Composition-theoretic series in partition theory

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q-Series and Related Topics
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Thinking about something that has nothing to do with partitions. . .

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where the c_n are written in terms of the a_n. We need $a_0 \neq 0$.

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The multiplicity $m_i = m_i(\lambda)$ of i in λ is the number of times that i appears as a part in λ.

It turns out you need partitions

$$c_n = \sum_{\lambda \vdash n} (-1)^{\ell(\lambda)} \frac{\ell(\lambda)!}{m_1! m_2! \cdots m_n!} a_0^{-\ell-1} a_1^{m_1} a_2^{m_2} \cdots a_n^{m_n}.$$

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This result appears in a different form in a paper by A. Salem (2011).

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Reciprocal power series coëfficients as a sum over compositions

Remember: $g(q) = \sum_{n=0}^{\infty} a_n q^n$ and $1/g(q) = \sum_{n=0}^{\infty} c_n q^n$.

Reciprocal power series coëfficients as a sum over compositions

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$$c_n = a_0^{-1} \sum_{\gamma \in \mathcal{C}_n} \left(-\frac{a_1}{a_0}\right)^{m_1} \left(-\frac{a_2}{a_0}\right)^{m_2} \cdots \left(-\frac{a_n}{a_0}\right)^{m_n},$$

where $\mathcal{C}_n$ is the set of all compositions of size n.

Ramanujan's notation and q-Pochhammer notation

$$f(a,b) := \sum_{n=-\infty}^{\infty} a^{n(n+1)/2} b^{n(n-1)/2}$$

$$= \prod_{j=0}^{\infty} (1 + a^{j+1} b^{j})(1 + a^{j} b^{j+1})(1 - (ab)^{j+1}).$$

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$(a_1, a_2, \cdots, a_r; q)_\infty := (a_1; q)_\infty (a_2; q)_\infty \cdots (a_r; q)_\infty$, we have

$$f(a,b) = (-a, -b, ab; ab)_\infty.$$

We exploit the fact that $f(-q, -q^2)$ is a lacunary series (most coëfficients are 0) and coëfficients are ± 1 at q^k where k is an extended pentagonal number, i.e. in the set $\{j(3j-1)/2 : j \in \mathbb{Z}\}$, We exploit the fact that $f(-q,-q^2)$ is a lacunary series (most coëfficients are 0) and coëfficients are ± 1 at q^k where k is an extended pentagonal number, i.e. in the set $\{j(3j-1)/2 : j \in \mathbb{Z}\}$, along with the fact that

$$\sum_{n=0}^{\infty} p(n)q^n = \frac{1}{f(-q,-q^2)},$$

where $p(n)$ denotes the number of partitions of size n.

Using $g(q) = f(q, -q^2)$

Let $\mathcal{C}_{\pentagon}$ denote the set of compositions where every part is an extended pentagonal number.

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$$p(n) = (-1)^n \sum_{\substack{\gamma \in \mathcal{C}_{\pentagon} \\ |\gamma| = n}} (-1)^{\ell^*(\gamma)},$$

where $\ell^*(\gamma)$ is the number of parts in γ that are of the form $n(3n-1)/2$ for $n \neq 0$ and $n \not\equiv 2 \pmod 3$.

Another related result

$$p(n) = \sum_{\substack{\gamma \in \mathcal{C}_{\pentagon} \\ |\gamma| = n}} (-1)^{\widehat{\ell}(\gamma)},$$

where $\widehat{\ell}(\gamma)$ is the number of parts in γ that are of the form $k(3k \pm 1)/2$ for k even and positive.

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$$pod(n) = (-1)^n \sum_{\substack{\gamma \in \mathcal{C}_\Delta \\ |\gamma| = n}} (-1)^{\ell(\gamma)},$$

where $\mathcal{C}_\Delta$ denotes the set of compositions where each part is a triangular number.

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The eight overpartitions of 3 are

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Let $\overline{p}(n)$ denote the number of overpartitions of size n.

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we deduce

$$\overline{p}(n) = (-1)^n \sum_{\substack{\gamma\in\mathcal{C}_{\square} \\ |\gamma|=n}} (-2)^{\ell(\gamma)},$$

where $\mathcal{C}_{\square}$ is the set of compositions where all parts are perfect squares.

Jacobi's identity

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$$p^{(3)}(n) = \sum_{\substack{\gamma \in \mathcal{C}_\Delta \\ |\gamma| = n}} 3^{m_1} (-5)^{m_3} 7^{m_6} (-9)^{m_{10}} 11^{m_{15}} \cdots$$

Now, Robert's turn. . .