Congruences for Ramanujan's Third Order Mock Theta Functions

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Seminar in Partition Theory, q-Series and Related Topics
Michigan Technological University
Department of Mathematical Sciences
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ABSTRACT: We study the action of weight 1/2 Hecke operators acting on harmonic Maaß forms associated with Ramanujan's third order mock theta functions f(q) and ω(q). We obtain new congruences which are extensions of previous works of Ono, Bruinier and others.

https://qseries.org/fgarvan/research.html

RESEARCH

Interests: **q**-series, partitions, congruences for modular functions, symbolic computation.

- CURRICULUM VITAE (Updated January 5, 2021)
- A q-series MAPLE package (Version 1.3, Released August 13, 2016)
  Related MAPLE packages:
  - RANK
  - CRANK
  - SPT-CRANK
  - T-CORE (Version 0.2, Released November 8, 2023; ONLY UF-APPS version)
  - BAILEY
  - ETA (Dedekind eta-function; Version 0.3, Released June 29, 2019)
  - THETAIDS (Theta-function identities; Version 1.1, Updated October 15, 2022)
  - RAMAROBINSIDS (Ramanujuan-Robins theta-function identities; Version 0.2, Released July 21, 2018)
  - MISC (Version 0.5a, Released May 1, 2020) (MANUAL, January 25, 2022)
  - ramamocktheta (Version 0.1, Released September 29, 2020; ONLY UF-APPS version)
  - qsOEIS (Version 0.2, Updated January 26, 2022; ONLY UF-APPS version)
  - etatheta (Version 0.2, Released October 5, 2020; ONLY UF-APPS version)
  - modforms (Version 0.2, Released January 30, 2021; ONLY UF-APPS version)
  - orank (Version 1.0, Released October 15, 2022; ONLY UF-APPS version)
  - ocrank (Version 1.0, Released October 15, 2022; ONLY UF-APPS version)
  - partitions (Version 0.2, Updated June 16, 2023; ONLY UF-APPS version)
  - RANK AND CRANK PACKAGES
    (New additions: m2rank, m2crank and ocrank2)
- PUBLICATIONS (Updated Friday, January 10, 2025)
- CITATIONS
- DATA FILES (Created June 24, 2009; Updated May 13, 2014)
- THE RAMANUJAN JOURNAL
- REFERENCE
  (original math papers you should read)
- MY READING COURSE on Partitions and q-Series
- MY READING COURSE on Modular Functions
- TALKS

The url of this page is **https://qseries.org/fgarvan/research.html**.
Created by F.G. Garvan (*fgarvan@ufl.edu*) on Thursday, May 10, 2018.
Last update made Fri Jan 10 13:00:58 CST 2025.

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```
> currentdir();
        "C:\cygwin64\home\fgarv\math\talks\MICHIGAN-PTNS-QS-SEMINAR-2025"      (1)
> with(misc):
> SaveAll("mod11-23-vars-01-23-25.m"):
> SaveAll("mod11-23-vars-01-22-25.m"):
> read "mod11-23-vars-01-23-25.m":
> read "mod11-23-vars-01-22-25.m":
> read prog2:
["PACKAGES:", "qseries", "ETA", "ramamocktheta", "qsOEIS"]
"DIM f3AMAT =", 100001, 2
"DIM omega3AMAT =", 500000, 2
"Checking coeffs of f3 up to ", 2001
Checking coeffs of omega3 up to qdegree(om3q)
{0}
["FUNCTIONS:", "f3gen", "om3gen2", "EH2", "H4", "EK2", "K4", "fWp",
```

```
"om3Wp"]
```

Warning, (in fWp) `n` is implicitly declared local |prog2:31|
Warning, (in altfWp) `n` is implicitly declared local |prog2:34|
Warning, (in om3Wp) `n` is implicitly declared local |prog2:37|
Warning, (in altom3Wp) `n` is implicitly declared local |prog2:38|

```
> aomega(500000-1);
```

3569303586622289481506555171881979795499233698985605980896078322409093015484165 4\
979412220539527936320536438255164680012451658319489789928650659554856660184 61\
411067617188702495809254329556975751903709467557621262388023715088896664379 10\
463612223764372993850714440801519249231065041221329391276905394509869919782 58\
188201218916838701460899579091751112700646457865024193209309438639053223342 93\
293612843741129714484242861853309908257194290915376100605618425434433091305 95\
134725400582117071000509951912907251150699518302496844468849054844981463615 75\
872966603920 **(2)**

```
> ifactor(%,easy);
```

$(2)^4\ (5)\ (7)\ (11)\ (29)\ \_c544\_1\ (28927)$ **(3)**

===
====

```
> seq(omega3AMAT[k,2],k=1..20);
```

1, 2, 3, 4, 6, 8, 10, 14, 18, 22, 29, 36, 44, 56, 68, 82, 101, 122, 146, 176 **(4)**

```
>
```

Search: **seq:1,2,3,4,6,8,10,14,18,22,29,36,44,56,68,82,101,122,146,176**

Displaying 1-2 of 2 results found. page 1

Sort: relevance | references | number | modified | created Format: long | short | data

A053253 Coefficients of the '3rd-order' mock theta function omega(q). +30 18

1, 2, 3, 4, 6, 8, 10, 14, 18, 22, 29, 36, 44, 56, 68, 82, 101, 122, 146, 176, 210, 248, 296, 350, 410, 484, 566, 660, 772, 896, 1038, 1204, 1391, 1602, 1846, 2120, 2428, 2784, 3182, 3628, 4138, 4708, 5347, 6072, 6880, 7784, 8804, 9940, 11208, 12630

(list; graph; refs; listen; history; text; internal format)

OFFSET 0,2

COMMENTS Empirical: a(n) is the number of integer partitions mu of 2n+1 such that the diagram of mu has an odd number of cells in each row and in each column. - John M. Campbell, Apr 24 2020

From Gus Wiseman, Jun 26 2022: (Start)

By Campbell's conjecture above that a(n) is the number of partitions of 2n+1 with all odd parts and all odd conjugate parts, the a(0) = 1 through a(5) = 8 partitions are (B = 11):

```
(1)  (3)    (5)      (7)        (9)          (B)
     (111)  (311)    (511)      (333)        (533)
            (11111)  (31111)    (711)        (911)
                     (1111111)  (51111)      (33311)
                                (3111111)    (71111)
                                (111111111)  (5111111)
                                             (311111111)
                                             (11111111111)
```

These partitions are ranked by A352143.

(End)

===
=====

```
> seq(f3AMAT[k,2],k=1..20);
```

1, 1, −2, 3, −3, 3, −5, 7, −6, 6, −10, 12, −11, 13, −17, 20, −21, 21, −27, 34 **(5)**

Search: **seq:1,1,-2,3,-3,3,-5,7,-6,6,-10,12,-11,13,-17,20,-21,21,-27,34**

Displaying 1-2 of 2 results found. page 1

Sort: relevance | references | number | modified | created Format: long | short | data

A000025 Coefficients of the 3rd-order mock theta function f(q). +30 21

(Formerly M0433 N0164)

1, 1, -2, 3, -3, 3, -5, 7, -6, 6, -10, 12, -11, 13, -17, 20, -21, 21, -27, 34, -33, 36, -46, 51, -53, 58, -68, 78, -82, 89, -104, 118, -123, 131, -154, 171, -179, 197, -221, 245, -262, 279, -314, 349, -369, 398, -446, 486, -515, 557, -614, 671, -715, 767, -845, 920, -977, 1046, -1148, 1244

(list; graph; refs; listen; history; text; internal format)

OFFSET 0,3

COMMENTS a(n) = number of partitions of n with even rank minus number with odd rank. The rank of a partition is its largest part minus the number of parts.

REFERENCES G. E. Andrews, The Theory of Partitions, Addison-Wesley, 1976, p. 82, Examples 4 and 5.
Srinivasa Ramanujan, Collected Papers, Chelsea, New York, 1962, pp. 354-355
Srinivasa Ramanujan, The Lost Notebook and Other Unpublished Papers, Narosa Publishing House, New Delhi, 1988, pp. 17, 31.
N. J. A. Sloane, A Handbook of Integer Sequences, Academic Press, 1973 (includes this sequence).
N. J. A. Sloane and Simon Plouffe, The Encyclopedia of Integer Sequences, Academic Press, 1995 (includes this sequence).

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CONGRUENCES

```
> with(qseries):
> f3GEN:=add(f3AMAT[k,2]*q^(k-1),k=1..50010):
> omega3GEN:=add(omega3AMAT[k,2]*q^(k-1),k=1..100010):
> findcong(f3GEN,100000,1000);
```

$$\{[299, 875, 7], [649, 875, 7]\} \quad (6)$$

```
>  a(p*n+d) = 0;
```

$$a(p\,n + d) = 0 \quad (7)$$

```
> ifactor(875);
```

$$(5)^3\,(7) \quad (8)$$

```
> sqrt(500000.);
```

$$707.1067812 \quad (9)$$

```
> findcong(omega3GEN,500000,1000,{2,4,8,16});
```

$$\{[6, 80, 5], [7, 56, 7], [14, 544, 17], [19, 88, 11], [23, 56, 7], [27, 40, 5], [31, 56, 7], [35, 40, 5], [51, 88, 11], [59, 88, 11], [66, 80, 5], [67, 88, 11], [78, 352, 11], [83, 88, 11], [110, 160, 5], [142, 160, 5], [142, 544, 17], [174, 544, 17], [186, 896, 7], [206, 352, 11], [238, 352, 11], [270, 352, 11], [302, 544, 17], [334, 352, 11], [366, 544, 17], [398, 544, 17], [442, 640, 5], [462, 544, 17], [494, 544, 17], [570, 640, 5], [641, 875, 7], [816, 875, 7]\} \quad (10)$$

MR4298484 - Congruences modulo powers of 5 for the rank parity function

Chen, Dandan; Chen, Rong; Garvan, Frank

Hardy-Ramanujan J. **43** (2020), 24–45.

(Reviewer: Dai, Haobo)

Rong Chen Dandan Chen and Frank Garvan, *Congruences modulo powers of 5 and 7 for the crank and rank parity functions and related mock theta functions*, (2024), submitted.

Theorem 1.3. *For all $\alpha \geq 3$ and all $n \geq 0$ we have*

$$a_f(5^\alpha n + \delta_\alpha) + a_f(5^{\alpha-2} n + \delta_{\alpha-2}) \equiv 0 \pmod{5^{\lfloor \frac{1}{2}\alpha \rfloor}}, \tag{1.6}$$

where δ_α satisfies $0 < \delta_\alpha < 5^\alpha$ and $24\delta_\alpha \equiv 1 \pmod{5^\alpha}$.

In [14], we also stated the following theorem without proof.

Theorem 1.4. *For all $\alpha \geq 3$ and all $n \geq 0$ we have*

$$a_f(7^\alpha n + \delta_\alpha) - a_f(7^{\alpha-2} n + \delta_{\alpha-2}) \equiv 0 \pmod{7^{\lfloor \frac{1}{2}(\alpha-1) \rfloor}}, \tag{1.7}$$

where δ_α satisfies $0 < \delta_\alpha < 7^\alpha$ and $24\delta_\alpha \equiv 1 \pmod{7^\alpha}$.

Theorem 1.5. (i) *For all $\alpha \geq 3$ and all $n \geq 0$ we have*

$$a_\omega(5^\alpha n + \delta_\alpha) + a_\omega(5^{\alpha-2} n + \delta_{\alpha-2}) \equiv 0 \pmod{5^{\lfloor \frac{1}{2}\alpha \rfloor}}, \tag{1.8}$$

where δ_α satisfies $0 < \delta_\alpha < 5^\alpha$ and $3\delta_\alpha + 2 \equiv 0 \pmod{5^\alpha}$.

(ii) *For all $\alpha \geq 3$ and all $n \geq 0$ we have*

$$a_\omega(7^\alpha n + \delta_\alpha) + a_\omega(7^{\alpha-2} n + \delta_{\alpha-2}) \equiv 0 \pmod{7^{\lfloor \frac{1}{2}(\alpha-1) \rfloor}}, \tag{1.9}$$

where δ_α satisfies $0 < \delta_\alpha < 7^\alpha$ and $3\delta_\alpha + 2 \equiv 0 \pmod{7^\alpha}$.

Remark. We note that Karl-Heine Fricke [19] independently observed (1.6)–(1.9) but without proof.

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RAMANUJAN'S LAST LETTER AND WATSON'S PAPER

$$2\phi(-q)-f(q)=f(q)+4\psi(-q)=\vartheta_4(0,\, q)\prod_{r=1}^{\infty}(1+q^r)^{-1},$$

$$4\chi(q)-f(q)=3\vartheta_4{}^2(0,\, q^3)\prod_{r=1}^{\infty}(1-q^r)^{-1},$$

$$2\rho(q)+\omega(q)=3[\tfrac{1}{2}q^{-\frac{3}{8}}\vartheta_2(0,\, q^{\frac{3}{2}})]^2\prod_{r=1}^{\infty}(1-q^{2r})^{-1},$$

$$\upsilon(\pm q)\pm q\omega(q^2)=\tfrac{1}{2}q^{-\frac{1}{4}}\vartheta_2(0,\, q)\prod_{r=1}^{\infty}(1+q^{2r}),$$

$$f(q^8)\pm 2q\omega(\pm q)\pm 2q^3\omega(-q^4)=\vartheta_3(0,\, \pm q)\vartheta_3{}^2(0,\, q^2)\prod_{r=1}^{\infty}(1-q^{4r})^{-2}.$$

```
> qetamake(theta4(q,100)/aqprod(-q,q,100),q,20);
```

$$\frac{\_E(q)^3}{\_E(q^2)^2} \tag{11}$$

$$E(q)=\prod_{n=1}^{\infty}(1-q^n) \tag{12}$$

Let

(1.3) $$\mathcal{B}(\tau)=\frac{\eta(\tau)^3}{\eta(2\tau)^2}=q^{-1/24}B(q),$$

where

$$B(q)=\frac{(q)_\infty^3}{(q^2;q^2)_\infty^2}=\sum_{n=0}^{\infty}\beta(n)q^n$$

and

$$\beta(n)=M_e(n)-M_o(n),$$

```
> qetamake(1/2*theta2(q,100)/q^(1/4)*aqprod(-q^2,q^2,100),q,10);
```

$$\frac{\_E(q^4)^3}{\_E(q^2)^2} \tag{13}$$

===

ZWEGERS AND WATSONS TRANSFORMATIONS

<table>
<tr><td>Date: January 16 2025</td><td>Title: Mock Theta Transformations without Watson</td></tr>
<tr><td>Speaker: Avi Mukhopadhyay</td><td rowspan="2">Abstract: Ramanujan in his last letter to G.H. Hardy introduced mock theta functions, provided examples and various relations between them. G.N. Watson found transformations for the third order mock theta functions $f(q)$ and $\omega(q)$. Zwegers in 2000 built on Watson's techniques to complete these mock theta functions and connected them to real analytic modular forms. We show how to derive these transformations using Lerch sums. To show the equivalence of the results involves some new q-series identities thus resulting in a new proof of Zwegers' theorem. This is joint work with Dr. Frank Garvan.

This talk was rescheduled from the cancelled October 2024 AMS Southeastern Sectional in Savannah.</td></tr>
<tr><td>Affiliation: UFL</td></tr>
<tr><td>Video:</td><td>Video</td></tr>
<tr><td>Slides:</td><td>Avi Mukhopadhyay slides</td></tr>
</table>

Define $F = (f_0, f_1, f_2)^T$ by

$$f_0(q) := q^{-1/24} f(q),$$
$$f_1(q) := 2q^{1/3}\omega(q^{1/2}),$$
$$f_2(q) := 2q^{1/3}\omega(-q^{1/2}).$$

Recall

- $f(q) = 1 + \frac{q}{(1+q)^2} + \frac{q^4}{(1+q)^2(1+q^2)^2} + \ldots = \sum_{n=0}^{\infty} \frac{q^{n^2}}{(-q;q)_n^2}.$
- $\omega(q) = \frac{1}{(1-q)^2} + \frac{q^4}{(1-q)^2(1-q^3)^2} + \ldots = \sum_{n=0}^{\infty} \frac{q^{2n^2+2n}}{(q;q^2)_{n+1}^2}.$

>

The G function

Define

$$g_0(z) := \sum_{n\in\mathbb{Z}} (-1)^n (n+1/3) e^{3\pi(n+1/3)^2 z},$$
$$g_1(z) := -\sum_{n\in\mathbb{Z}} (n+1/6) e^{3\pi(n+1/6)^2 z},$$
$$g_2(z) := \sum_{n\in\mathbb{Z}} (n+1/3) e^{3\pi(n+1/3)^2 z}.$$

We then define

$$G(\tau) := 2i\sqrt{3} \int_{-\tau}^{i\infty} \frac{(g_1(z), g_0(z), g_2(z))^T}{\sqrt{-i(z+\tau)}}\, dz.$$

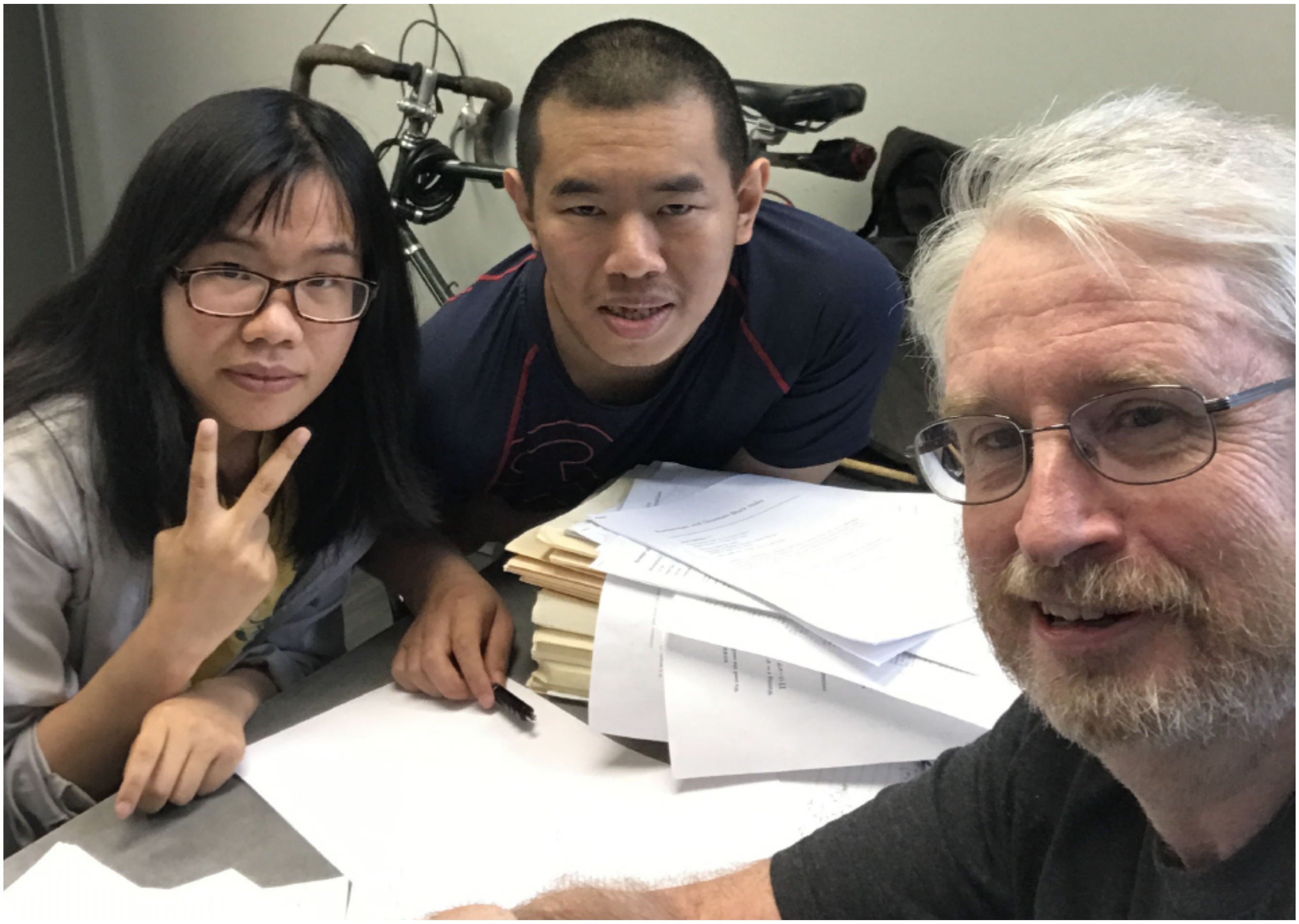
Theorem (Zwegers (2000))

The function H defined by $H(\tau) := F(\tau) - G(\tau)$ is a vector valued real analytic modular form of weight $1/2$ satisfying

- $H(\tau+1) = \begin{pmatrix} \zeta_{24}^{-1} & 0 & 0 \\ 0 & 0 & \zeta_3 \\ 0 & \zeta_3 & 0 \end{pmatrix} H(\tau),$
- $\frac{1}{\sqrt{-i\tau}} H(\frac{-1}{\tau}) = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} H(\tau).$

$$
\begin{aligned}
H(\tau) &= (h_0(\tau), h_1(\tau), h_2(\tau)) \\
&= (f_0(q), f_1(q), f_2(q))^T - 2i\sqrt{3} \int^{i\infty} \frac{(g_1(z), g_0(z), g_2(z))^T}{\sqrt{-i(z+\tau)}}\, dz.
\end{aligned}
$$

WEIGHT ZERO VERSION (Rong Chen and Dandan Chen)



Let

$$\widetilde{H} = (H_0, H_1, H_2) := \left(\frac{\eta(2\tau)^2}{\eta(\tau)^3} h_0(\tau), \frac{\eta(\tau/2)^2}{2\eta(\tau)^3} h_1(\tau), \frac{\eta(\tau)^3}{2\eta(\tau/2)^2 \eta(\tau)^2} h_2(\tau) \right). \tag{1.5}$$

THEN

$$\widetilde{H}(\tau+1) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & i \\ 0 & i & 0 \end{pmatrix} \widetilde{H}(\tau).$$

$$\widetilde{H}(-1/\tau) = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \widetilde{H}(\tau).$$

Lemma 1.4. *For $\gamma \in \Gamma_0(2)$,*

$$H_0(\gamma\tau) = v(\gamma)\, H_0(\tau).$$

WHERE

$$v\,:\,\Gamma_0(2) \longrightarrow \mathbb{C}$$

by

$$v\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = \begin{cases} +1 & \text{if } c \equiv 0 \pmod 4 \\ -1 & \text{if } c \equiv 2 \pmod 4 \end{cases}$$

is a character.

Let

Corollary 1.5. *For $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(2)$ we have*

$$h_0(\gamma\tau) = v(\gamma)\,\nu_{\mathcal{B}}(\gamma)\,\sqrt{c\tau + d}\,h_0(\tau).$$

Corollary 1.7. *If $p > 3$ is prime the function*

$$U_p^*(\tau) = \frac{h_0(p^2\tau)}{h_0(\tau)}$$

satisfies

$$U_p^*(\gamma\tau) = U_p^*(\tau),$$

for $\gamma \in \Gamma_0(2p^2)$.

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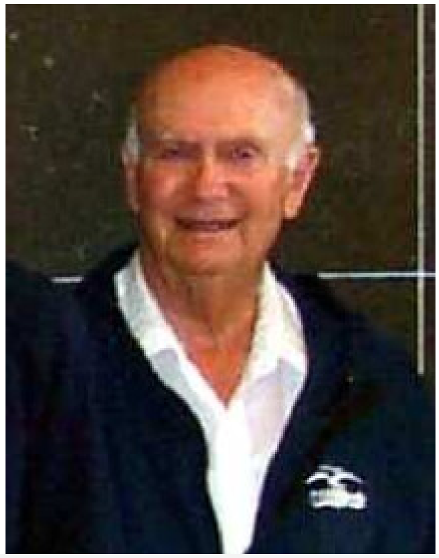
NEWMAN'S APPROACH TO HALF-INTEGER WEIGHT HECKE OPERATORS



MR0146150 - Modular forms whose coefficients possess multiplicative properties. II

Newman, Morris
Ann. of Math. (2) **75** (1962), 242–250.
(Reviewer: Rankin, R. A.)

We follow the method of Newman. Let $\Gamma_0 < \Gamma_1$ be subgroups of finite index in $\mathrm{SL}_2(\mathbb{Z})$. Suppose the function $f(\tau)$ is on Γ_0 then the function

$$\mathrm{Tr}(\Gamma_0, \Gamma_1)(f) := \sum_{j=1}^{n} f(R_n \tau)$$

is a function on the group Γ_1, where R_1, R_2, $\ldots$, R_n are right coset representatives of Γ_0 in Γ_1. We will calculate the following traces:

$$V_p^*(\tau) = \mathrm{Tr}(\Gamma_0(2p^2), \Gamma_0(2p))(U_p^*(\tau)), \quad \text{and} \quad W_p^*(\tau) = \mathrm{Tr}(\Gamma_0(2p), \Gamma_0(2))(V_p^*(\tau)).$$

==

Theorem 1.8. *Let $p > 3$ be prime. Then*

$$V_p^*(\tau) = \mathrm{Tr}(\Gamma_0(2p^2), \Gamma_0(2p))(U_p^*(\tau))$$
$$= U_p^*(\tau) + \frac{1}{\sqrt{p}} \exp\left(\frac{-\pi i(p-1)}{4}\right) \sum_{n=1}^{p-1} \left(\frac{n}{p}\right) \exp\left(\frac{-2\pi i p n}{24}\right) \frac{h_0(\tau - n/p)}{h_0(\tau)}.$$

Theorem 1.9. *Let $p > 3$ be prime and $s_p = \frac{1}{24}(p^2-1)$. Then*

$$V_p(\tau) = \frac{1}{B(q)} \sum_{n=-s_p}^{\infty} \left(a_f\left(\frac{n+s_p}{p^2}\right) + \left(\frac{3}{p}\right)\left(\frac{24n-1}{p}\right) a_f(n) - \left(\frac{-3}{n}\right) a_f(n) \right) q^n$$

is a weakly holomorphic modular function on $\Gamma_0(4p)$ that satisfies

$$V_p(\gamma\tau) = \upsilon(\gamma)\, V_p(\tau),$$

for $\gamma \in \Gamma_0(2p)$.

Examples. We illustrate Theorem 1.9 Vp with some examples. $V_5(\tau)$ is a modular function on $\Gamma_0(20)$. We find

$$V_5(\tau) = \frac{1}{B(q)} \sum_{n=-1}^{\infty} \left(a_f\left(\frac{n+1}{25}\right) - \left(\frac{24n-1}{5}\right) a_f(n) + a_f(n) \right) q^n$$
$$= 6 + \frac{\eta(5\tau)\eta(\tau)^3}{\eta(10\tau)^3\eta(2\tau)} + 8\frac{\eta(20\tau)^3\eta(5\tau)^2\eta(4\tau)\eta(2\tau)}{\eta(10\tau)^5\eta(\tau)^2}.$$

Theorem 1.9 Vp also implies $V_5(\tau)^2$ is a modular function on $\Gamma_0(10)$. We find

$$V_5(\tau)^2 = 52 + \frac{1}{H_{10}(\tau)^2} + 12\frac{1}{H_{10}(\tau)^2} + 80 H_{10}(\tau),$$

where

$$H_{10}(\tau) = \frac{\eta(10\tau)^3\eta(2\tau)}{\eta(5\tau)\eta(\tau)^3},$$

is a Hauptmodul for $\Gamma_0(10)$.

Theorem 1.10. *Let $p > 3$ be prime and $s_p = \frac{1}{24}(p^2-1)$. Then*

$$W_p(\tau) = \frac{1}{B(q)} \sum_{n=-s_p}^{\infty} \left(p a_f(p^2 n - s_p) + \left(\frac{3}{p}\right)\left(\frac{24n-1}{p}\right) a_f(n) + a_f\left(\frac{n+s_p}{p^2}\right) - (p+1)\left(\frac{-3}{p}\right) a_f(n) \right) q^n$$

is a weakly holomorphic modular function on $\Gamma_0(4)$ that satisfies

$$W_p(\gamma\tau) = \upsilon(\gamma)\, W_p(\tau),$$

for $\gamma \in \Gamma_0(2)$.

Examples. We illustrate Theorem 1.10 Wp with some examples. $W_5(\tau)$ is a modular function on $\Gamma_0(4)$. We find

$$\begin{aligned} W_5(\tau) &= \frac{1}{B(q)} \sum_{n=-1}^{\infty} \left(5\, a_f(25n-1) - \left(\frac{24n-1}{5}\right) a_f(n) + a_f\left(\frac{n+1}{25}\right) + 6\, a_f(n) \right) q^n \\ &= H_4(\tau) - \frac{256}{H_4(\tau)}, \end{aligned}$$

where

$$H_4(\tau) = \frac{\eta(2\tau)^{24}}{\eta(4\tau)^{16}\eta(\tau)^8}$$

is a Hauptmodul for $\Gamma_0(4)$. Theorem 1.10 Wp also implies that $W_5(\tau)^2$ is a modular function on $\Gamma_0(2)$. We find

$$W_5(\tau)^2 = H_2(\tau)^2 + 64H_2(\tau),$$

where

$$H_2(\tau) = \frac{\eta(\tau)^{24}}{\eta(2\tau)^{24}}$$

is a Hauptmodul for $\Gamma_0(2)$.

Theorem 2.7. *Let $p > 3$ be prime and $N_p = \frac{1}{6}(p^2-1)$. Then*

$$\begin{aligned} Z_p(\tau) = \frac{q^{1/2}}{A(q)} \sum_{n=0}^{\infty} \Bigg(& p a_\omega(p^2 n + 2(p^2-1)/3) + \left(\frac{-1}{p}\right)\left(\frac{N_p + \frac{1}{2}(p-1) - n}{p}\right) a_\omega(n) + a_\omega\left(\frac{n-4N_p}{p^2}\right) \\ & - \left(\frac{-3}{p}\right)(p+1) a_\omega(n) \Bigg) q^n \end{aligned}$$

is a weakly holomorphic weight zero modular form that satisfies

$$Z_p(\gamma\tau) = w(\gamma)\, Z_p(\tau),$$

for $\gamma \in \Gamma_0(2)$. Further, $(Z_p(\tau))^2$ is a modular function on $\Gamma_0(2)$ and $Z_p(2\tau)$ is a modular function on $\Gamma_0(4)$.

Examples. We illustrate Theorem 2.7 Zp with some examples. $Z_5(2\tau)$ is a modular function on $\Gamma_0(4)$. We find

$$Z_5(2\tau) = \frac{1}{A(q^2)} \sum_{n=0}^{\infty} \left(5\, a_\omega(25n+16) + \left(\frac{n-1}{5}\right) a_\omega(n) + a_\omega\left(\frac{n-16}{25}\right) + 6\, a_\omega(n) \right) q^{2n+1}$$
$$= 16\, K_4(\tau) - \frac{16}{K_4(\tau)},$$

where

$$K_4(\tau) = \frac{\eta(2\tau)^{24}}{\eta(4\tau)^8 \eta(\tau)^{16}}$$

is a Hauptmodul for $\Gamma_0(4)$. Theorem 2.7 Zp also implies that $Z_5(\tau)^2$ is a modular function on $\Gamma_0(2)$. We find

$$Z_5(\tau)^2 = 2^{18}\, H_2(\tau)^{-1} + 2^{24}\, H_2(\tau)^{-2},$$

where

$$H_2(\tau) = \frac{\eta(\tau)^{24}}{\eta(2\tau)^{24}}$$

is our Hauptmodul for $\Gamma_0(2)$.

G. (2024)

Corollary

Suppose $p > 3$ is prime and $k_p = \frac{2}{3}(p^2 - 1)$. Then

$$p\, a_\omega(p^2 n + k_p) + \left(\frac{3}{p}\right)\left(\frac{3n+2}{p}\right) a_\omega(n) + a_\omega\left(\frac{n-k_p}{p^2}\right)$$
$$\equiv \left(\frac{p}{3}\right)(p+1) a_\omega(n) \pmod{2^9}$$

BRUINIER and ONO (2010)

MR2739208 - Identities and congruences for Ramanujan's $\omega(q)$
Bruinier, Jan H.; Ono, Ken
Ramanujan J. 23 (2010), no. 1-3, 151–157.
(Reviewer: Marks, Christopher Edwin)

In particular, if $p \geq 5$ is prime, then

$$a_\omega\left(\frac{2p^2-2}{3}\right) \equiv \begin{cases} \left(\frac{p}{3}\right) \pmod{512} & \text{if } p \equiv 1, 3 \pmod 8, \\ \left(\frac{p}{3}\right)(1+2p^{255}) \pmod{512} & \text{if } p \equiv 5, 7 \pmod 8. \end{cases}$$

EXAMPLE

$$a_\omega(31^3 n + 19860) \equiv 0 \pmod{2^5}$$

provided $n \not\equiv 15 \pmod{31}$.

$$\begin{aligned} &a_\omega(19860) \\ &= 1824237368919873794351998596780121431213 \cdots 557120 \\ &= 2^6 \cdot 5 \cdot 11 \cdot 59 \cdot 157 \cdot 293 \cdot c_{99} \end{aligned}$$

EXOTIC CONGRUENCES MOD 11 AND 23

Berg, Jen (1-TX); Castillo, Abel (1-ILCC-MS); Grizzard, Robert (1-TX); Kala, Vítězslav (1-PURD); Moy, Richard (1-NW); Wang, Chongli (1-CLMB)
Congruences for Ramanujan's f and ω functions via generalized Borchards products. (English summary)
Ramanujan J. **35** (2014), no. 2, 327–338.

Corollary 1.4 *We have*

$$a_\omega\left(\frac{2(5^{2M}-1)}{3}\right) \equiv (-5)^{-m-\varepsilon} \pmod{23}$$

for each positive integer $M = 2m + \varepsilon$, $\varepsilon \in \{0, 1\}$.

ALTERNATIVE FORM OF HECKE OPERATOR

```
> bomega:=n->if type(n,integer)  and  n>=0 then if modp(n,6)=4 then
  aomega((n-4)/6) else 0 fi else 0 fi;
```

$$bomega := n \mapsto \textbf{if } type(n, integer) \textbf{ and } 0 \le n \textbf{ then if } modp(n, 6) = 4 \textbf{ then } aomega\left(\frac{n}{6} - \frac{2}{3}\right) \textbf{ else } 0 \textbf{ end if else } 0 \textbf{ end if} \quad (14)$$

```
> leg:=(a,b)->NumberTheory[LegendreSymbol](a,b):
> wbar:=n->if n>=0 and type(n,integer) then 5*bomega(25*n) +leg(n,
  5)*bomega(n) + bomega(n/25)+6*bomega(n) else 0 fi;
```

$$wbar := n \mapsto \textbf{if } 0 \le n \textbf{ and } type(n, integer) \textbf{ then } 5 \cdot bomega(25 \cdot n) + leg(n, 5) \cdot bomega(n) + bomega\left(\frac{n}{25}\right) + 6 \cdot bomega(n) \textbf{ else } 0 \textbf{ end if} \quad (15)$$

```
> series(add(wbar(6*n+4)*q^(2*n+1),n=0..500)-16*EK2*(K4-1/K4),q,
  500);
```

$$O(q^{500}) \quad (16)$$

```
> gbar:=n-> if n>=0 and type(n,integer) then 5*wbar(5^2*n) +leg(n,
  5)*wbar(n) + wbar(n/25) else 0 fi;
```

$$gbar := n \mapsto \textbf{if } 0 \le n \textbf{ and } type(n, integer) \textbf{ then } 5 \cdot wbar(25 \cdot n) + leg(n, 5) \cdot wbar(n) + wbar\left(\frac{n}{25}\right) \textbf{ else } 0 \textbf{ end if} \quad (17)$$

```
> fsolve((5^4*x-4)/6-500000,x);
```

$$4800.006400 \quad (18)$$

```
> gbarGEN:=add(gbar(n)*q^n,n=0..4800):
> gbarGENA:=sift(gbarGEN,q,6,4,4800):
> qdegree(gbarGENA);
```

$$799 \quad (19)$$

```
> basmake:=(m,T)->map(f->series(f,q,T),[seq(EK2*(K4^j-1/K4^j),j=1..
  m)]);
Warning, (in basmake) `j` is implicitly declared local
```

(20)

$$basmake := (m, T) \mapsto map\left(f \mapsto series(f, q, T), \left[seq\left(EK2 \cdot \left(K4^j - \frac{1}{K4^j}\right), j = 1..m\right)\right]\right) \quad (20)$$

```
> symbasmake:=m->[seq(_EK2*(_K4^j-1/_K4^j),j=1..m)];
```

Warning, (in symbasmake) `j` is implicitly declared local

$$symbasmake := m \mapsto \left[seq\left(\_EK2 \cdot \left(\_K4^j - \frac{1}{\_K4^j}\right), j = 1..m\right)\right] \quad (21)$$

```
> bas1:=basmake(27,500):
> bas2:=map(f->sift(f,q,2,1,500),bas1):
> RELgbar:=findlincombo(gbarGENA,bas2,symbasmake(27),q,0):
> modp(RELgbar,23);
```

$$5\,\_EK2\left(\_K4 + \frac{22}{\_K4}\right) + \_EK2\left(\_K4^2 + \frac{22}{\_K4^2}\right) + 10\,\_EK2\left(\_K4^3 + \frac{22}{\_K4^3}\right) + 16\,\_EK2\left(\_K4^4 + \frac{22}{\_K4^4}\right) + 16\,\_EK2\left(\_K4^5 + \frac{22}{\_K4^5}\right) + 16\,\_EK2\left(\_K4^6 + \frac{22}{\_K4^6}\right) + 16\,\_EK2\left(\_K4^7 + \frac{22}{\_K4^7}\right) + 9\,\_EK2\left(\_K4^8 + \frac{22}{\_K4^8}\right) + 14\,\_EK2\left(\_K4^9 + \frac{22}{\_K4^9}\right) + 10\,\_EK2\left(\_K4^{10} + \frac{22}{\_K4^{10}}\right) + 15\,\_EK2\left(\_K4^{11} + \frac{22}{\_K4^{11}}\right) + 9\,\_EK2\left(\_K4^{12} + \frac{22}{\_K4^{12}}\right) + 20\,\_EK2\left(\_K4^{13} + \frac{22}{\_K4^{13}}\right) + 5\,\_EK2\left(\_K4^{14} + \frac{22}{\_K4^{14}}\right) + 17\,\_EK2\left(\_K4^{15} + \frac{22}{\_K4^{15}}\right) + 7\,\_EK2\left(\_K4^{16} + \frac{22}{\_K4^{16}}\right) + 4\,\_EK2\left(\_K4^{17} + \frac{22}{\_K4^{17}}\right) + 21\,\_EK2\left(\_K4^{18} + \frac{22}{\_K4^{18}}\right) + 19\,\_EK2\left(\_K4^{19} + \frac{22}{\_K4^{19}}\right) + 6\,\_EK2\left(\_K4^{20} + \frac{22}{\_K4^{20}}\right) + 4\,\_EK2\left(\_K4^{21} + \frac{22}{\_K4^{21}}\right) + 9\,\_EK2\left(\_K4^{22} + \frac{22}{\_K4^{22}}\right) + 20\,\_EK2\left(\_K4^{24} + \frac{22}{\_K4^{24}}\right) + 4\,\_EK2\left(\_K4^{25} + \frac{22}{\_K4^{25}}\right) + 9\,\_EK2\left(\_K4^{26} + \frac{22}{\_K4^{26}}\right) \quad (22)$$

```
> checkgbarcong:=proc(p,ep1)
> local S,T,LT,m,n:
> S:={}:
> T:=500000:
> LT:=floor( (T-4)/6/5^4)-1: #print("LT = ",LT);
> for n from 0 to LT do
>    m:=6*n+4:
>    if leg(m,p)=ep1 then S:=S union {modp(gbar(m),p)}: fi:
> od:
> RETURN(S): end:
> checkgbarcong(5,-1);
```

$$\{0\} \quad (23)$$

```
> checkgbarcong(11,-1);
```

$$\{0\} \tag{24}$$

```
> checkgbarcong(23,1);
```

$$\{0\} \tag{25}$$

CONJECTURES

- **gbar(n) == 0 (mod 23) if leg(n,23)=1**
- **gbar(n) == 0 (mod 11) if leg(n,11)=-1**

PROVING $a_\omega\left(\frac{2}{3}(5^{2n}-1)\right) \equiv 3^n(2-(-1)^n) \pmod{23}$

Let $b(n) = a_\omega\left(\frac{n-4}{6}\right)$, $b(6n+4) = a_\omega(n)$

$$\bar{\omega}(n) = 5\, b(5^2 n) + \left(\frac{n}{5}\right) b(n) + b\left(\frac{n}{25}\right) + 6\, b(n)$$

$$\bar{g}(n) = 5\, \bar{\omega}(5^2 n) + \left(\frac{n}{5}\right) \bar{\omega}(n) + \bar{\omega}\left(\frac{n}{25}\right)$$

$$\bar{g}(n) \equiv 0 \pmod{23} \text{ if } \left(\frac{n}{23}\right) = +1$$

Let $v(n) = b(5^{2n} \cdot 4)$

Then

$$\bar{g}(4) = 5^2 v(2) + 35\, v(1) + 12\, v(0) \equiv 0 \ (23)$$

$$\bar{g}(5^2 \cdot 4) = 5^2 v(3) + 30\, v(2) + 10\, v(1) + 7 v(0) \equiv 0 \ (23)$$

$$\bar{g}(5^4 \cdot 4) = 5^2 v(4) + 30\, v(3) + 10 v(2) + 6\, v(1) + v(0) \equiv 0 \ (23)$$

$$\bar{g}(5^6 \cdot 4) = 5^2 v(5) + 30\, v(4) + 10\, v(3) + 6\, v(2) + v(1) \equiv 0 \ (23)$$

$$\bar{g}(5^{2n-4} \cdot 4) = 5^2 v(n) + 30\, v(n-1) + 10\, v(n-2) + 6\, v(n-3) + v(n-4) \equiv 0 \pmod{23},$$

for $n \geq 4$

SOLUTION $v(n) \equiv 3^n(2-(-1)^n) \pmod{23}$

```
> fsolve(2/3*(5^(2*x)-1)=500000,x);
```

4.202656628 **(26)**

```
> [seq(aomega(2/3*(5^(2*n)-1))- 3^n*(2-(-1)^n),n=1..4)];
```

[92, 147019574355940, **(27)**

6105528781789826255338616657367489071828343984275529702652915791815662460457\
20,
4713504317755769699658386490848529902877317903609618663661969973372815286672\
2865738619072627326914457022419421615118007256010585973847924915814619231828\
9815530589074254347821822773223303943277462010996573525810144795782340537005\
0991197816548299225987091881940041827267379817078651918570357671359629992867\
9963588710300979543276318001547509333358824720658613828230463949020720074861\
0599262211650315368]

```
> modp(%,23);
```

$$[0, 0, 0, 0] \tag{28}$$

Suppose $5 \nmid n$ & $\left(\frac{n}{23}\right) = +1$

$$25\, b(5^4 n) + 5\left(6 + \left(\frac{n}{5}\right)\right) b(5^2 n) + 6\left(1 + \left(\frac{n}{5}\right)\right) b(n) \equiv 0 \pmod{23}$$

$$b(5^4 n) + b(5^2 n) \equiv 0 \pmod{23}$$

if $\left(\frac{n}{23}\right) = +1$ & $\left(\frac{n}{5}\right) = -1$.

```
>
```

$$a_\omega(5^4 n + 416) + a_\omega(5^2 n + 16) \equiv 0 \pmod{23}$$

if $\left(\frac{6n+4}{23}\right) = 1$, $\left(\frac{6n+4}{5}\right) = -1$

```
> x1:=5^4*8+416; x2:=25*8+16;
```

$$x1 := 5416$$

$$x2 := 216 \quad (29)$$

```
> aomega(x1)+aomega(x2);
```

$$320568654523160872717424077204180831428115027358493184 00 \quad (30)$$

```
> ifactor(%);
```

$$(2)^{11} (5)^{2} (11) (23) (43) (367) (1063) (3991899479717) (1665202974538050473102183) (22193) \quad (31)$$