Some Legendre theorems for partitions with early conditions by

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Online Talk

9 February, 2023

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Definition

A partition of n shall be taken as a representation $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_m)$ where $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_m \geq 1$ such that $\sum_{i=1}^{m} \lambda_i = n$

- Multiplicity notation: $(\mu_1^{m_1}, \mu_2^{m_2}, \mu_3^{m_3}, \cdots, \mu_\ell^{m_\ell})$ in which m_i denotes the multiplicity of the part μ_i and $\mu_1 > \mu_2 > \cdots > \mu_\ell$. If $m_i = 1$ for all i, we have a partition into distinct parts.
- $d(n)$: the number of partitions of n into distinct parts. Let $d^e(n)$ (resp. $d^o(n)$) be the number of $d(n)$-partitions with even (resp. odd) length.

It turns out (by A.M. Legendre) that proved

$$d^e(n) - d^o(n) = \begin{cases} (-1)^j, & \text{if } n = j(3j \pm 1)/2, j \geq 0; \\ 0, & \text{otherwise.} \end{cases} \tag{1}$$

Hence, any partition theorem in the *shape* of Theorem 1 shall be called a Legendre theorem.
Another class of interest is the class of partitions with initial repetitions. This class was introduced by George Andrews.

Definition

Let k be a positive integer. A partition of n with initial k-repetitions is one in which if j is repeated at least k times, every positive integer less than j is repeated at least k times (this means that all parts $> j$ have multiplicities stricly less than k)

For example, $(3^3 2^4, 1^5)$ is a partition with initial 3-repetitions. It is also a partition with initial 4-repetitions

Theorem (Andrews, 2009)

The number of partitions of n with initial k-repetitions is equal to the number of partitions in which each part appears not more than $2k-1$ times.

Proof:

$$\sum_{n=0}^{\infty} \frac{q^{k(1+2+3+\cdots+n)}}{(1-q)(1-q^2)\cdots(1-q^n)} \prod_{j=n+1}^{\infty} (1+q^j+\cdots+q^{(k-1)j})$$

is equal to

$$\prod_{n=1}^{\infty} \frac{1-q^{2kn}}{1-q^n}.$$

A bijective proof of this theorem was given by W. Keith.

Notation:

$f_e(n,m)$: the number of partitions of n with initial 2-repetitions with m *different* parts and an even number of distinct parts

$f_o(n,m)$: the number of partitions of n with initial 2-repetitions with m *different* parts and an odd number of distinct parts.

We have the following Legendre theorem.

Theorem (Andrews, 2009)

$$f_e(n,m) - f_o(n,m) = \begin{cases} (-1)^j, & \textit{if } m = j,\ n = \frac{j(j+1)}{2}; \\ 0, & \textit{otherwise.} \end{cases}$$

This theorem has a combinatorial proof.

A broader class called partitions with early conditions were studied.

Question: Can we find more of Legendre theorems in the category of partitions with early conditions.

Tools:

We assume that $q \in \mathbb{C}$ and $|q| < 1$:

$$\sum_{n=-\infty}^{\infty} z^n q^{n(n+1)/2} = \prod_{n=1}^{\infty} (1-q^n)(1+zq^n)(1+z^{-1}q^{n-1}) \quad (2)$$

where $z \neq 0$,

$$\sum_{n=-\infty}^{\infty} (-1)^n q^{n^2} = \prod_{n=1}^{\infty} \frac{1-q^n}{1+q^n}. \quad (3)$$

Main Results

• On Andrews partitions with initial 2-repetitions:
The following factorisations hold:

Lemma

$$\sum_{n=1}^{\infty} \frac{q^{2n^2-n}}{(q;q)_{2n-1}} = (-q;q)_\infty \sum_{n=1}^{\infty} (-1)^{n+1} q^{n^2}, \tag{4}$$

$$\sum_{n=0}^{\infty} \frac{q^{2n^2+n}}{(q;q)_{2n}} = (-q;q)_\infty \sum_{n=0}^{\infty} (-1)^{n} q^{n^2}. \tag{5}$$

PROOF:

$d(n) = d^e(n) - d^o(n) + 2d^o(n)$ so that

$$\sum_{n=0}^{\infty} d(n)q^n = \sum_{n=0}^{\infty}(d^e(n) - d^o(n))q^n + 2\sum_{n=0}^{\infty} d^o(n)q^n.$$

$$\begin{aligned} 2\sum_{n=0}^{\infty} d^o(n)q^{\infty} &= (-q;q)_\infty - \sum_{n=0}^{\infty}(d^e(n) - d^o(n))q^n \\ &= (-q;q)_\infty - (q;q)_\infty \\ &= (-q;q)_\infty \left(1 - \frac{(q;q)_\infty}{(-q;q)_\infty}\right) \\ &= (-q;q)_\infty \left(1 - \sum_{n=-\infty}^{\infty} (-1)^n q^{n^2}\right) \quad \text{(by (3))} \end{aligned}$$

$$= 2(-q;q)_\infty \sum_{n=1}^{\infty}(-1)^{n+1}q^{n^2}.$$

It is not difficult to see that

$$\sum_{n=0}^{\infty} d^o(n)q^n = \sum_{n=1}^{\infty} \frac{q^{n(2n-1)}}{(q;q)_{2n-1}}$$

from which (4) follows. For (5), we have

$$\sum_{n=0}^{\infty} d^e(n)q^n = \sum_{n=0}^{\infty} d(n)q^n - \sum_{n=0}^{\infty} d_o(n)q^n$$

$$= (-q;q)_\infty + (-q;q)_\infty \sum_{n=1}^{\infty}(-1)^n q^{n^2} \text{ (by (4))}$$

$$= (-q;q)_\infty \sum_{n=0}^{\infty} (-1)^n q^{n^2}$$

It can easily be shown that

$$\sum_{n=0}^{\infty} d^e(n) q^n = \sum_{n=0}^{\infty} \frac{q^{n(2n+1)}}{(q;q)_{2n}}$$

and so (5) follows.

We have the following theorem.

Theorem

Let $b^e(n)$ be the number of partitions of n with initial 2-repetitions in which either all parts are distinct or the largest repeated part is even. Similarly, let $b^o(n)$ denote the number of partitions of n with initial 2-repetitions in which at least one part is repeated and the largest repeated part is odd. Then

$$b^e(n) - b^o(n) = \begin{cases} 1, & \text{if } n = \frac{j(j+1)}{2}, j \geq 0; \\ 0 & \text{otherwise.} \end{cases}$$

PROOF:

Note that

$$\sum_{n=0}^{\infty} b^e(n)q^n = (-q;q)_\infty +$$

$$+\sum_{m=1}^{\infty} \frac{q^{2(1+2+3+\ldots+2m)}}{(q;q)_{2m}} \prod_{j=2m+1}^{\infty} (1+q^j)$$

$$= \sum_{m=0}^{\infty} \frac{q^{2(1+2+3+\ldots+2m)}}{(q;q)_{2m}} \prod_{j=2m+1}^{\infty} (1+q^j)$$

$$= \sum_{m=0}^{\infty} \frac{q^{2m(2m+1)}}{(q;q)_{2m}} \prod_{j=2m+1}^{\infty} \frac{(1-q^{2j})}{(1-q^j)}$$

$$= \sum_{m=0}^{\infty} \frac{q^{2m(2m+1)}}{(q;q)_{2m}(q^{2m+1};q)_\infty} \prod_{j=2m+1}^{\infty} (1-q^{2j})$$

$$= \frac{1}{(q;q)_\infty} \sum_{m=0}^{\infty} q^{2m(2m+1)} \frac{\prod_{j=1}^{\infty}(1-q^{2j})}{\prod_{j=1}^{2m}(1-q^{2j})}$$

$$= \frac{(q^2;q^2)_\infty}{(q;q)_\infty} \sum_{m=0}^{\infty} \frac{q^{2m(2m+1)}}{(q^2;q^2)_{2m}}$$

and

$$\sum_{n=0}^{\infty} b^o(n)q^n = \sum_{m=1}^{\infty} \frac{q^{2(1+2+3+\ldots+2m-1)}}{(q;q)_{2m-1}} \prod_{j=2m}^{\infty} (1+q^j)$$

$$= \sum_{m=1}^{\infty} \frac{q^{2(1+2+3+\ldots+2m-1)}}{(q;q)_{2m-1}} \prod_{j=2m}^{\infty} \frac{1-q^{2j}}{1-q^{j}}$$

$$= \sum_{m=1}^{\infty} \frac{q^{2m(2m-1)}}{(q;q)_{2m-1}(q^{2m};q)_{\infty}} \prod_{j=2m}^{\infty} (1-q^{2j})$$

$$= \frac{1}{(q;q)_{\infty}} \sum_{m=1}^{\infty} q^{2m(2m-1)} \frac{\prod_{j=1}^{\infty}(1-q^{2j})}{\prod_{j=1}^{2m-1}(1-q^{2j})}$$

$$= \frac{(q^2;q^2)_\infty}{(q;q)_\infty} \sum_{m=1}^{\infty} \frac{q^{2m(2m-1)}}{(q^2;q^2)_{2m-1}}.$$

Thus

$$\sum_{n=0}^{\infty} (b^e(n) - b^o(n)) q^n =$$

$$= \frac{(q^2;q^2)_\infty}{(q;q)_\infty} \left(\sum_{n=0}^{\infty} \frac{q^{2n(2n+1)}}{(q^2;q^2)_{2n}} - \sum_{n=1}^{\infty} \frac{q^{2n(2n-1)}}{(q^2;q^2)_{2n-1}} \right)$$

$$= \frac{(q^4;q^4)_\infty}{(q;q)_\infty} \left(\sum_{n=0}^{\infty} (-1)^n q^{2n^2} - \sum_{n=1}^{\infty} (-1)^{n+1} q^{2n^2} \right)$$

(by (4) and (5))

$$
\begin{aligned}
&= \frac{(q^4;q^4)_\infty}{(q;q)_\infty}\left(\sum_{n=-\infty}^{\infty}(-1)^n q^{2n^2}\right) \\
&= \frac{(q^4;q^4)_\infty}{(q;q)_\infty}\frac{(q^2;q^2)_\infty}{(-q^2;q^2)_\infty} \text{ (by (3))} \\
&= \frac{(q^4;q^4)_\infty(-q;q)_\infty}{(-q^2;q^2)_\infty} \\
&= (q^4;q^4)_\infty(-q;q^4)_\infty(-q^3;q^4)_\infty \\
&= \sum_{n=0}^{\infty} q^{2n^2+n} \text{ (by (2))}
\end{aligned}
$$

$$= \sum_{n=0}^{\infty} q^{n(n+1)/2}.$$

In the following section we demonstrate how to give partition theoretic interpretation of numerous identities of Rogers-Ramanujan identities due to Lucy J. Slater (1952). We use the following identity for demonstration. The interpretation is given in terms of a Legendre theorem.

$$(q^2;q^2)_\infty \sum_{n=0}^{\infty} \frac{q^{2n(n+1)}}{(q;q)_{2n+1}} = \prod_{n=1}^{\infty} (1-q^{8n})(1+q^{8n-1})(1+q^{8n-7}) \tag{6}$$

For more examples, see the article on arxiv.org (Legendre Theorems for partitions with initial repetitions, D. Nyirenda & B. Mugwangwavari). Let $c_2(n)$ denote the number of partitions of n in which either

(a) even parts are distinct and the only odd part is 1 or

(b) there exists $j \geq 1$ such that an even part $2j$ appears twice, all positive even integers $< 2j$ appear twice, any even part $> 2j$ is distinct and the largest odd part is at most $2j + 1$.

Furthermore, let $c_{2,e}(n)$ (resp. $c_{2,o}(n)$) be $c_2(n)$-partitions with an even (resp. odd) number of distinct even parts.

Then we have the following.

Theorem

For all $n \geq 0$,

$$c_{2,e}(n) - c_{2,o}(n) = \begin{cases} 1, & n = 4j^2 + 3j, j \in \mathbb{Z}; \\ 0, & \textit{otherwise.} \end{cases}$$

PROOF:

$$\sum_{n=0}^{\infty} c_2(n)q^n$$
$$= \frac{(-q^2;q^2)_\infty}{1-q}$$
$$+ \sum_{n=1}^{\infty} \frac{q^{2+2+4+4+\cdots+2n+2n}}{(1-q)(1-q^3)\ldots(1-q^{2n+1})}(-q^{2n+2};q^2)_\infty$$
$$= \sum_{n=0}^{\infty} \frac{q^{2+2+4+4+\cdots+2n+2n}}{(1-q)(1-q^3)\ldots(1-q^{2n+1})}(-q^{2n+2};q^2)_\infty,$$

So we have

$$
\begin{aligned}
&\sum_{n=0}^{\infty}(c_{2,e}(n)-c_{2,o}(n))q^n \\
&=\sum_{n=0}^{\infty}\frac{q^{2n(n+1)}(q^{2n+2};q^2)_\infty}{(q;q^2)_{n+1}} \\
&=\sum_{n=0}^{\infty}\frac{q^{2n(n+1)}(q^2;q^2)_\infty}{(q;q^2)_{n+1}(q^2;q^2)_n} \\
&=(q^2;q^2)_\infty\sum_{n=0}^{\infty}\frac{q^{2n(n+1)}}{(q;q^2)_{n+1}(q^2;q^2)_n} \\
&=(q^2;q^2)_\infty\sum_{n=0}^{\infty}\frac{q^{2n(n+1)}}{(q;q)_{2n+1}} \quad \text{(by (6))}
\end{aligned}
$$

$$= \prod_{n=1}^{\infty} \left(1 + q^{8n-1}\right) \left(1 + q^{8n-7}\right) \left(1 - q^{8n}\right)$$

$$= \sum_{n=-\infty}^{\infty} q^{4n^2+3n}.$$

Example

Consider $n = 10$.

The $c_2(10)$-partitions are:

$$(10), (8, 2), (8, 1^2), (6, 4), (6, 2, 1^2), (6, 1^4), (5^2), (4, 2, 1^4),$$

$$(4, 1^6), (2, 1^8), (1^{10}).$$

From the above, note that $c_{2,e}(10)$-partitions are: $(8, 2), (5^2), (1^{10}), (6, 4), (6, 2, 1^2), (4, 2, 1^4)$ and $c_{2,o}(10)$-partitions are: $(10), (8, 1^2), (6, 1^4), (4, 1^6), (2, 1^8)$. Thus,

$$c_{2,e}(10) - c_{2,o}(10) = 1.$$

This agrees with the theorem because $10 = 4(-2)^2 + 3(-2)$.

References

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THANK YOU!