Copartitions and Their Connections to Classical Partition-Theoretic Objects

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Online Partitions Seminar

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Outline

Outline

Definition

A **copartition** is a combination of
a partition γ
a partition σ
along with a rectangle ρ with dimensions

$$(\text{number of parts of } \gamma) \times (\text{number of parts of } \sigma)$$

uniting them.

Definition

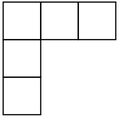
An (a, b, m)-**copartition** is a combination of
a partition γ into parts $\equiv a \pmod{m}$,
a partition σ into parts $\equiv b \pmod{m}$,
along with a rectangle ρ of $m's$ with dimensions

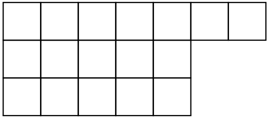
$$(\text{number of parts of } \gamma) \times (\text{number of parts of } \sigma)$$

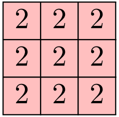
uniting them.

We define $\mathrm{cp}(a, b, m; n)$ to be the number of (a, b, m)-copartitions of n.

Example of a $(1, 1, 2)$-copartition

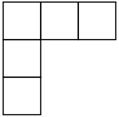
$\sigma :$  $3 + 1 + 1$

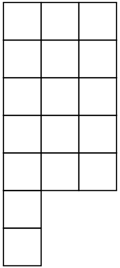
$\gamma :$  $7 + 5 + 5$

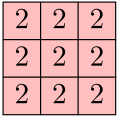
$\rho :$ 

3×3 rectangle

Example of a $(1, 1, 2)$-copartition

σ :  $3 + 1 + 1$

γ' :  $7 + 5 + 5$ conjugated

ρ : 

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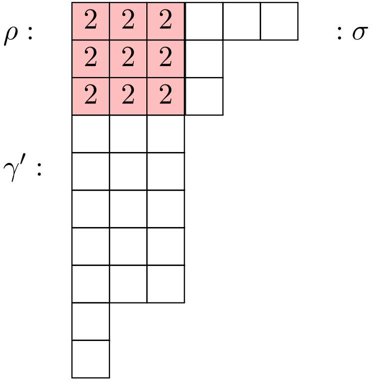
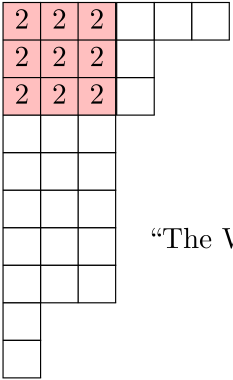


Example of a $(1, 1, 2)$-copartition

$\rho :$

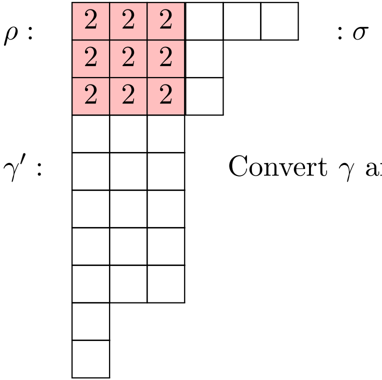


$: \sigma$ "The Sky"

$\gamma' :$

"The Woods" or "The Ground"

Diagram of a $(1, 1, 2)$-copartition



Convert γ and σ to their 2-modular diagrams.

Diagram of a $(1, 1, 2)$-copartition

ρ :	2	2	2	2	1	: σ
	2	2	2	1		
	2	2	2	1		
	2	2	2			
γ' :	2	2	2			
	2	1	1			
	1					

Convert γ and σ to their 2-modular diagrams.

Diagram of a $(1, 1, 2)$-copartition

$$
\begin{array}{l|c|c|c|c|c|l}
\rho : & 2 & 2 & 2 & 1 & 2 & : \sigma \\
 & 2 & 2 & 2 & 1 & & \\
 & 2 & 2 & 2 & 1 & & \\
 & 1 & 1 & 1 & & & \\
\gamma' : & 2 & 2 & 2 & & & \\
 & 2 & 2 & 2 & & & \\
 & 2 & & & & &
\end{array}
$$

Convert γ and σ to their 2-modular diagrams.
Move the 1's in to border rectangle ρ.

This is the **copartition diagram** of
$(\gamma, \rho, \sigma) = (7 + 5 + 5, 2^{3\times 3}, 3 + 1 + 1)$

Diagram of an (a, b, m)-copartition

$$
\begin{array}{r|c|c|c|c|c|l}
\rho : & m & m & m & b & m & : \sigma \\
 & m & m & m & b & & \\
 & m & m & m & b & & \\
 & a & a & a & & & \\
\gamma' : & m & m & m & & & \\
 & m & m & m & & & \\
 & m & & & & &
\end{array}
$$

Convert γ and σ to their m-modular diagrams.
Move the a's and b's in to border rectangle ρ.

This is the **copartition diagram** of

$$(\gamma, \rho, \sigma) = ((3m + a) + (2m + a) + (2m + a), m^{3\times 3}, (m + b) + b + b)$$

Partitions with evens below odds *

Definition (Andrews)

$\mathcal{EO}^*(n)$ counts the number of partitions of n such that

- all even parts are smaller than all odd parts
- the only part that appears an odd number of times is the largest even part

Example

$$9^2 + 7^4 + 6^5 + 2^2 = 80$$

is a partition counted by $\mathcal{EO}^*(80)$

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Connection between $\mathrm{cp}(n)$ and $\mathcal{EO}^*(2n)$

Theorem (Burson-E.)

For every integer $n \geq 0$,

$$\mathrm{cp}(1,1,2;n) = \mathcal{EO}^*(2n).$$

Remark

The symmetry of $(1,1,2)$-copartitions is easy to see.

In contrast, the symmetry of partitions counted by $\mathcal{EO}^*$ was not immediately apparent.

Connection between $\mathrm{cp}(n)$ and $\mathcal{EO}^*(2n)$

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Definition (Garvan, Schlosser)

An (m,k)-capsid partition is a partition π such that

- all parts of π are of size $m-k$ or congruent to 0 or $k \mod m$;
- if there are no parts of size $m-k$, then all parts are congruent to $k \mod m$;
- if $m-k$ is a part, then it is the smallest part, all parts $\equiv 0 \mod m$ have size $\leq m \cdot f_{m-a}$, and all parts $\equiv k \mod m$ have size $> m \cdot f_{m-k}$.

Connection to (m, k)-capsids

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They use (m, k)-capsids and t-cores to give a combinatorial interpretation of Ramanujan's tau function.

Theorem (Burson-E.)

For every integer $n \geq 0$, $\mathrm{cp}(k, m-k, m; n)$ is equal to the number of (m, k)-capsids of n.

Remark

The symmetry between $(k, m-k, m)$-copartitions and $(m-k, k, m)$-copartitions is easy to see.

In contrast, the symmetry between (m, k)-capsids and $(m, m-k)$-capsids is, "not at all combinatorially obvious."

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Theorem (Burson, E.)

$$\mathbf{cp}(a, b, m; q) := \sum_{n=0}^{\infty} \mathrm{cp}(a, b, m; n) q^n$$

Generating function

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$$\mathbf{cp}(a,b,m;q) := \sum_{n=0}^{\infty} \mathrm{cp}(a,b,m;n)q^n$$
$$= \sum_{w=0}^{\infty}\sum_{s=0}^{\infty} \frac{q^{mws+aw+bs}}{(q^m;q^m)_w(q^m;q^m)_s}$$

Generating function

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$$= \sum_{w=0}^{\infty}\sum_{s=0}^{\infty} \frac{q^{mws+aw+bs}}{(q^m;q^m)_w(q^m;q^m)_s}$$
$$= \frac{(q^{a+b};q^m)_\infty}{(q^a;q^m)_\infty(q^b;q^m)_\infty}$$

Generating function

Our combinatorial proof is a direct proof of

$$\frac{1}{(q^a;q^m)_\infty(q^b;q^m)_\infty} = \frac{1}{(q^{a+b};q^m)_\infty}\sum_{n=0}^{\infty}\mathrm{cp}(a,b,m;n)q^n$$

Open Problem

Give a direct combinatorial proof of

$$\frac{(q^{a+b};q^m)_\infty}{(q^a;q^m)_\infty(q^b;q^m)_\infty} = \sum_{n=0}^{\infty}\mathrm{cp}(a,b,m;n)q^n$$

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Outline

Three partition-theoretic objects

Ordinary partitions	**2-colored partitions**	**$(1,1,2)$-Copartitions**
$p(5n+4) \equiv 0 \pmod 5$	$c_2(5n+2) \equiv 0 \pmod 5$ $c_2(5n+3) \equiv 0 \pmod 5$ $c_2(5n+4) \equiv 0 \pmod 5$	

Three partition-theoretic objects

Ordinary partitions

$$p(5n+4) \equiv 0 \pmod 5$$

2-colored partitions

$$c_2(5n+2) \equiv 0 \pmod 5$$
$$c_2(5n+3) \equiv 0 \pmod 5$$
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$(1,1,2)$-Copartitions

A Ramanujan-type congruence

Theorem (Andrews)

$$\mathcal{EO}^*(10n+8) \equiv 0 \pmod 5$$

A Ramanujan-type congruence

Theorem (Andrews)

$$\mathcal{EO}^*(10n+8) \equiv 0 \pmod 5$$

that is,

$$\mathrm{cp}(1,1,2;5n+4) \equiv 0 \pmod 5$$

Three partition-theoretic objects

Ordinary partitions	**2-colored partitions**	**$(1,1,2)$-Copartitions**
$p(5n+4) \equiv 0 \pmod 5$	$c_2(5n+2) \equiv 0 \pmod 5$ $c_2(5n+3) \equiv 0 \pmod 5$ $c_2(5n+4) \equiv 0 \pmod 5$	$\mathrm{cp}_{1,1,2}(5n+4) \equiv 0 \pmod 5$
witnessed by Dyson's rank Largest part - # of parts		

Three partition-theoretic objects

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witnessed by Dyson's rank Largest part - # of parts		

Partitions of 9 separated by their rank modulo 5

Rank = Largest part - # of parts

$0 \pmod 5$	$1 \pmod 5$	$2 \pmod 5$	$3 \pmod 5$	$4 \pmod 5$
$7+2$	$8+1$	$6+1^3$	9	$7+1^2$
$5+1^4$	$5+2+1^2$	$5+3+1$	$6+2+1$	$6+3$
$4+3+1^2$	4^2+1	$5+2^2$	$5+4$	$4+2+1^3$
$4+2^2+1$	$4+3+2$	$3+2+1^4$	$4+1^5$	3^2+2+1
3^3	$3+1^6$	2^4+1	3^2+1^3	$3+2^3$
$2+1^7$	2^2+1^5	2^3+1^3	$3+2^2+1^2$	1^9

Three partition-theoretic objects

<u>**Ordinary partitions**</u>	<u>**2-colored partitions**</u>	<u>$(1,1,2)$**-Copartitions**</u>
$p(5n+4) \equiv 0 \pmod 5$	$c_2(5n+2) \equiv 0 \pmod 5$ $c_2(5n+3) \equiv 0 \pmod 5$ $c_2(5n+4) \equiv 0 \pmod 5$	$\mathrm{cp}_{1,1,2}(5n+4) \equiv 0 \pmod 5$
witnessed by	witnessed by	
Dyson's rank Largest part - # of parts	Birank # red parts - # blue parts	

2-colored partitions of 3 separated by their birank modulo 5

Birank = # red parts - # blue parts

$0 \pmod 5$	$1 \pmod 5$	$2 \pmod 5$	$3 \pmod 5$	$4 \pmod 5$
2 + 1	3	2+1	2+1	3
2+1	1+1+1	1+1+1	1+1+1	1+1+1

Three partition-theoretic objects

Ordinary partitions	**2-colored partitions**	**$(1,1,2)$-Copartitions**
$p(5n+4) \equiv 0 \pmod 5$	$c_2(5n+2) \equiv 0 \pmod 5$ $c_2(5n+3) \equiv 0 \pmod 5$ $c_2(5n+4) \equiv 0 \pmod 5$	$\mathrm{cp}_{1,1,2}(5n+4) \equiv 0 \pmod 5$
witnessed by	witnessed by	witnessed by
Dyson's rank Largest part - # of parts	Birank # red parts - # blue parts	Copartition crank # ground parts - # sky parts

Definition

(Andrews) For each partition π enumerated by $\mathcal{EO}^*$, the **even-odd crank** of π is

$$eoc(\pi) = \text{largest even part } - \# \text{ (odd parts of } \pi)$$

(Burson, E.)
If $\lambda = (\gamma, \rho, \sigma)$ is a copartition, the **copartition crank** of λ is

$$\begin{aligned}\text{cp-crank}(\lambda) &= \text{(number of parts of } \gamma\text{) - (number of parts of } \sigma\text{)}\\ &= \text{“the rank of the rectangle” } \rho.\end{aligned}$$

The congruence is witnessed by a crank

Theorem (Andrews)

The even-odd crank separates the partitions enumerated by $\mathcal{EO}^*(10n+8)$ into five equinumerous sets.

Equivalently, the copartition crank separates the partitions enumerated by $\mathrm{cp}(1,1,2;5n+4)$ into five equinumerous sets.

Three partition-theoretic objects

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$p(5n+4) \equiv 0 \pmod 5$	$c_2(5n+2) \equiv 0 \pmod 5$ $c_2(5n+3) \equiv 0 \pmod 5$ $c_2(5n+4) \equiv 0 \pmod 5$	$\mathrm{cp}_{1,1,2}(5n+4) \equiv 0 \pmod 5$
witnessed by	witnessed by	witnessed by
Dyson's rank Largest part - # of parts	Birank # red parts - # blue parts	Copartition crank # ground parts - # sky parts

Open Problem

Give a combinatorial proof that Dyson's rank, the birank, or the copartition crank witness their associated congruences.

Three partition-theoretic objects

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Three partition-theoretic objects

Ordinary partitions	**2-colored partitions**	**$(1,1,2)$-Copartitions**
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witnessed by	witnessed by	witnessed by
Dyson's rank Largest part - # of parts	Birank # red parts - # blue parts	Copartition crank # ground parts - # sky parts
Rank identities		

Definition (Dyson)

Let $N(r, 5, n)$ be the number of partitions of n with rank congruent to $r \pmod 5$.

Theorem (Atkin, Swinnerton-Dyer)

For every integer $n \geq 0$,

$$N(1,5,5n+1) = N(2,5,5n+1) = N(3,5,5n+1) = N(4,5,5n+1)$$
$$N(0,5,5n+2) = N(2,5,5n+2) = N(3,5,5n+2)$$
$$N(0,5,5n+4) = N(1,5,5n+4) = N(2,5,5n+4) = N(3,5,5n+4) = N(4,5,5n+4)$$

Three partition-theoretic objects

Ordinary partitions	**2-colored partitions**	**$(1,1,2)$-Copartitions**
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witnessed by	witnessed by	witnessed by
Dyson's rank Largest part - # of parts	Birank # red parts - # blue parts	Copartition crank # ground parts - # sky parts
Rank identities	Birank identities	

Identities for the birank

Definition (Hammond, Lewis)

Let $R(r, 5, n)$ be the number of 2-colored partitions of n with birank congruent to $r \pmod 5$.

Theorem (Hammond, Lewis)

For every integer $n \geq 0$,

$$R(1,5,5n) = R(2,5,5n) = R(3,5,5n) = R(4,5,5n)$$
$$R(0,5,5n+1) = R(2,5,5n+1) = R(3,5,5n+1)$$
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Three partition-theoretic objects

Ordinary partitions	**2-colored partitions**	**$(1,1,2)$-Copartitions**
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witnessed by	witnessed by	witnessed by
Dyson's rank Largest part - # of parts	Birank # red parts - # blue parts	Copartition crank # ground parts - # sky parts
Rank identities	Birank identities	Crank identities

Definition

Let $M(r, 5, n)$ be the number of $(1, 1, 2)$-copartitions of n with copartition crank congruent to $r \pmod 5$.

Theorem (Burson-E.)

For every integer $n \geq 0$,

$$\begin{aligned}
&M(1,5,5n) = M(2,5,5n) = M(3,5,5n) = M(4,5,5n) \\
&M(0,5,5n+1) = M(1,5,5n+1) = M(4,5,5n+1) \\
&M(0,5,5n+2) = M(2,5,5n+2) = M(3,5,5n+2) \\
&M(1,5,5n+3) = M(2,5,5n+3) = M(3,5,5n+3) = M(4,5,5n+3)
\end{aligned}$$

Identities for the copartition crank

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Let $M(r, 5, n)$ be the number of $(1, 1, 2)$-copartitions of n with copartition crank congruent to $r \pmod 5$.

Theorem (Burson-E.)

For every integer $n \geq 0$,

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$$M(0,5,5n+2) = M(2,5,5n+2) = M(3,5,5n+2)$$
$$M(1,5,5n+3) = M(2,5,5n+3) = M(3,5,5n+3) = M(4,5,5n+3)$$

Andrews:

$$M(0,5,5n+4) = M(1,5,5n+4) = M(2,5,5n+4) = M(3,5,5n+4) = M(4,5,5n+4)$$

Three partition-theoretic objects

Ordinary partitions	**2-colored partitions**	**$(1,1,2)$-Copartitions**
$p(5n+4) \equiv 0 \pmod 5$	$c_2(5n+2) \equiv 0 \pmod 5$ $c_2(5n+3) \equiv 0 \pmod 5$ $c_2(5n+4) \equiv 0 \pmod 5$	$\mathrm{cp}_{1,1,2}(5n+4) \equiv 0 \pmod 5$
witnessed by	witnessed by	witnessed by
Dyson's rank Largest part - # of parts	Birank # red parts - # blue parts	Copartition crank # ground parts - # sky parts
Rank identities	Birank identities	Crank identities

Open Problem

Open Problem

Give an explanation or a heuristic for why ordinary partitions, 2-colored partitions, and $(1, 1, 2)$-copartitions share these properties.

Outline

Conjugation

2	2	2	1	2	2	2
2	2	2	1	2		
1	1	1				
2	2					
2	2					

$$\longrightarrow$$

2	2	1	2	2
2	2	1	2	2
2	2	1		
1	1			
2	2			
2				
2				

$$(\gamma, \rho, \sigma) \qquad \longrightarrow \qquad (\sigma, \rho', \gamma)$$

A congruence modulo 2

Theorem (Burson-E.)

$$\mathrm{cp}(1,1,2;2n+1) \equiv 0 \pmod 2$$

A congruence modulo 2

Theorem (Burson-E.)

$$\mathrm{cp}(1,1,2;2n+1) \equiv 0 \pmod 2$$

Proof.
Copartitions of odd size can be paired by conjugation,
and the rectangle cannot be a square,
so there are no self-conjugate partitions.

Conjugation

$$
\begin{array}{|c|c|c|c|c|c|c|}
\hline
2 & 2 & 2 & 1 & 2 & 2 & 2 \\
\hline
2 & 2 & 2 & 1 & 2 \\
\cline{1-5}
1 & 1 & 1 \\
\cline{1-3}
2 & 2 \\
\cline{1-2}
2 & 2 \\
\cline{1-2}
\end{array}
\longrightarrow
\begin{array}{|c|c|c|c|c|}
\hline
2 & 2 & 1 & 2 & 2 \\
\hline
2 & 2 & 1 & 2 & 2 \\
\hline
2 & 2 & 1 \\
\cline{1-3}
1 & 1 \\
\cline{1-2}
2 & 2 \\
\cline{1-2}
2 \\
\cline{1-1}
2 \\
\cline{1-1}
\end{array}
$$

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$$\mathrm{cp}(1,1,2;2n+1) \equiv 0 \pmod{2}$$

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Copartitions of odd size can be paired by conjugation.
and there are no self-conjugate partitions. □

A congruence modulo 2

Theorem

(Chern)

$$\mathcal{EO}^*(4n+2) \equiv 0 \pmod 2$$

(Burson, E.)

$$\mathrm{cp}(1,1,2;2n+1) \equiv 0 \pmod 2$$

Proofs:

Chern: An identity between two sub-types of $\mathcal{EO}^*$ partitions

Burson, E.: Conjugation, which matches completely different sub-types

Conjugation of an (a, b, m)-copartition

$$
\begin{array}{|c|c|c|c|c|c|c|}
\hline
m & m & m & b & m & m & m \\
\hline
m & m & m & b & m \\
\cline{1-5}
a & a & a \\
\cline{1-3}
m & m \\
\cline{1-2}
m & m \\
\cline{1-2}
m \\
\cline{1-1}
\end{array}
\quad \longrightarrow \quad
\begin{array}{|c|c|c|c|c|c|}
\hline
m & m & a & m & m & m \\
\hline
m & m & a & m & m \\
\cline{1-5}
m & m & a \\
\cline{1-3}
b & b \\
\cline{1-2}
m & m \\
\cline{1-2}
m \\
\cline{1-1}
m \\
\cline{1-1}
\end{array}
$$

$$(\gamma, \rho, \sigma) \quad \longrightarrow \quad (\sigma, \rho', \gamma)$$

Theorem (Burson-E.)

$$\mathrm{cp}(a, b, m; n) = \mathrm{cp}(b, a, m; n)$$

Proof.

Conjugation. $\square$

Another congruence modulo 2

Theorem (Burson-E.)

For even m,

$$\mathrm{cp}(a, a, m; 2n+1) \equiv 0 \pmod 2$$

Conjugation of an (a, a, m)-copartition

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\begin{array}{ccccccc}
m & m & m & a & m & m & m \\
m & m & m & a & m & & \\
a & a & a & & & & \\
m & m & & & & & \\
m & m & & & & & \\
m & & & & & &
\end{array}
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\begin{array}{cccccc}
m & m & a & m & m & m \\
m & m & a & m & m & \\
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Proof.
Copartitions of odd size can be paired by conjugation,
and the rectangle cannot be a square,
so there are no self-conjugate partitions.

Outline

Example of a $(1, 1, 1)$-copartition

1	1	1	1	1	1	1
1	1	1	1	1		
1	1	1				
1	1					
1	1					
1						

Where is the rectangle?

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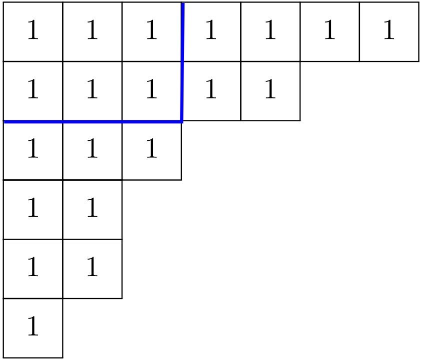

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Where is the rectangle?

Definition

The **diversity** of a partition λ is the number of different part sizes that appear in λ.

We denote **diversity** of a partition λ as $\mathrm{dv}(\lambda)$.

Example of a $(1,1,1)$-copartition

1	1	1	1	1	1	1
1	1	1	1	1		
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1	1					
1	1					
1						

Where is the rectangle?

There are $\mathrm{dv}(\lambda)+1$ choices for the rectangle of a copartition with shape λ.

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which is also the total number of 1s among all partitions of $n + 1$.

Example of a $(0, 1, 1)$-copartition

1	1	1	1	1	1	1
1	1	1	1	1		
0	0					
1	1					
1	1					
1						

We have to insist that σ be non-empty.

Example of several $(0, 1, 1)$-copartitions

1	1	1	1	1	1	1
1	1	1	1	1		
0	0	0				
1	1					
1	1					
1						

Example of several $(0, 1, 1)$-copartitions

1	1	1	1	1	1	1
1	1	1	1	1		
0	0	0	0			
1	1					
1	1					
1						

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1	1	1	1	1	1	1
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All of these have the same set of 1s.

The partition with this set of 1s can be realized once for every square in the first row.

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Thus $\mathrm{cp}(0,1,1;n)$ is the total number of parts among all partitions of n.

Also,

$$\mathrm{cp}(0,1,1;n) = \sum_{k=0}^{n-1} p(k)d(n-k),$$

where $d(n)$ is the number of divisors of n.

$\mathrm{cp}(0, 0, 1; n)$

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$$\mathrm{cp}(0, 0, 1; n) = 2\mathrm{cp}(0, 1, 1; n) - p(n) = -p(n) + 2\sum_{k=0}^{n-1} p(k)d(n-k),$$

where $d(n)$ is the number of divisors of n.

Summary of Interesting Special Cases

- $\mathrm{cp}(1,1,2;n) = \mathcal{EO}^*(2n)$
- $\mathrm{cp}(1,1,1;n)$ is the total number of 1s among all partitions of $n+1$.
- $\mathrm{cp}(0,1,1;n)$ is the total number of parts among all partitions of n.
- $\mathrm{cp}(0,0,1;n)$ is sum of the perimeters of all partitions of n.
- $$\frac{1}{(q;q^5)_\infty(q^4;q^5)_\infty} = \frac{1}{(q^5;q^5)_\infty}\sum_{n=0}^{\infty}\mathrm{cp}(1,4,5;n)q^n$$
    - A new characterization of the Rogers-Ramanujan partitions
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Conclusion

- $\mathrm{cp}(1,1,2;n)$ shares an alarming number of properties with $p(n)$ and $c_2(n)$ regarding congruences modulo 5 witnessed by crank statistics.
- The symmetries of $\mathrm{cp}(a,b,m;n)$ are very clear, and they shed light on the symmetries of $\mathcal{EO}^*$ and (m,k)-capsid partitions.
- Conjugation allows us to prove some simple congruences modulo 2.
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Thanks!