d(q^n)}{(1-q^n)^{d+1}}$$
$$= \prod_{n=1}^{\infty} \left(\sum_{j=0}^{\infty} (j+1)^d q^{jn} \right).$$

Schmidt type simplification

This simplified generating function allows us to prove a wide variety of congruences for $s_d(n)$, such as the infinite family

$$s_d(2n+1) \equiv 0 \pmod{2^d}$$

for all $d \geq 1$ and all $n \geq 0$.

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Much more on congruences next week!

Thank you! Questions?