Generating functions for fixed points of the Mullineux map

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Introduction to Representation Theory

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$$\psi : G \to GL_n(F).$$

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ψ is irreducible or simple if 0 and F^n are the only G-invariant subspaces.

A two-dimensional (irreducible?) representation of Σ_3

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Moral: The characteristic of the field matters!

Isomorphism Classes of Irreducible Representations

Theorem

Let G be a finite group.

1. *The number of irreducible representations over $\mathbb{C}$ is the number of conjugacy classes of G. Every finite dimensional representation is a direct sum of irreducibles.*
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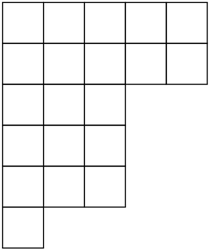
What does this mean for symmetric groups?

Partitions and Young diagrams

Example: Let $\lambda = (5, 5, 3, 3, 3, 1) \vdash 19$.

$\lambda =$



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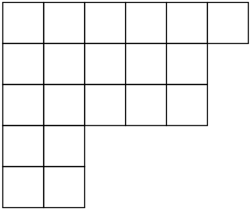

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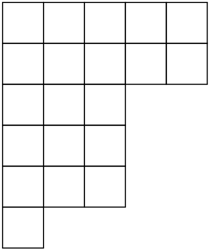
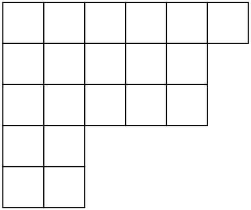
$\lambda =$ 

$\lambda' =$ 

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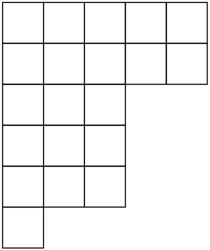
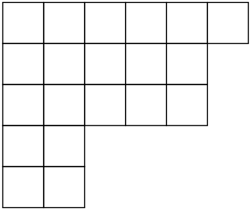
$\lambda =$ 

$\lambda' =$ 

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λ is e-regular for $e \geq 4$.

Modules for the symmetric group Σ_n

Irreducible $\mathbb{C}\Sigma_n$ modules are Specht modules:

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Mullineux Problem:

$$D^\lambda \otimes \text{sgn} \cong D^{m(\lambda)}.$$

What is the involution $m(\lambda)$?

History of Mullineux Problem

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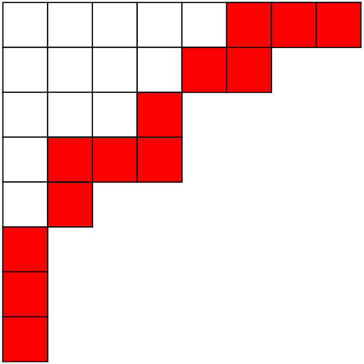
Remark: Consider $e \geq 2$. Our work extends Andrews-Olsson and Bessenrodt-Olsson to arbitrary e odd (straightforward) and e even (different). There are representation theory interpretations for Hecke algebras at eth roots of unity.

The Mullineux symbol

$$\lambda = (8, 6, 4, 4, 2, 1, 1, 1) \vdash 27 \text{ and } e = 5:$$

The e-rim contains 13 boxes spanning 8 rows.



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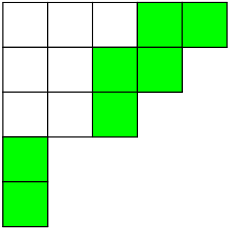

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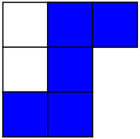

$$\begin{pmatrix} 13 & 7 & 5 \\ 8 & 5 & 2 \end{pmatrix}$$

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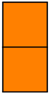

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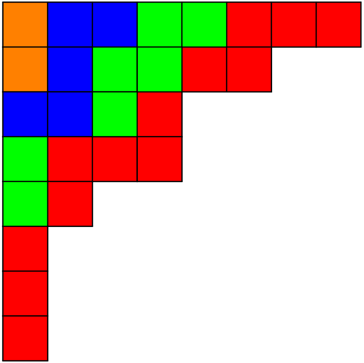

$$G(\lambda) = \begin{pmatrix} 13 & 7 & 5 & 2 \\ 8 & 5 & 3 & 2 \end{pmatrix}$$

The Mullineux map

Given

$$G(\lambda) = \begin{pmatrix} a_1 & a_2 & \cdots & a_t \\ r_1 & r_2 & \cdots & r_t \end{pmatrix}$$

replace r_i with

$$a_i - r_i + \epsilon_i$$

where $\epsilon_i = 1$ if $e \nmid a_i$ and $\epsilon_i = 0$ else.

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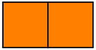
$$\begin{pmatrix} 13 & 7 & 5 & 2 \\ 8 & 5 & 3 & 2 \end{pmatrix} \Rightarrow \begin{pmatrix} 13 & 7 & 5 & 2 \\ 6 & 3 & 2 & 1 \end{pmatrix}$$

Reconstructing $m(\lambda)$



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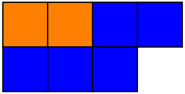

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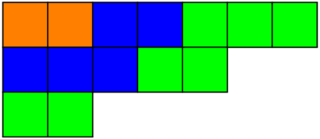

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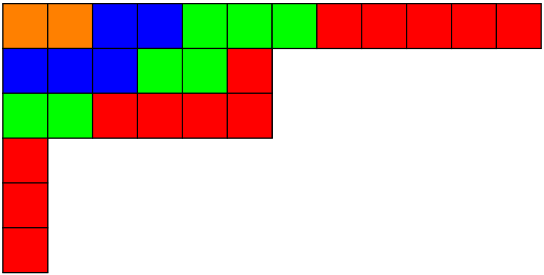

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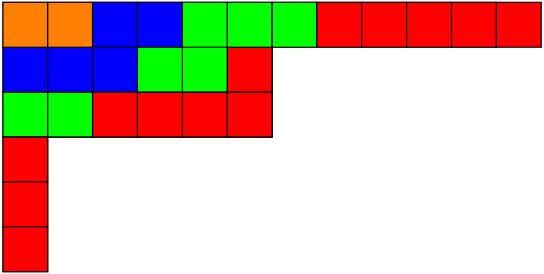

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$$m(8,6,4,4,2,1,1,1) = (12,6,6,1,1,1).$$

Counting Mullineux fixed points

1. Mullineux gave conditions on $(a_1, a_2, \ldots, a_t) \vdash n$ and $\{r_1, r_2, \ldots, r_t\}$ to be a Mullineux symbol $G(\lambda)$.

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4. Count fixed points or, for blockwise version, use e-bar quotients.

Theorem (Mullineux 1979)

An array $\begin{pmatrix} a_1 & a_2 & \cdots & a_k \\ r_1 & r_2 & \cdots & r_k \end{pmatrix}$ *is the Mullineux symbol of an e-regular partition of* $n = \sum a_i$ *if and only if:*

1. $0 \leq r_i - r_{i+1} \leq e;$
2. $r_i - r_{i+1} + \epsilon_{i+1} \leq a_i - a_{i+1} \leq r_i - r_{i+1} + \epsilon_{i+1} + e$
3. $r_i = r_{i+1} \to e \mid a_i$
4. $r_i - r_{i+1} = e \to e \nmid a_i$
5. $0 \leq a_k - r_k < e$
6. $1 \leq r_k \leq e$
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Remark: Given the top row $(a_1, a_2, \ldots, a_k)$, there is at most one corresponding partition fixed under the Mullineux map.

Idea: Count eligible partitions $\{(a_1, a_2, \ldots, a_k) \vdash n\}$.

Definition

Define $\mathcal{M}_e(n)$ to be the set of partitions $(a_1, a_2, \ldots, a_k) \vdash n$ satisfying:

(i) $2 \mid a_i \iff e \mid a_i$

(ii) $0 \leq a_i - a_{i+1} \leq 2e$

(iii) If $a_i = a_{i+1}$ then a_i is even.

(iv) If $a_i - a_{i+1} = 2e$ then a_i is odd.

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True for any e.

Andrews-Olsson 1991

Theorem 2. *Let* $A = \{a_1, a_2, \ldots, a_r\}$ *be a set of* r *distinct positive integers arranged in increasing order, and let* N *be an integer larger than* a_r. *Let* $P_1(A; N, n)$ *denote the number of partitions of* n *into distinct parts each of which is congruent to some* a_i modulo N. *Let* $P_2(A; N, n)$ *denote the number of partitions of* n *into parts each of which is congruent to* 0 *or to some* a_i modulo N, *in addition only parts divisible by* N *may be repeated, the smallest part is* $< N$, *the difference between successive parts is at most* N *and strictly less than* N *if either part is divisible by* N. *Then for each* $n \geqq 0$,

$$P_1(A; N, n) = P_2(A; N, n). \tag{1.1}$$

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For e odd and $N = 2e$, $A = \{1, 3, 5, \dots, e-2, e+2, \dots, 2e-1\}$, we get that $\mathcal{M}_e(n)$ is $P_2(A; 2e, n)$.

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The generating function for P_1 is easy to describe.

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$(3,3,2,1,1)$

or for partitions with distinct parts:

$$P_d(q) = (1+q)(1+q^2)(1+q^3)\cdots .$$

Mullineux fixed points

Let $mf_e(n)$ be the number of e-regular partitions of n fixed by the Mullineux map and let

$$MF_e(q) = \sum_{n=0}^{\infty} mf_e(n) q^n.$$

Our first result determines this generating function.

Theorem

(a) *(Andrews-Olsson for odd primes) Let* e *be odd. Then* $mf_e(n)$ *is the number of partitions of* n *into distinct odd parts not divisible by* e*. Thus:*

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Check this agrees with $e = 2$ answer.

A mysterious coincidence?

For e arbitrary:

$$MF_e(-q) = \sum_{n=0}^{\infty} (-1)^n mf_e(n) q^n = \prod_{k=1}^{\infty} \frac{1+q^{ek}}{1+q^k}.$$

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$e = 4$ OEIS Entry
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Blocks of Group Algebras

The group algebra FG has primitive central idempotents

$$1 = e_1 + e_2 + \cdots + e_r.$$

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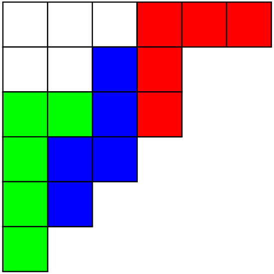
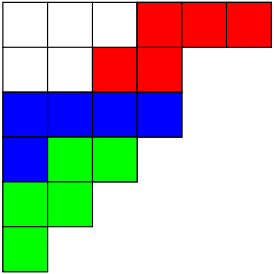
e-blocks and e-cores



$e = 5, w = 3, \tilde{\lambda} = (3, 2)$



If $m(\lambda) = \lambda$ then $\tilde{\lambda}$ is self-conjugate.

Blocks and residue sequences

Fact: Two partitions $\lambda \vdash n$ and $\mu \vdash n$ are in the same e-block (have the same e-core) if and only if their boxes have the same set of residues mod e.

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0	1	2	3	0	1	2
3	0	1	2			
2	3	0				
1	2	3				
0						

0	1	2	3	0
3	0	1	2	
2	3	0		
1	2	3		
0	1	2		

Two different $\lambda, \mu \vdash 18$ with the same 4-core $(3, 2, 1)$ and 4-weight 3.

Table: Mullineux fixed points for $e = 4$ by weight

n	$w = 0$	$w = 1$	$w = 2$	$w = 3$	$w = 4$	$w = 5$
0	1					
1	1					
2	0					
3	1					
4	1	1				
5	1	1				
6	1	0				
7	1	1				
8	0	1	3			
9	0	1	3			
10	2	1	0			
11	0	1	3			
12	1	0	3	4		
13	1	0	3	4		
14	1	2	3	0		
15	2	0	3	4		
16	0	1	0	4	9	
17	0	1	0	4	9	
18	1	1	6	4	0	
19	1	2	0	4	9	
20	0	0	3	0	9	12

The sequence $\{1, 1, 3, 4, 9, 12 \ldots\}$ counts *cubic partitions*.
The first column counts *self-conjugate 4-cores*.

Fixed points in a block

Definition: Let e be even. Define $f_e(w)$ by:

$$\sum_{w=0}^{\infty} f_e(w)q^w \quad := \quad \prod_{k=1}^{\infty} \frac{1}{(1-q^{2k})^{e/2}} \frac{1}{(1-q^{2k-1})}$$

Let e be odd. Define $g_e(w)$ by:

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$f_e(w)$ and $g_e(w)$ will count fixed points in a block with a self-conjugate e-core and weight w.

Theorem

(a) *Suppose e is even. Then:*

$$mf_{e,w}(n) = f_e(w)sc_e(n - ew).$$

(b) *Suppose e is odd. Then $mf_{e,w}(n)$ is zero unless w is even in which case*

$$mf_{e,w}(n) = g(\frac{w}{2})sc_e(n - ew).$$

Joint generating function

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The generating function $MF_e(x,q) = \sum_{n,w} mf_{e,w}(n)x^w q^n$ is:

$$MF_e(x,q) = \begin{cases} \frac{(-q,q^2)_\infty (q^{2e},q^{2e})_\infty^{e/2}}{(q^{2e}x^2,q^{2e}x^2)_\infty^{e/2}(q^e x,q^{2e}x^2)_\infty} & \text{if } e \text{ is even} \\ \frac{(-q,q^2)_\infty (q^{2e},q^{2e})_\infty^{e-1/2}}{(q^{2e}x^2,q^{2e}x^2)_\infty^{(e-1)/2}(-q^e,q^{2e})_\infty} & \text{if } e \text{ is odd} \end{cases}$$

Computing $f_e(w)$

Given a partition λ with distinct parts, we form an abacus with runners lying north to south labelled $\{0, 1, 2, \ldots, q-1\}$. The bead positions are labelled as below:

$$\begin{array}{ccccc} 0 & 1 & \cdots & q-2 & q-1 \\ q & q+1 & \cdots & 2q-2 & 2q-1 \\ \vdots & \vdots & & \vdots & \vdots \end{array}$$

Place a bead on the abacus corresponding to each part of λ.

$$\begin{array}{cccccc} 0 & 1 & 2 & 3 & 4 & 5 \\ \circ & \circ & \circ & \circ & \circ & \circ \\ \circ & \bigcirc & \circ & \bigcirc & \circ & \bigcirc \\ \circ & \bigcirc & \circ & \circ & \circ & \bigcirc \\ \circ & \circ & \circ & \bigcirc & \circ & \bigcirc \end{array}$$

Figure: Abacus display for $\lambda = (23, 21, 17, 13, 11, 9, 7)$ and $q = 6$.

q-bar cores

On the abacus display of a bar partition (i.e. distinct parts) we can do the following:

- Slide a bead one step up its runner to a vacant position, beads at position zero disappear.
- Remove a pair of beads in position a and $q - a$ for $a = 1, 2, \ldots, \lfloor (q-1)/2 \rfloor$.

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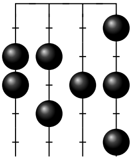
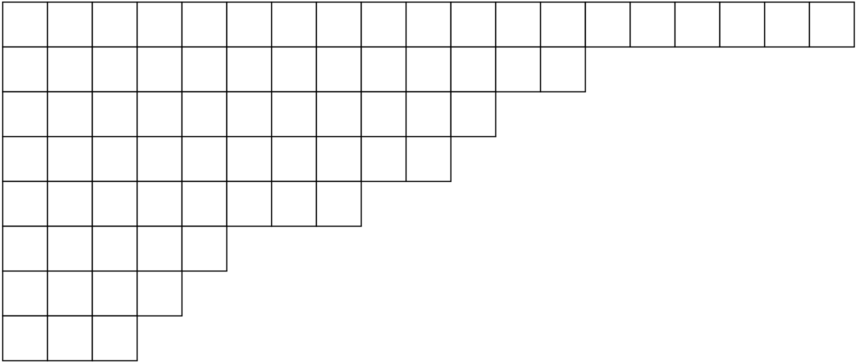
Eventually we reach the abacus diagram of the q-bar core of λ.

Computing the 4-bar core of $\lambda = (19, 13, 11, 10, 8, 5, 4, 3)$





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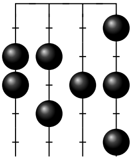

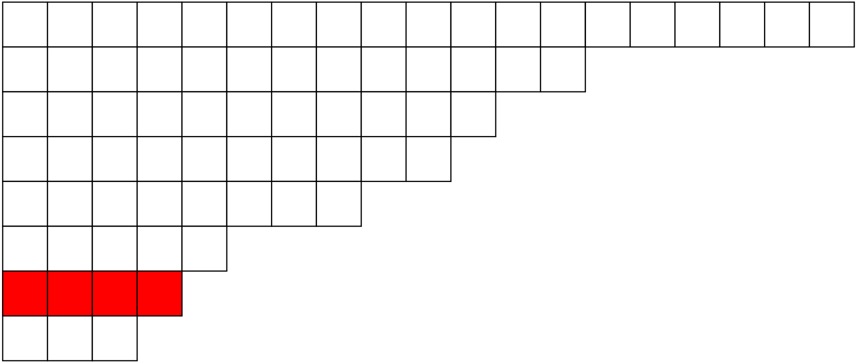

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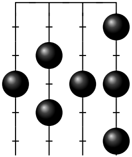

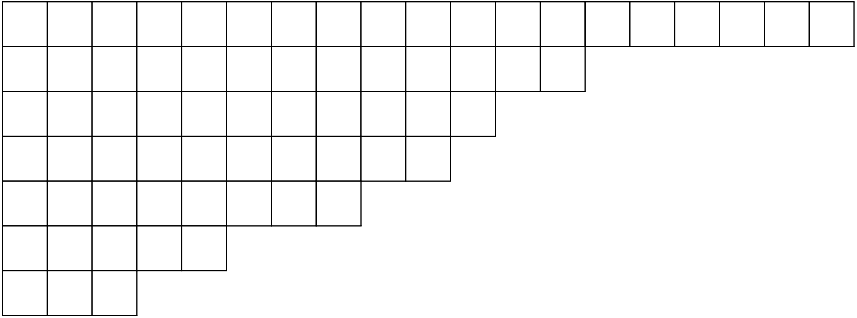

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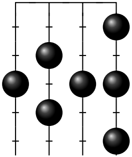

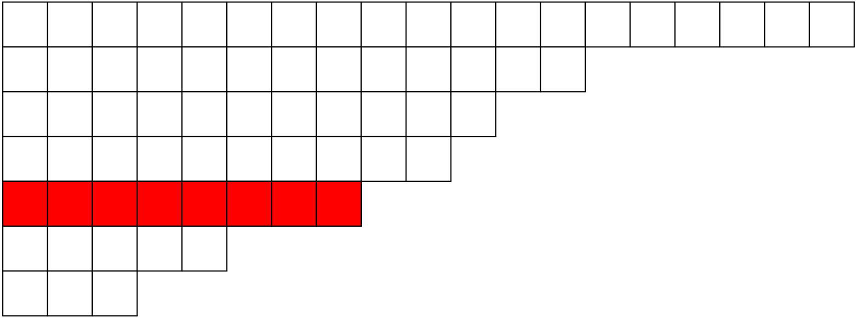

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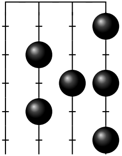

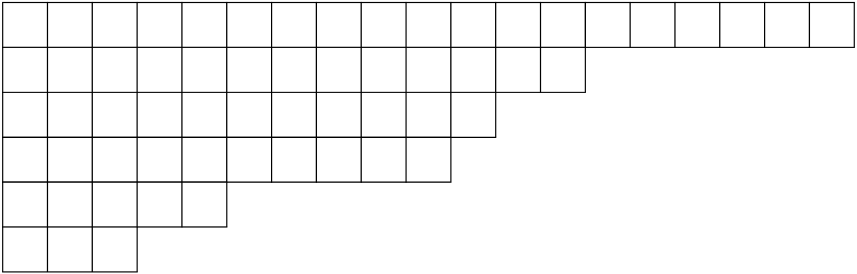

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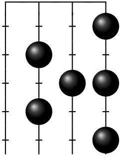

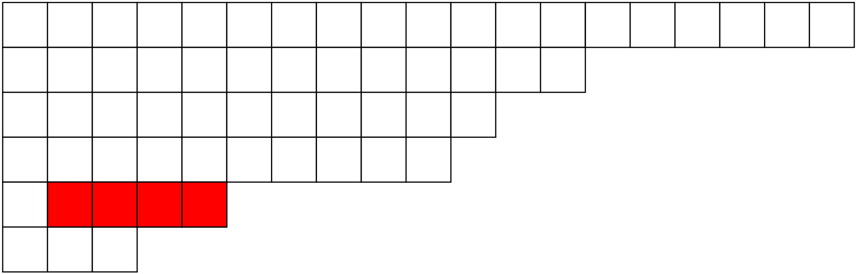

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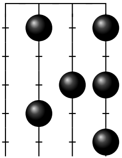

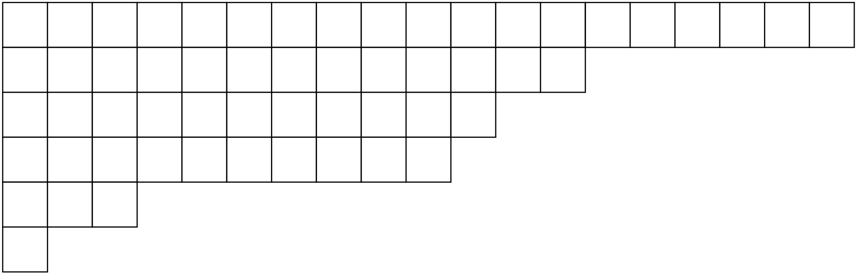

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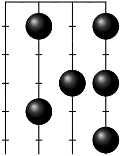

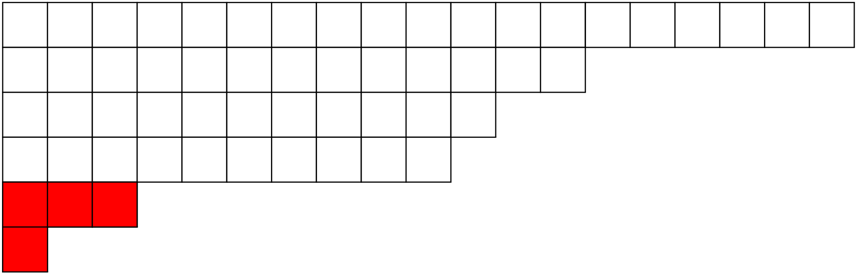

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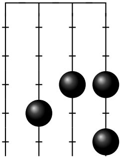

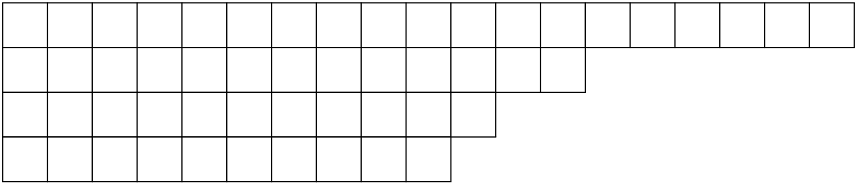

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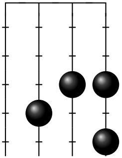

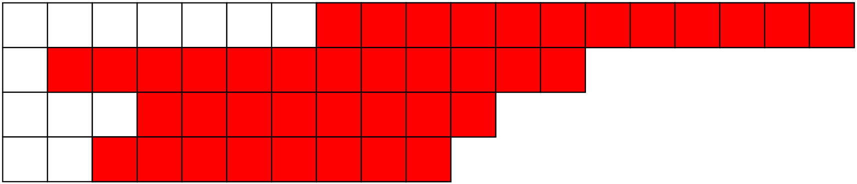

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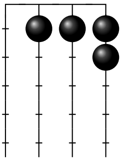

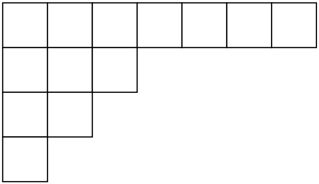

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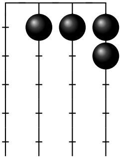

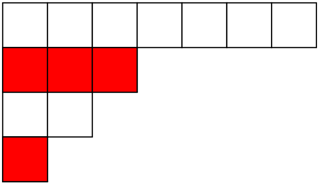

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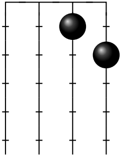

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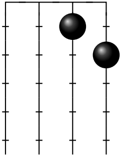

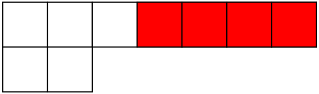

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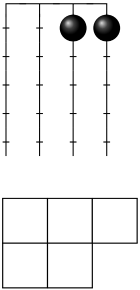

Final steps

Let e be even and suppose $\lambda \vdash n$ has e weight w and self-conjugate core $\lambda_{(e)}$. Apply Bessenrodt's bijection to get a pair $\{\tau, e\gamma\}$ where $\tau = (c_1, c_2, \ldots, c_k)$ has distinct odd parts and γ has all odd parts. Verify all the τ have the same 2e-bar core.

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