# Berkovich and Uncu's Conjectures regarding partition inequalities

Damanvir Singh Binner IISER Mohali, India

Joint work with A. Rattan (SFU, Canada)

Specialty Seminar in Partition Theory, q-Series and Related Topics, Michigan Technological University. Nov. 4, 2021.

# Notation

- Let $\pi$ be a partition. We write $\pi = (1^{f_1}, 2^{f_2}, \ldots)$, where $f_i$ is the number of times a part $i$ occurs in $\pi$, also known as the frequency of $i$.
- Thus $(4, 4, 2, 2, 1)$ is expressed as $(1^1, 2^2, 4^2)$.
- In this notation, it is clear that

$$|\pi| = \sum_{i \geq 1} i f_i.$$

# First Conjecture

- $C_{L,s,1}$ denotes the set of partitions where the smallest part is $s$, all parts are $\leq L+s$ and $L+s-1$ does not appear as a part.
- $C_{L,s,2}$ is the set of partitions with parts in $\{s+1,\ldots,L+s\}$.

Conjecture (Berkovich and Uncu (2019))

*There exists an $M$, which only depends on $s$, such that*

$$|\{\pi \in C_{L,s,1} : |\pi| = N\}| \geq |\{\pi \in C_{L,s,2} : |\pi| = N\}|$$

*for every $N \geq M$.*

# Second Conjecture

- if $L \geq s+1$, $C^*_{L,s,1}$ denotes the set of partitions where the smallest part is $s$, all parts are $\leq L+s$, and $L$ does not appear as a part.

Conjecture (Berkovich and Uncu (2019))

*For positive integers $L \geq 3$ and $s$, there exists an $M$, which only depends on $s$, such that*

$$|\{\pi \in C^*_{L,s,1} : |\pi| = N\}| \geq |\{\pi \in C_{L,s,2} : |\pi| = N\}|,$$

*for every $N \geq M$.*

# Third Conjecture

- The *q-Pochhammer symbol* is defined as

$$(a;q)_n = (1-a)(1-aq)\cdots(1-aq^{n-1}).$$

- The series $H_{L,s,k}(q)$ is defined as

$$H_{L,s,k}(q) = \frac{q^s(1-q^k)}{(q^s;q)_{L+1}} - \left(\frac{1}{(q^{s+1};q)_L} - 1\right).$$

**Conjecture (Berkovich and Uncu (2019))**

*For* $k \geq s+1$, $H_{L,s,k}(q)$ *is eventually positive.*

# Our Main Result

**Theorem**

*For positive integers $L$, $s$ and $k$, with $L \geq 3$ and $k \geq s+1$, the coefficient of $q^N$ in $H_{L,s,k}(q)$ is positive whenever $N \geq \Gamma(s)$, where $\Gamma(s)$ can be written explicitly in terms of $s$ only.*

- If $L \geq 3s+3$ and $k \geq 2s+2$, the bound is $O(s^5)$.
- If $L \geq 3s+3$ and $k \leq 2s+1$, the bound is $O(s^{10})$.
- If $L \leq 3s+2$, the bound is $O\left((6s)^{(6s)^{18s}}\right)$.

# Fourth Conjecture

- The series $G_{L,1}(q)$ is defined as

$$G_{L,1}(q) = \sum_{\substack{\pi\in\mho,\\ s(\pi)=1,\\ l(\pi)-s(\pi)\leq L}} q^{|\pi|} - \sum_{\substack{\pi\in\mho,\\ s(\pi)\geq 2,\\ l(\pi)-s(\pi)\leq L}} q^{|\pi|},$$

  where $s(\pi)$ and $l(\pi)$ denote the smallest and largest parts of $\pi$, respectively, and $\mho$ denotes the set of partitions $\pi$ with $|\pi| > 0$.

- The series $G_{L,2}(q)$ is defined as

$$G_{L,2}(q) = \sum_{\substack{\pi\in\mho,\\ s(\pi)=2,\\ l(\pi)-s(\pi)\leq L}} q^{|\pi|} - \sum_{\substack{\pi\in\mho,\\ s(\pi)\geq 3,\\ l(\pi)-s(\pi)\leq L}} q^{|\pi|},$$

**Theorem (Berkovich and Uncu (2019))**

*For $L \geq 1$,*

$$G_{L,1}(q) = \frac{H_{L,1,L}(q)}{1-q^L} \succeq 0,$$
$$G_{L,2}(q) = \frac{H_{L,2,L}(q)}{1-q^L}.$$

**Conjecture (Berkovich and Uncu (2019))**

*For $L = 3$,*

$$G_{L,2}(q) + q^3 + q^9 + q^{15} \succeq 0,$$

*for $L = 4$,*

$$G_{L,2}(q) + q^3 + q^9 \succeq 0,$$

*and for $L \geq 5$,*

$$G_{L,2}(q) + q^3 \succeq 0.$$

Theorem (Berkovich and Uncu (2019))

*For $L \geq 1$,*

$$G_{L,1}(q) = \frac{H_{L,1,L}(q)}{1-q^L} \succeq 0,$$

$$G_{L,2}(q) = \frac{H_{L,2,L}(q)}{1-q^L}.$$

Conjecture (Berkovich and Uncu (2019))

*For $L = 3$,*

$$G_{L,2}(q) + q^3 + q^9 + q^{15} \succeq 0,$$

*for $L = 4$,*

$$G_{L,2}(q) + q^3 + q^9 \succeq 0,$$

*and for $L \geq 5$,*

$$G_{L,2}(q) + q^3 \succeq 0.$$

Proved (B-Rattan 2020)

# Helping Results

### Lemma (Sylvester (1882))

*For natural numbers a and b such that* $\gcd(a,b) = 1$*, the equation* $ax + by = n$ *has a solution* $(x, y)$*, with x and y nonnegative integers, whenever* $n \geq (a-1)(b-1)$*.*

### Lemma

*Let s and n be positive integers such that* $n \geq s+1$*. Then, the equation*

$$n = (s+1)X_{s+1} + (s+2)X_{s+2} + \cdots + (2s+1)X_{2s+1}$$

*has a solution* $(X_{s+1}, X_{s+2}, \ldots, X_{2s+1})$*, where* $X_i$ *is a nonnegative integer for all* $i$*.*

# Proofs of Zang and Zeng

- Zang and Zeng also gave proofs of the first three conjectures.
- Their proofs are analytic for some cases and combinatorial for other cases, whereas our methods are entirely combinatorial.
- While their methods are somewhat more straightforward than ours, they produce results that are asymptotic and therefore do not give explicit bounds.
- In contrast, our methods produce explicit bounds on when $H_{L,s,k}(q)$ has positive coefficients and also lead to a proof of the fourth conjecture.

# Proof of First Conjecture for $L \geq s+3$

- $F(s) = (10s-2)(15s-3) + 8s$;
- $\kappa(s) = (12s-1)\Big((s+1) + (s+2) + \cdots (F(s)-1)\Big) + 1.$

**Theorem**

*If s and L are positive integers with $L \geq s+3$ and $N \geq \kappa(s)$, then*

$$|\{\pi \in C_{L,s,1} : |\pi| = N\}| \ \geq \ |\{\pi \in C_{L,s,2} : |\pi| = N\}|.$$

Sketch of Proof: We construct an injective map

$$\phi : \{\pi \in C_{L,s,2} : |\pi| = N\} \ \to \ \{\pi \in C_{L,s,1} : |\pi| = N\}.$$

# Strategy

- Recall that $C_{L,s,2}$ consists of partitions with all parts lying between $s+1$ and $L+s$.
- Any $\pi \in C_{L,s,2}$ has the form

$$\pi = \left((s+1)^{f_{s+1}}, \ldots, (L+s-1)^{f_{L+s-1}}, (L+s)^{f_{L+s}}\right).$$

- $C_{L,s,1}$ denotes the set of partitions where the smallest part is $s$, all parts are $\leq L+s$ and $L+s-1$ does not appear as a part.
- To map $\pi$ to a partition in $C_{L,s,1}$, we need to remove all parts of $L+s-1$ (if any) and add some parts of $s$, while ensuring that it still remains a partition of $N$.
- We consider several different cases depending on the frequency of $L+s-1$ in $\pi$, which we denote by $f$.

- Our strategy for ensuring that $\phi$ is injective is to construct the map in such a way that in different cases, the partitions in the image have different frequencies of $s$.

| Case | Possible frequencies of $s$ |
|---|---|
| 1(a) | Odd numbers other than 15 |
| 1(b) | 14 |
| 2(a) | Multiples of 12 |
| 2(b)(i) | 15 |
| 2(b)(ii) | 20 |
| 2(b)(iii) | 2,4,6,8 |

Table: The frequency of $s$ in the image of a partition under the function $\phi$ in the different cases.

Case 1: Suppose that $f \geq 1$.

- Remove the $f$ parts of $L + s - 1$ and to compensate add back $2f - 1$ parts of $s$.
- Then we further need to add the number

$$(L + s - 1)f - s(2f - 1) = (L - s - 1)f + s.$$

Case 1: Suppose that $f \geq 1$.

- Remove the $f$ parts of $L+s-1$ and to compensate add back $2f-1$ parts of $s$.
- Then we further need to add the number

$$(L+s-1)f - s(2f-1) = \underbrace{(L-s-1)}_{\geq 1}\underbrace{f}_{\geq 1} + s \geq s+1.$$

- By an application of the division algorithm, this number can be added by adding some parts of $s+1, s+2, \ldots, 2s+1$.
- The frequency of $s$ in the image is $2f-1$ and thus $f$ can be recovered from there.

- Case 2: Suppose that $f = 0$.
- Case 2(a): Suppose there exists $m < F(s)$ with $f_m \geq 12s$. Let $m_0$ be the least such number. Then define

$$\phi(\pi) = \left(s^{12m_0}, (s+1)^{f_{s+1}}, \dots, m_0^{f_{m_0}-12s}, \dots\right).$$

- From the frequency of $s$ in the image, we can recover $m_0$.
- Case 2(b): Suppose that for every $m < F(s)$, $f_m < 12s$. Then,

$$\pi = \left(\underbrace{(s+1)^{f_{s+1}}, \dots, (F(s)-1)^{f_{F(s)-1}}}_{low\ \ freq.}, \dots, \right).$$

- Case 2: Suppose that $f = 0$.
- Case 2(a): Suppose there exists $m < F(s)$ with $f_m \geq 12s$. Let $m_0$ be the least such number. Then define

$$\phi(\pi) = \left( s^{12m_0}, (s+1)^{f_{s+1}}, \ldots, m_0^{f_{m_0}-12s}, \ldots \right).$$

- From the frequency of $s$ in the image, we can recover $m_0$.
- Case 2(b): Suppose that for every $m < F(s)$, $f_m < 12s$. Then,

$$\pi = \left( \underbrace{(s+1)^{f_{s+1}}, \ldots, (F(s)-1)^{f_{F(s)-1}}}_{low \quad freq.}, \ldots, \right).$$

- Since $N \geq \kappa(s)$, there must exist an $h \geq F(s)$ such that $f_h > 0$. Let $l$ be the least such number.

- Thus, we can write $\pi$ as

$$\pi = \Big((s+1)^{f_{s+1}}, \ldots, (F(s)-1)^{f_{F(s)-1}}, \ldots, l^{f_l}, \ldots\Big).$$

- Since $l \geq F(s)$, so $l - 8s \geq (10s-2)(15s-3)$, which is the Frobenius number of $10s-1$ and $15s-2$. Thus,

$$l - 8s = (10s-1)x_l + (15s-2)y_l.$$

- If we define

$$\phi(\pi) = \Big(s^8, \ldots, (10s-1)^{x_l + f_{10s-1}}, \ldots, (15s-2)^{y_l + f_{15s-2}},$$
$$\ldots, (F(s)-1)^{f_{F(s)-1}}, l^{f_l - 1}, \ldots\Big).$$

- Thus, we can write $\pi$ as

$$\pi = \Big( (s+1)^{f_{s+1}}, \ldots, (F(s)-1)^{f_{F(s)-1}}, \ldots, l^{f_l}, \ldots \Big).$$

- Since $l \geq F(s)$, so $l - 8s \geq (10s-2)(15s-3)$, which is the Frobenius number of $10s-1$ and $15s-2$. Thus,

$$l - 8s = (10s-1)x_l + (15s-2)y_l.$$

- If we define

$$\phi(\pi) = \Big( s^8, \ldots, (10s-1)^{x_l}, \ldots, (15s-2)^{y_l}, \\ \ldots, (F(s)-1)^{f_{F(s)-1}}, l^{f_l - 1}, \ldots \Big).$$

- Case 2(b)(i): If $f_{5s+1} \geq 1$ and $f_{10s-1} \geq 1$, then define

$$\phi(\pi) = \left(s^{15}, \ldots, (5s+1)^{f_{5s+1}-1}, \ldots, (10s-1)^{f_{10s-1}-1}, \ldots\right).$$

- Case 2(b)(ii): If $f_{5s+2} \geq 1$ and $f_{15s-2} \geq 1$, then define

$$\phi(\pi) = \left(s^{20}, \ldots, (5s+2)^{f_{5s+2}-1}, \ldots, (15s-2)^{f_{15s-2}-1}, \ldots\right).$$

- Case 2(b)(iii): If $f_{5s+1} = 0$ or $f_{10s-1} = 0$ and $f_{5s+2} = 0$ or $f_{15s-2} = 0$. Then, at least one of the following statements is true:

- $\star$ $T_1$: $f_{5s+1} = 0$ and $f_{5s+2} = 0$;
- $\star\star$ $T_2$: $f_{5s+1} = 0$ and $f_{15s-2} = 0$;
- $\star\star\star$ $T_3$: $f_{10s-1} = 0$ and $f_{5s+2} = 0$;
- $\star\star\star\star$ $T_4$: $f_{10s-1} = 0$ and $f_{15s-2} = 0$.

Suppose $T_4$ is true.

- Using Frobenius numbers,

$$l - 8s = (10s - 1)x_l + (15s - 2)y_l.$$

- Define

$$\phi(\pi) = \Big(s^8, (s+1)^{f_{s+1}}, \dots, (10s-1)^{x_l}, \dots, (15s-2)^{y_l}, \\ \dots, (F(s)-1)^{f_{F(s)-1}}, \dots, l^{f_l - 1}, \dots\Big).$$

- From $x_l$ and $y_l$, we can recover $l$.
- This completes the proof of the first conjecture in the given case $L \geq s + 3$.

# Proof for $L \leq s+2$

Allowed interval:

$$\{s+1, s+2, \cdots, L+s\} \subset \{s+1, s+2, \cdots, 2s+2\}.$$

- $S_s = (s+1) + (s+2) + \cdots + (2s+2)$;
- $P_s = (s+1)(s+2)\cdots(2s+2)$;
- $Q_s = \left(P_s^2 - 1\right)(s+2) + 2$;
- $\gamma(s) = S_s \left(P_s^{Q_s} + (Q_s - 4)\, P_s\right)$.

**Theorem**

*If $s$ and $L$ are positive integers with $L \geq s+3$ and $N \geq \gamma(s)$, then*

$$|\{\pi \in C_{L,s,1} : |\pi| = N\}| \geq |\{\pi \in C_{L,s,2} : |\pi| = N\}|.$$

# Sketch of Proof

Set $f = f_{L+s-1}$, so a partition in the domain has the form

$$\pi = ((s+1)^{f_{s+1}}, \ldots, (L+s-1)^{f}, (L+s)^{f_{L+s}}).$$

Case 1: Suppose $f = 0$. Since $N \geq \gamma(s)$ is large enough, there is an $m$ such that $s+1 \leq m \leq L+s$ and $f_m \geq s$. Let $m_0$ be the least such number. Then define

$$\psi(\pi) = (s^{m_0}, (s+1)^{f_{s+1}}, \ldots, m_0^{f_{m_0}-s}, \ldots).$$

The frequency of $s$ in a partition in its image is in the set

$$U_1 = \{s+1, \ldots, L+s\} \subset \{s+1, \ldots, 2s+2\}.$$

Case 2: Suppose that $f \neq 0$ in $\pi$.

# Creating Gaps

Case 2(a): Suppose $\pi$ has $f \geq P_s^2$. First suppose $P_s^2 \leq f < P_s^3$. Then, we use

$$(L+s-1)f = s(f-P_s)+(s+1)x_f+(s+2)y_f.$$

Next suppose $P_s^3 \leq f < P_s^4$. Then, we use

$$(L+s-1)f = s(f)+(s+1)x_f+(s+2)y_f.$$

Next suppose $P_s^4 \leq f < P_s^5$. Then, we use

$$(L+s-1)f = s(f+P_s)+(s+1)x_f+(s+2)y_f.$$

Allowed because the difference

$$(L+s-1)f - s(f+P_s) = (L-1)f - sP_s$$

is still a large number for $P_s^4 \leq f < P_s^5$.

# The Gaps

| Values of $f$ | Possible frequencies of $s$ in $\psi(\pi)$ | Gaps created |
|---|---|---|
| $[P_s^2, P_s^3)$ | $[P_s^2 - P_s, P_s^3 - P_s)$ | – |
| $[P_s^3, P_s^4)$ | $[P_s^3, P_s^4)$ | $[P_s^3 - P_s, P_s^3)$ |
| $[P_s^4, P_s^5)$ | $[P_s^4 + P_s, P_s^5 + P_s)$ | $[P_s^4, P_s^4 + P_s)$ |
| $[P_s^5, P_s^6)$ | $[P_s^5 + 2P_s, P_s^6 + 2P_s)$ | $[P_s^5 + P_s, P_s^5 + 2P_s)$ |

Table: The gaps created by suitably choosing frequencies of $s$ in the image.

Thus the gaps are: $[(P_s^n + (n-4)P_s), (P_s^n + (n-3)P_s))$ $\forall n \geq 3$.
We will only require the gaps $P_s^n + (n-4)P_s$ for our purpose.

Case 2(b): Suppose that $1 < f < P_s^2$ in $\pi$. For any $0 < i < P_s^2$ and $1 \leq h \leq L$, set

$$m_{i,h} = P_s^{(i-1)L+(h+2)} + ((i-1)L + (h-2))P_s.$$

Since $N \geq \gamma(s)$ is large enough, there exists an $h$ such that $1 \leq h \leq L$ and

$$f_{s+h} \geq m_{f,h}.$$

Let $p$ be the least integer $1 \leq h \leq L$ for which this equation is satisfied. Then, $f_{s+p} \geq m_{f,p}$. Notice that $m_{f,p}$ is divisible by $P_s$, and thus also by $(s+p)$; hence, we can define $j_{f,p}$ by

$$j_{f,p} = \frac{s m_{f,p}}{s+p}.$$

Note that $f_{s+p} \geq j_{f,p}$. Our idea is to remove $j_{f,p}$ parts of $s+p$ and compensate by adding $m_{f,p}$ parts of $s$.

# Adjusting $L + s - 1$ terms

We remove all the $f$ parts of $L + s - 1$ and compensate in the following way:

- If $f$ is even, we add $L + s - 2$ and $L + s$ both with a frequency of $\frac{f}{2}$.
- If $f \geq 3$ is odd, we add $L + s - 3$, $L + s - 2$ and $L + s$ with a frequency of $1$, $\frac{f-3}{2}$ and $\frac{f+1}{2}$ respectively.

# References

Andrews, G., Beck, M. and Robbins, N., 2015. Partitions with fixed differences between largest and smallest parts. Proceedings of the American mathematical society, 143(10), pp.4283-4289.

Binner, D.S. and Rattan, A., 2021. On Conjectures Concerning the Smallest Part and Missing Parts of Integer Partitions. Annals of Combinatorics, 25:697–728.

Berkovich, A. and Uncu, A.K., 2019. Some elementary partition inequalities and their implications. Annals of Combinatorics, 23(2), pp.263-284.

Zang, W.J. and Zeng, J., 2020. Gap between the largest and smallest parts of partitions and Berkovich and Uncu's conjectures. arXiv, pp.arXiv-2004.

THANK YOU