

Asymptotic Enumeration of Sets of Primes and MacMahon's Partition Statistics

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Setting

$\mathcal{P}$ is a set of primes.

$N_{\mathcal{P}}$ is the set of all integers that can be written as a product of primes from $\mathcal{P}$, where repetitions are allowed.

Examples

$$\mathcal{P} = \{2, 3, 5\} \implies N_{\mathcal{P}} = \{2^a 3^b 5^c : a, b, c \geq 0 \text{ integers}\}$$

$\mathcal{P} = \text{all primes except 2 and 3} \implies N_{\mathcal{P}} = \text{all positive odd integers not divisible by 3}$

$$= \{1, 5, 7, 11, 13, 17, 19, 23, 25, \dots\}$$

Counting Functions

$$\mathcal{P} = \emptyset \implies N_{\mathcal{P}} = \{1\},$$

$$\mathcal{P} = \text{all primes} \implies N_{\mathcal{P}} = \mathbb{N} = \{1, 2, 3, \dots\}$$

Let

$$\pi_{\mathcal{P}}(x) = |\mathcal{P} \cap [1, x]|, \quad N_{\mathcal{P}}(x) = |N_{\mathcal{P}} \cap [1, x]|$$

be the counting functions of $\mathcal{P}$ and $N_{\mathcal{P}}$.

Main Problem

Main Problem. Study interactions between $N_{\mathcal{P}}(x)$ and $\pi_{\mathcal{P}}(x)$.

Obvious or trivial bounds:

$$\pi_{\mathcal{P}}(x) \leq N_{\mathcal{P}}(x) \leq [x] = x + O(1).$$

If $\mathcal{P}$ is finite, then $N_{\mathcal{P}}(x) \asymp (\log x)^{|\mathcal{P}|}$.

If $\mathcal{P}$ is all primes, then $N_{\mathcal{P}}(x) = [x]$.

$N_{\mathcal{P}}(x)$ ranges between a log-power and x (long range).

Classical Results

$N_{\mathcal{P}}(x) = [x] \implies \mathcal{P}$ is the set of all primes

$$\implies \pi_{\mathcal{P}}(x) = \pi(x) \asymp \frac{x}{\log x}$$

(Chebyshev, 1850's)

Finer conclusions led to big breakthroughs or notorious open problems.

$$\pi(x) \sim \frac{x}{\log x} \quad (\text{PNT, 1896})$$

$$\pi(x) = \text{li}(x) + O_{\varepsilon}(x^{\frac{1}{2}+\varepsilon}) \quad \text{for all } \varepsilon > 0$$

(RH, 1860)

A Common Mistake

$$\text{li}(x) = \int_2^x \frac{1}{\log t} dt \implies \text{logarithmic integral.}$$

It is a common mistake to write

$$\pi(x) = \frac{x}{\log x} + O_\varepsilon(x^{\frac{1}{2}+\varepsilon})$$

for RH. Indeed

$$\pi(x) = \frac{x}{\log x} + O\left(\frac{x}{(\log x)^2}\right).$$

Smaller Regimes

$$N_{\mathcal{P}}(x) = cx + O_{\gamma}\left(\frac{x}{(\log x)^{\frac{3}{2}+\gamma}}\right) \implies \pi_{\mathcal{P}}(x) \sim \frac{x}{\log x}$$

for some constants $0 < c \leq 1$ and $\gamma > 0$ (General form of PNT, Beurling, 1937)

This includes the classical PNT by taking $c = 1$ and any $\gamma > 0$.

$$N_{\mathcal{P}}(x) = cx + o\left(\frac{x}{(\log x)^2}\right) \implies \pi_{\mathcal{P}}(x) \sim \frac{x}{\log x}$$

(General Elementary Proof of PNT, Selberg, 1950's)

Other important contributions by Malliavin, Bateman, Diamond,

...

Weaker Expectations

What if $N_{\mathcal{P}}(x) \sim cx$?

No asymptotic follows in general for $\pi_{\mathcal{P}}(x)$. This puts an end to the PNT type results.

The problem is still interesting for weaker expectations.

From a weak assumption on $N_{\mathcal{P}}(x)$, can we obtain

$$\pi_{\mathcal{P}}(x) \geq f(x)$$

for explicit functions $f(x)$?

Still a Smaller Regime

(Wirsing, 1956)

$$N_{\mathcal{P}}(x) \sim \frac{cx}{(\log x)^{1-\tau}} \quad \text{for some } c > 0, 0 < \tau < 1$$

$$\iff \sum_{\substack{p \leq x \\ p \in \mathcal{P}}} \frac{1}{p} = \tau \log \log x + B_{\mathcal{P}} + o(1) \quad \text{as } x \rightarrow \infty,$$

i.e. $\mathcal{P}$ satisfies an expected Mertens type formula with the Mertens–Meissel constant $B_{\mathcal{P}}$.

Classical Mertens formula:

$$\sum_{p \leq x} \frac{1}{p} = \log \log x + B + o(1), \quad B = 0.2614 \dots$$

(Mertens, 1870)

Primes in Progressions

Wirsing criterion does not give a direct conclusion on the size of $\pi_{\mathcal{P}}(x)$. Boundary case $\tau = 1$ already gives no universal conclusion on $\pi_{\mathcal{P}}(x)$ like a PNT. But there are cases where this happens.

$\mathcal{P} = \{\text{all primes in the progression } \equiv \ell \pmod{k}\}$, $(\ell, k) = 1$, $k \geq 3$, $1 \leq \ell < k$. Then

$$\sum_{\substack{p \leq x \\ p \in \mathcal{P}}} \frac{1}{p} = \frac{1}{\phi(k)} \log \log x + B_{\ell, k} + o(1) \quad (\text{Dirichlet})$$

$$\iff N_{\mathcal{P}}(x) \sim \frac{C_{\mathcal{P}} x}{(\log x)^{1 - \frac{1}{\phi(k)}}}.$$

A First Appetizer

$$N_{\mathcal{P}}(x) \sim \frac{cx}{(\log x)^{1-\tau}} \quad (\text{Wirsing regime})$$

Then

$$\sum_{\substack{p \leq x \\ p \in \mathcal{P}}} \frac{1}{p} = \tau \log \log x + B_{\mathcal{P}} + o(1),$$

so that for every small $\varepsilon > 0$,

$$\tau \log \frac{1}{1-\varepsilon} + o(1) = \sum_{\substack{x^{1-\varepsilon} < p \leq x \\ p \in \mathcal{P}}} \frac{1}{p} \leq \frac{\pi_{\mathcal{P}}(x)}{x^{1-\varepsilon}}$$

and

$$\pi_{\mathcal{P}}(x) \geq \left(\tau \log \frac{1}{1-\varepsilon} + o(1) \right) x^{1-\varepsilon} = f_{\varepsilon}(x)$$

achieves our weaker expectation.

A Universal Lower Bound for $\pi_{\mathcal{P}}(x)$

Assume $x^\varepsilon \leq N_{\mathcal{P}}(x) \leq x$ for some $\varepsilon > 0$ as $x \rightarrow \infty$.

By UFP, any $n \in N_{\mathcal{P}} \cap [1, x]$ can be written as a product

$$n = \prod_{p \in \mathcal{P} \cap [1, x]} p^{\nu_p}, \quad \nu_p \text{ nonnegative exponents.}$$

Clearly, $\nu_p \leq \left\lceil \frac{\log x}{\log p} \right\rceil \leq 2 \log x$ for all $p \in \mathcal{P} \cap [1, x]$.

The number of all such products cannot exceed $(2 \log x)^{\pi_{\mathcal{P}}(x)}$. Since all $n \in N_{\mathcal{P}} \cap [1, x]$ are among these,

$$x^\varepsilon \leq N_{\mathcal{P}}(x) = |N_{\mathcal{P}} \cap [1, x]| \leq (2 \log x)^{\pi_{\mathcal{P}}(x)}$$

and

$$\pi_{\mathcal{P}}(x) \gg \frac{\log x}{\log \log x}$$

(Entropy type bound for any regime which is not very small.)

Beating the Entropy Bound

Any bound which beats the entropy type bound by assuming only a regime on $N_{\mathcal{P}}(x)$ is of interest.

In the Wirsing regime, $x^{1-\varepsilon}$ is much larger than $\frac{\log x}{\log \log x}$. But Wirsing regime assumes an asymptotic formula on $N_{\mathcal{P}}(x)$.

Erdős' Book Proof

(Erdős, 1938)

$$\pi_{\mathcal{P}}(x) \gg \log x \quad \text{whenever } x^\varepsilon \leq N_{\mathcal{P}}(x) \leq x.$$

Slightly beats the entropy bound on a much weaker hypothesis on $N_{\mathcal{P}}(x)$.

(One of the “book” proofs in Aigner and Ziegler.)

Weaker Form of Wirsing Regime

One sided hypothesis. Assume

$$N_{\mathcal{P}}(x) \geq \frac{(C_{\mathcal{P}} + o(1))x}{(\log x)^{1-\alpha}}, \quad C_{\mathcal{P}} > 0, \quad 0 < \alpha \leq 1. \quad (\text{H})$$

Thm 1. Under (H), $\pi_{\mathcal{P}}(x) \geq (\log x)^M \Rightarrow$ any log-power, as $x \rightarrow \infty$.

Thm 2. Under (H), $\pi_{\mathcal{P}}(x) \geq e^{\kappa_{\mathcal{P}}(\log x)^{\alpha}}, \quad \kappa_{\mathcal{P}} > 0$.

Taking $\alpha = 1$, $\pi_{\mathcal{P}}(x) \geq x^{\kappa_{\mathcal{P}}}$, $\pi(x) \geq x^{\kappa}$.

Proofs use only weighted counting and Eratosthenian counting, namely the Inclusion–Exclusion principle.

MacMahon's q -Series

$$\sum_{n=1}^{\infty} M_a(n) q^n = \sum_{0 < s_1 < \dots < s_a} \frac{q^{s_1 + \dots + s_a}}{(1 - q^{s_1})^2 \dots (1 - q^{s_a})^2}$$

(MacMahon q -series, 1920)

$$M_a(n) = \sum_{\substack{0 < s_1 < \dots < s_a \\ n = m_1 s_1 + \dots + m_a s_a}} \left(\prod_{i=1}^a m_i \right)$$

$s_1, \dots, s_a$ are the constituents, $m_1, \dots, m_a$ are the multiplicities.

These are heavily additive objects.

Analytic combinatorics, analytic theory of partitions.

Craig–van Ittersum–Ono

Certain explicit polynomial combinations of $M_a(n)$ capture primes as their integral zeros.

Thm 3

Thm 3. Assume (H). Then as $x \rightarrow \infty$, the number of $n \in N_{\mathcal{P}} \cap (1, x]$ with

$$\begin{aligned} & (25n^4 - 171n^3 + 423n^2 - 447n + 170) M_1(n) \\ & + (300n^3 - 3554n^2 + 12900n - 14490) M_2(n) \\ & + (2400n^2 - 60480n + 214080) M_3(n) - 725760 M_4(n) = 0 \end{aligned}$$

is at least $e^{\kappa_{\mathcal{P}}(\log x)^{\alpha}}$.

Proof of Thm 3

By Craig–van Ittersum–Ono, this polynomial combination is 0 for $n > 1$

$$\iff n \text{ is prime.}$$

But the only primes in $N_{\mathcal{P}}$ are the primes in $\mathcal{P}$, and Thm 2 completes the proof.