Holomorphic projection and identities for Ramanujan's mock theta functions

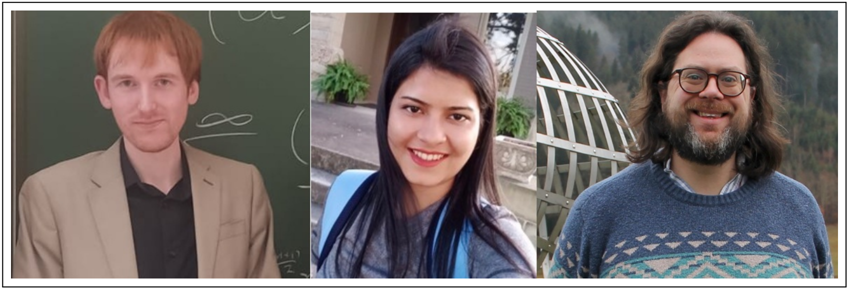
F.G. Garvan
url: qseries.org/fgarvan

University of Florida

Seminar in Partition Theory, q-Series and Related Topics
Michigan Technological University
Department of Mathematical Sciences
Thursday, September 17, 2026

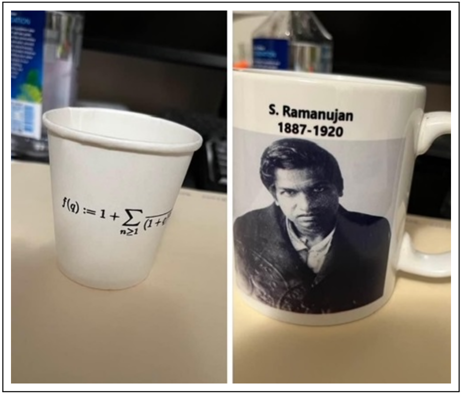
ABSTRACT

In 2014 Imamoglu, Raum and Richter obtained some recursion relations for the coefficients of Ramanujan's third mock theta functions $f(q)$ and $\omega(q)$, using holomorphic projection of a certain tensor product. We apply this method to other third order mock theta functions. Our recursions are of a different type since they involve Pell equations $x^2 - dy^2 = n$ where d is not a square. The identities involve Hecke-Rogers type series of weight 2, and solve a conjecture of the speaker. This is joint work with Jonathan Bradley-Thrush, Jayashree Kalita and Larry Rolen.



WHAT IS A MOCK THETA FUNCTION?

RAMANUJAN (1887 – 1920)



RAMANUJAN'S DEFINITION OF A MOCK THETA FUNCTION

A mock theta function is a function f of a complex variable q, defined by a q-series of a particular type (Eulerian form), which converges for $|q| < 1$ and satisfies the following conditions:

- infinitely many roots of unity are exponential singularities;
- for every root of unity ξ, there is a theta function $\vartheta_\xi(q)$ such the difference $f(q) - \vartheta_\xi(q)$ is bounded as $q \to \xi$ radially; and
- f is not the sum of two functions, one of which is a theta function and the other a function that is bounded radially toward all roots of unity.

RAMANUJAN'S DEFINITION OF A MOCK THETA FUNCTION

A mock theta function is a function f of a complex variable q, defined by a q-series of a particular type (Eulerian form), which converges for $|q| < 1$ and satisfies the following conditions:

- infinitely many roots of unity are exponential singularities;
- for every root of unity ξ, there is a theta function $\vartheta_\xi(q)$ such the difference $f(q) - \vartheta_\xi(q)$ is bounded as $q \to \xi$ radially; and
- f is not the sum of two functions, one of which is a theta function and the other a function that is bounded radially toward all roots of unity.

RAMANUJAN'S DEFINITION OF A MOCK THETA FUNCTION

A mock theta function is a function f of a complex variable q, defined by a q-series of a particular type (Eulerian form), which converges for $|q| < 1$ and satisfies the following conditions:

- infinitely many roots of unity are exponential singularities;
- for every root of unity ξ, there is a theta function $\vartheta_\xi(q)$ such the difference $f(q) - \vartheta_\xi(q)$ is bounded as $q \to \xi$ radially; and
- f is not the sum of two functions, one of which is a theta function and the other a function that is bounded radially toward all roots of unity.

$$(a)_n = (a;q)_n = \prod_{j=1}^{n}(1 - aq^{j-1}) = (1-a)(1-aq)\cdots(1-aq^{n-1})$$

$$(a)_\infty = (a;q)_\infty = \prod_{j=1}^{\infty}(1 - aq^{j-1})$$

when $|q| < 1$.

$$
\begin{aligned}
f(q) &= \sum_{n=0}^{\infty} \frac{q^{n^2}}{(-q;q)_n^2} \\
&= 1 + \frac{q}{(1+q)^2} + \frac{q^4}{(1+q)^2(1+q^2)^2} + \cdots \\
&= 1 + q - 2q^2 + 3q^3 - 3q^4 + 3q^5 - 5q^6 + 7q^7 - 6q^8 + 6q^9 + \cdots \\
\omega(q) &= \sum_{n=0}^{\infty} \frac{q^{2n(n+1)}}{(q;q^2)_{n+1}^2} \\
&= \frac{1}{(1-q)^2} + \frac{q^4}{(1-q)^2(1-q^3)^2} + \cdots \\
&= 1 + 2q + 3q^2 + 4q^3 + 6q^4 + 8q^5 + 10q^6 + 14q^7 + 18q^8 + \cdots
\end{aligned}
$$

Ramanujan (1919) and Watson (1935)

Theorem

Let

$$b(q) = (q;q^2)_\infty \sum_{n=-\infty}^{\infty} (-1)^n q^{n^2} = \frac{(q;q)_\infty^3}{(q^2;q^2)_\infty^2},$$

and assume ζ is a primitive kth root of unity. Then as $q \to \zeta$ (radially) we have

(1) $f(q) = O(1)$, *if k is odd,*

(2) $f(q) + b(q) = O(1)$, *if $k \equiv 2 \pmod 4$, and*

(3) $f(q) - b(q) = O(1)$, *if $k \equiv 0 \pmod 4$.*

ZWEGERS AND THE MODERN DEFINITION OF A MOCK THETA FUNCTION

Sander Zwegers

THE MODERN DEFINITION OF A MOCK THETA FUNCTION (Zagier and Zwegers)

A mock theta function is a q-series

$$H(q) = \sum_{n=0}^{\infty} a_n q^n$$

such that there exists a rational number λ and a unary theta function

$$g(z) = \sum_{n} b_n q^n$$

of weight k, such that

$$h(z) = q^{\lambda} H(q) + g^*(z)$$

is a **nonholomorphic modular form** of weight $2-k$, where

$$g^*(z) = \sum_{n} n^{2-k} \overline{b_n} \Gamma(k-1, 4\pi ny) q^{-n}$$

and $\Gamma(w,t) = \int_t^{\infty} u^{w-1} e^{-u} du$ is the incomplete Gamma function.

THE MODERN DEFINITION OF A MOCK THETA FUNCTION (Zagier and Zwegers)

A mock theta function is a q-series

$$H(q) = \sum_{n=0}^{\infty} a_n q^n$$

such that there exists a rational number λ and a unary theta function

$$g(z) = \sum_{n} b_n q^n$$

of weight k, such that

$$h(z) = q^{\lambda} H(q) + g^*(z)$$

is a **nonholomorphic modular form** of weight $2-k$, where

$$g^*(z) = \sum_{n} n^{2-k} \overline{b_n} \Gamma(k-1, 4\pi n y) q^{-n}$$

and $\Gamma(w,t) = \int_t^{\infty} u^{w-1} e^{-u} du$ is the incomplete Gamma function.

The function g is called the *shadow*

$k = 1/2$ or $k = 3/2$

$$g^*(z) = \int_{-\bar{z}}^{i\infty} \frac{\sum_n \overline{b_n} q^n}{(-i(w+z))^{2-k}}\, dw.$$

The function g is called the *shadow*

$k = 1/2$ or $k = 3/2$

$$g^*(z) = \int_{-\overline{z}}^{i\infty} \frac{\sum_n \overline{b_n} q^n}{(-i(w+z))^{2-k}}\, dw.$$

The function g is called the *shadow*

$k = 1/2$ or $k = 3/2$

$$g^*(z) = \int_{-\overline{z}}^{i\infty} \frac{\sum_n \overline{b_n} q^n}{(-i(w+z))^{2-k}}\, dw.$$

The function g is called the *shadow*

$k = 1/2$ or $k = 3/2$

$$g^*(z) = \int_{-\bar{z}}^{i\infty} \frac{\sum_n \overline{b_n} q^n}{(-i(w+z))^{2-k}}\, dw.$$

THEOREM **Zwegers(2000)** Define $F(z) = (f_0, f_1, f_2)^T$ by

$$f_0(z) = q^{-1/24} f(q), \quad f_1(z) = 2q^{1/3}\omega(q^{1/2}), \quad f_2(z) = 2q^{1/3}\omega(-q^{1/2}),$$

where $q = \exp(2\pi i z), \quad z \in \mathfrak{h}$. Define

$$G(z) = 2i\sqrt{3}\int_{-\bar{z}}^{i\infty} \frac{(g_1(\tau), g_0(\tau), -g_2(\tau))^T}{\sqrt{-i(\tau + z)}}\, d\tau,$$

where
$$g_0(z) = \sum_n (-1)^n \left(n + \frac{1}{3}\right) q^{3(n+1/3)^2/2},$$
$$g_1(z) = -\sum_n \left(n + \frac{1}{6}\right) q^{3(n+\frac{1}{6})^2/2},$$
$$g_2(z) = \sum_n \left(n + \frac{1}{3}\right) q^{3(n+1/3)^2/2}.$$

Then

$$H(z) = F(z) - G(z)$$

is a (vector-valued) real analytic modular form of weight 1/2 satisfying

$$H(z+1) = \begin{pmatrix} \zeta_{24}^{-1} & 0 & 0 \\ 0 & 0 & \zeta_3 \\ 0 & \zeta_3 & 0 \end{pmatrix} H(z),$$

$$\frac{1}{\sqrt{-iz}} H\left(\frac{-1}{z}\right) = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} H(z).$$

VECTOR-VALUED MODULAR FORMS

$$\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \tau := \frac{a\tau + b}{c\tau + d}.$$

The metaplectic double cover $\mathrm{Mp}_2(\mathbf{Z})$ of $\mathrm{SL}_2(\mathbf{Z})$ is the set of pairs

$$(\gamma, \varepsilon(\tau)) \quad \text{where} \quad \gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbf{Z})$$

and $\varepsilon(\tau)$ is a holomorphic choice of a square root of $c\tau + d$.
Group law:

$$(\gamma_1, \varepsilon_1(\tau)) \cdot (\gamma_2, \varepsilon_2(\tau)) := (\gamma_1\gamma_2, \varepsilon_1(\gamma_2(\tau))\varepsilon_2(\tau)).$$

Let ρ be a finite-dimensional unitary representation of $\mathrm{Mp}_2(\mathbf{Z})$.
We write a vector-valued modular form $F(\tau)$ as

$$F =: \sum_{j=1}^{d} F_j \mathfrak{e}_j.$$

For $(\gamma, \varepsilon(\tau)) \in \mathrm{Mp}_2(\mathbf{Z})$, and weight $k \in \frac{1}{2}\mathbf{Z}$ we define

$$\left(F\big|_{k,\rho}\gamma\right)(\tau) := \rho(\gamma)^{-1}\varepsilon(\tau)^{-2k}F(\gamma\tau).$$

We say $F(\tau)$ is a **vector-valued modular form of weight k and type ρ** if

$$F\big|_{k,\rho}\gamma = F \qquad \text{for all } \gamma \in \mathrm{Mp}_2(\mathbf{Z}).$$

The weight k hyperbolic Laplacian

$$\Delta_k := y^{2-k}\frac{\partial}{\partial\tau}y^k\frac{\partial}{\partial\overline{\tau}}$$

A vector-valued modular form F of weight k for ρ is a **harmonic Maass form of weight k for ρ** if

- $\Delta_k(F_j) = 0$ for each j
- Each F_j satisfies certain growth conditions at infinity

Each harmonic Maass form F_j can decomposed into a holomorphic part and non-holomorphic part

$$F_j = F_j^+ + F_j^-,$$

where

$$F_j^+ = \sum_{n \gg -\infty} a_{F_j^+}(n) q^n, \quad F_j^- = \sum_{n \ll \infty} a_{F_j^-}(n) \Gamma(1-k, -4\pi n v)\, q^n,$$

where

$$\Gamma(s,z) := \int_z^\infty e^{-t} t^s \frac{dt}{t}$$

is the *(upper) incomplete gamma function*.

WHAT IS HOLOMORPHIC PROJECTION?

The space of weight k holomorphic cusp forms $S_k(\Gamma)$ is a Hilbert space with inner product

$$\langle f, g \rangle := \int_{\Gamma\backslash\mathbb{H}} f(\tau)\overline{g(\tau)} v^k \frac{du\, dv}{v^2}.$$

If $F(\tau)$ is a weight k modular form (satisfying certain growth conditions), but which is not necessarily holomorphic, the linear functional

$$f \mapsto \langle f, F \rangle,$$

is defined on $S_k(\Gamma)$. By the *Riesz representation theorem* there exists a unique function $g \in S_k(\Gamma)$ such that

$$\langle f, F \rangle = \langle f, g \rangle \quad \text{for all } f \in S_k(\Gamma).$$

This defines the holomorphic projection of F:

$$\pi_{\text{hol}}(F) := g,$$

If

$$F(\tau) = \sum_{m \in \mathbb{Q}} c_F(m, v) q^m,$$

satisfies the following:

- There is an $\varepsilon > 0$ and a $c_0 \in V$ such that as $v \to \infty$, we have
$$F(\tau) = c_0 + O\left(v^{-\varepsilon}\right).$$
- For all $m > 0$, we have as $v \to 0$,
$$c_F(m, v) = O(v^{2-k}).$$

Then

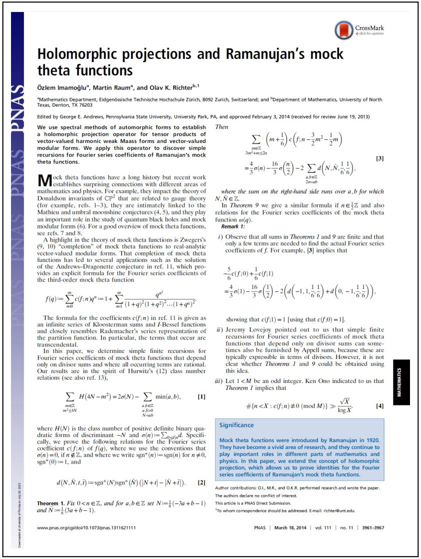
$$\pi_{\mathrm{hol}}^{(k)}(F)(\tau) = c_0 + \sum_{0 < m \in \mathbb{Q}} c_m \, q^m,$$

where for $m > 0$

$$c_m := \frac{(4\pi m)^{k-1}}{\Gamma(k-1)} \int_0^\infty c_F(m, v) e^{-4\pi m v} v^{k-2} \, dv,$$

and Γ is the Gamma-function.

Ö. Imamoḡlu, M. Raum, and O. K. Richter



CrossMark

Holomorphic projections and Ramanujan's mock theta functions

Özlem Imamoğlu[a], Martin Raum[a], and Olav K. Richter[b,1]

[a]Mathematics Department, Eidgenössische Technische Hochschule Zürich, 8092 Zurich, Switzerland; and [b]Department of Mathematics, University of North Texas, Denton, TX 76203

Edited by George E. Andrews, Pennsylvania State University, University Park, PA, and approved February 3, 2014 (received for review June 19, 2013)

We use spectral methods of automorphic forms to establish a holomorphic projection operator for tensor products of vector-valued harmonic weak Maass forms and vector-valued modular forms. We apply this operator to discover simple recursions for Fourier series coefficients of Ramanujan's mock theta functions.

Mock theta functions have a long history but recent work establishes surprising connections with different areas of mathematics and physics. For example, they impact the theory of Donaldson invariants of $\mathbb{CP}^2$ that are related to gauge theory (for example, refs. 1–3), they are intimately linked to the Mathieu and umbral moonshine conjectures (4, 5), and they play an important role in the study of quantum black holes and mock modular forms (6). For a good overview of mock theta functions, see refs. 7 and 8.

A highlight in the theory of mock theta functions is Zwegers's (9, 10) "completion" of mock theta functions to real-analytic vector-valued modular forms. That completion of mock theta functions has led to several applications such as the solution of the Andrews–Dragonette conjecture in ref. 11, which provides an explicit formula for the Fourier series coefficients of the third-order mock theta function

$$f(q) := \sum_{n=0}^{\infty} c(f;n) q^n := 1 + \sum_{n=1}^{\infty} \frac{q^{n^2}}{(1+q)^2(1+q^2)^2\cdots(1+q^n)^2}.$$

The formula for the coefficients $c(f;n)$ in ref. 11 is given as an infinite series of Kloosterman sums and I-Bessel functions and closely resembles Rademacher's series representation of the partition function. In particular, the terms that occur are transcendental.

In this paper, we determine simple finite recursions for Fourier series coefficients of mock theta functions that depend only on divisor sums and where all occurring terms are rational. Our results are in the spirit of Hurwitz's (12) class number relations (see also ref. 13),

$$\sum_{\substack{m\in\mathbb{Z}\\ m^2\le 4N}} H\left(4N-m^2\right) = 2\sigma(N) - \sum_{\substack{a,b\in\mathbb{Z}\\ a,b>0\\ N=ab}} \min(a,b), \qquad [1]$$

where $H(N)$ is the class number of positive definite binary quadratic forms of discriminant $-N$ and $\sigma(n) := \sum_{0<d|n} d$. Specifically, we prove the following relations for the Fourier series coefficient $c(f;n)$ of $f(q)$, where we use the conventions that $\sigma(n)=0$, if $n\notin\mathbb{Z}$, and where we write $\mathrm{sgn}^+(n) := \mathrm{sgn}(n)$ for $n\neq 0$, $\mathrm{sgn}^+(0) := 1$, and

$$d\left(N,\tilde{N},t,\tilde{t}\right) := \mathrm{sgn}^+(N)\,\mathrm{sgn}^+\left(\tilde{N}\right)\left(|N+t| - \left|\tilde{N}+\tilde{t}\right|\right). \qquad [2]$$

Theorem 1. *Fix $0<n\in\mathbb{Z}$, and for $a,b\in\mathbb{Z}$ set $N := \frac{1}{6}(-3a+b-1)$ and $\tilde{N} := \frac{1}{6}(3a+b-1)$. Then*

$$\sum_{\substack{m\in\mathbb{Z}\\ 3m^2+m\le 2n}} \left(m+\frac{1}{6}\right) c\left(f; n-\frac{3}{2}m^2-\frac{1}{2}m\right) = \frac{4}{3}\sigma(n) - \frac{16}{3}\sigma\left(\frac{n}{2}\right) - 2\sum_{\substack{a,b\in\mathbb{Z}\\ 2n=ab}} d\left(N,\tilde{N},\frac{1}{6},\frac{1}{6}\right), \qquad [3]$$

where the sum on the right-hand side runs over a,b for which $N,\tilde{N}\in\mathbb{Z}$.

In *Theorem 9* we give a similar formula if $n\in\frac{1}{2}\mathbb{Z}$ and also relations for the Fourier series coefficients of the mock theta function $\omega(q)$.

Remark 1:

i) Observe that all sums in *Theorems 1* and *9* are finite and that only a few terms are needed to find the actual Fourier series coefficients of f. For example, [**3**] implies that

$$-\frac{5}{6}c(f;0) + \frac{1}{6}c(f;1) = \frac{4}{3}\sigma(1) - \frac{16}{3}\sigma\left(\frac{1}{2}\right) - 2\left(d\left(-1,1,\frac{1}{6},\frac{1}{6}\right) + d\left(0,-1,\frac{1}{6},\frac{1}{6}\right)\right),$$

showing that $c(f;1)=1$ [using that $c(f;0)=1$].

ii) Jeremy Lovejoy pointed out to us that simple finite recursions for Fourier series coefficients of mock theta functions that depend only on divisor sums can sometimes also be furnished by Appell sums, because these are typically expressible in terms of divisors. However, it is not clear whether *Theorems 1* and *9* could be obtained using this idea.

iii) Let $1<M$ be an odd integer. Ken Ono indicated to us that *Theorem 1* implies that

$$\#\{n<X : c(f;n)\not\equiv 0 \pmod{M}\} \gg \frac{\sqrt{X}}{\log X}. \qquad [4]$$

Significance

Mock theta functions were introduced by Ramanujan in 1920. They have become a vivid area of research, and they continue to play important roles in different parts of mathematics and physics. In this paper, we extend the concept of holomorphic projection, which allows us to prove identities for the Fourier series coefficients of Ramanujan's mock theta functions.

Author contributions: O.I., M.R., and O.K.R. performed research and wrote the paper.

The authors declare no conflict of interest.

This article is a PNAS Direct Submission.

[1]To whom correspondence should be addressed. E-mail: richter@unt.edu.

MATHEMATICS

www.pnas.org/cgi/doi/10.1073/pnas.1311621111

PNAS | March 18, 2014 | vol. 111 | no. 11 | 3961–3967

THEOREM [Imamoğlu, Raum, and Richter (2014)] Fix $2 \leq k \in \frac{1}{2}\mathbf{Z}$ and a representation ρ of $\mathrm{Mp}_2(\mathbf{Z})$. Let $F \colon \mathbb{H} \to V(\rho)$ be a continuous function satisfying the following conditions:

- The function F transforms as a modular form of weight k for the representation ρ.
- We have the growth condition
$$F(\tau) = c_0 + O\left(v^{-\varepsilon}\right),$$
for some $\varepsilon > 0$, $c_0 \in V(\rho)$, and as $v \to \infty$.

If $k > 2$, then $\pi_{\mathrm{hol}}(F) \in M_k(\rho)$, and if $k = 2$, then $\pi_{\mathrm{hol}}(F) \in \widetilde{M_2}(\rho)$.

THEOREM [Imamoğlu, Raum, and Richter (2014)]

$$\sum_{\substack{m\in\mathbf{Z}\\ 3m^2+m\leq 2n}} \left(m+\frac{1}{6}\right) c_f\left(n-\frac{3}{2}m^2-\frac{1}{2}m\right)$$
$$=\frac{4}{3}\sigma_1(n)-\frac{16}{3}\sigma_1\left(\frac{n}{2}\right)-2\sum_{\substack{a,b\in\mathbb{Z}\\ 2n=ab}} d\left(N_1,N_2,\frac{1}{6},\frac{1}{6}\right),$$

where $N_1 := (-3a+b-1)/6$, $N_2 := (3a+b-1)/6$, the sum over $a,b\in\mathbb{Z}$ is restricted to those pairs of integers for which $N_1, N_2 \in \mathbb{Z}$, and

$$d(N_1,N_2,t_1,t_2) := \operatorname{sign}^+(N_1)\operatorname{sign}^+(N_2)\left(|N_1+t_1|-|N_2+t_2|\right)$$

$$\operatorname{sign}^+(n) := \begin{cases}\operatorname{sign}(n) & \text{if } n\neq 0,\\ 1 & \text{if } n=0.\end{cases}$$

NEW IDENTITIES FOR THE THIRD ORDER MOCK THETA FUNCTIONS

$$\phi(q) := \sum_{n\geq 0} \frac{q^{n^2}}{(-q^2;q^2)_n}, \quad \psi(q) := \sum_{n>0} \frac{q^{n^2}}{(q;q^2)_n}, \quad \nu(q) := \sum_{n\geq 0} \frac{q^{n^2+n}}{(-q;q^2)_{n+1}}$$

THEOREM [Bradley-Thrush, G., Kalita, Rolen (2026)]

- $$(q;q)_\infty^3\,\psi(q) = \frac{1}{6}\,\frac{(q^2;q^2)_\infty^7}{(q^4;q^4)_\infty^3} - \frac{1}{2}\sum_{m=1}^{\infty}\sum_{n=1}^{\lfloor\frac{3}{2}m\rfloor}\left(\frac{-4}{m}\right)\left(\frac{n}{6}\right)\left(m-\frac{2}{3}n\right)q^{\frac{m^2}{8}-\frac{n^2}{24}-\frac{1}{12}},$$
- $$(q^4;q^4)_\infty^3\,\psi(q) = \frac{1}{3}\,\frac{(q^2;q^2)_\infty^7}{(q;q)_\infty^3} - \frac{1}{2}\sum_{m=1}^{\infty}\sum_{n=1}^{3m}\left(\frac{-4}{m}\right)\left(\frac{n}{6}\right)\left(m-\frac{1}{3}n\right)q^{\frac{m^2}{2}-\frac{n^2}{24}-\frac{11}{24}},$$
- $$(q;q)_\infty^3\,\psi(-q) = \frac{1}{4}\,\frac{(q;q)_\infty^6}{(q^2;q^2)_\infty^2} - \frac{1}{2}\sum_{m=1}^{\infty}\sum_{n=1}^{\lfloor\frac{3}{2}m\rfloor}\left(\frac{-4}{m}\right)\left(\frac{n}{12}\right)\left(m-\frac{1}{2}n\right)q^{\frac{m^2}{8}-\frac{n^2}{24}-\frac{1}{12}},$$

- $$(q;q)_\infty^3\,\nu(q) = \frac{4}{3}\,\frac{(q;q)_\infty^3\,(q^4;q^4)_\infty^3}{(q^2;q^2)_\infty^2} - \sum_{m=1}^{\infty}\sum_{n=1}^{\left\lfloor \frac{3}{8}m \right\rfloor}(-1)^n\left(\frac{-4}{m}\right)\left(\frac{n}{3}\right)\left(m-\frac{8}{3}n\right)q$$
- $$(q^4;q^4)_\infty^3\,\nu(q) = \frac{(q^4;q^4)_\infty^6}{(q^2;q^2)_\infty^2} - \sum_{m=1}^{\infty}\sum_{n=1}^{\left\lfloor \frac{3m}{4} \right\rfloor}(-1)^n\left(\frac{-4}{m}\right)\left(\frac{n}{3}\right)(m-n)q$$

$$(q;q)_\infty^3 = \sum_{n=0}^{\infty} (-1)^n (2n+1) q^{n(n+1)/2}.$$

$$(q;q)_\infty^3 \, \psi(q) = \sum_{n=0}^{\infty} \left(\sum_{\substack{m \geq 0 \\ m(m+1) \leq 2n}} (-1)^n (2n+1) c_\psi \left(n - \frac{1}{2} m(m+1) \right) \right) q^n$$

David Klein and Jennifer Kupka

Define

$$\theta_{N,a}(\tau) := \sum_{n \equiv a \mod 2N} n \exp\left(\frac{2\pi i n^2 \tau}{4N}\right).$$

$$F_{(3)}(\tau) := \begin{pmatrix} q^{-1/48}\,\phi(q^{1/2}) \\ q^{-1/48}\,\phi(-q^{1/2}) \\ 2\,q^{-1/48}\,\psi(q^{1/2}) \\ 2\,q^{-1/48}\,\psi(-q^{1/2}) \\ \sqrt{2}\,q^{1/6}\,\nu(q^{1/2}) \\ \sqrt{2}\,q^{1/6}\,\nu(-q^{1/2}) \end{pmatrix}.$$

$$G_{(3)}(\tau) := -\frac{i}{2\sqrt{6}}\int_{-\overline{\tau}}^{i\infty} \frac{g_{(3)}(z)}{\sqrt{-i(z+\tau)}}dz,$$

$$g_{(3),0}(\tau) := -\left(\theta_{12,1}(\tau)+\theta_{12,5}(\tau)+\theta_{12,7}(\tau)+\theta_{12,11}(\tau)\right),$$

$$g_{(3),1}(\tau) := -\left(\theta_{12,1}(\tau)-\theta_{12,5}(\tau)+\theta_{12,7}(\tau)-\theta_{12,11}(\tau)\right),$$

$$g_{(3),2}(\tau) := \theta_{12,1}(\tau)+\theta_{12,5}(\tau)+\theta_{12,7}(\tau)+\theta_{12,11}(\tau),$$

$$g_{(3),3}(\tau) := \theta_{12,1}(\tau)-\theta_{12,5}(\tau)+\theta_{12,7}(\tau)-\theta_{12,11}(\tau),$$

$$g_{(3),4}(\tau) := -\sqrt{2}\,\left(\theta_{12,4}(\tau)+\theta_{12,8}(\tau)\right),$$

$$g_{(3),5}(\tau) := \sqrt{2}\,\left(\theta_{12,4}(\tau)+\theta_{12,8}(\tau)\right).$$

THEOREM [Klein and Kupka (2021)] The sum $H_{(3)} := F_{(3)} + G_{(3)}$ is a vector-valued harmonic Maass form of weight $1/2$ on $\mathrm{SL}_2(\mathbf{Z})$ with representation $\rho_{(3)}$ defined by its action on generators:

$$\rho_{(3)}(T) := \dots \qquad \rho_{(3)}(S) := \sqrt{-i}\begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Define

$$\eta(\tau) := q^{\frac{1}{24}}(q;q)_\infty.$$

PROPOSITION The function

$$N(\tau) := \begin{pmatrix} \frac{1}{2}\, \frac{\eta^9(\tau)}{\eta^3(\tau/2)\,\eta^3(2\tau)} \\ \frac{1}{2}\,\eta^3(\tau/2) \\ \sqrt{2}\,\eta^3(2\tau) \end{pmatrix},$$

is a vector-valued cusp form of weight $3/2$ on $\mathrm{SL}_2(\mathbf{Z})$ with representation ρ defined by its action on generators:

$$\rho(T) := \begin{pmatrix} 0 & \zeta_{16} & 0 \\ \zeta_{16} & 0 & 0 \\ 0 & 0 & \zeta_4 \end{pmatrix}, \qquad \rho(S) := \zeta_8^{-3} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

Recall

$$H_{(3)} = (H_0, H_1, H_2, H_3, H_4, H_5)^T, \qquad N = (N_0, N_1, N_2)^T.$$

The tensor product

$$H_{(3)} \otimes N = \left(H_0 N_0,\ H_0 N_1,\ H_0 N_2,\ H_1 N_0,\ \ldots,\ H_5 N_2\right)^T.$$

PROPOSITION

$$\pi_{\mathrm{hol}}(H_{(3)} \otimes N) \in \widetilde{M}_2(\rho_{(3)} \otimes \rho).$$

DECOMPOSITION INTO IRREDUCIBLES

We find that

$$\begin{aligned}
H_0 N_0(\tau+1) &= \zeta_{24}\, H_1 N_1(\tau), \\
H_1 N_1(\tau+1) &= \zeta_{24}\, H_0 N_0(\tau), \\
H_0 N_0\left(-\tfrac{1}{\tau}\right) &= -\tau^2\, H_2 N_0(\tau), \\
H_2 N_0\left(-\tfrac{1}{\tau}\right) &= -\tau^2\, H_0 N_0(\tau).
\end{aligned}$$

We represent this in the diagram

$$H_2 N_0 \xleftrightarrow{\quad S \quad} H_0 N_0 \xleftrightarrow{\quad T \quad} H_1 N_1.$$

Similarly, we have

$$\begin{tikzcd}
H_0N_0 \arrow[r, leftrightarrow, "S"] \arrow[d, leftrightarrow, "T"'] & H_2N_0 \arrow[d, leftrightarrow, "T"] \\
H_1N_1 \arrow[d, leftrightarrow, "S"'] & H_3N_1 \arrow[d, leftrightarrow, "S"] \\
H_5N_2 \arrow[r, leftrightarrow, "T"'] & H_4N_2.
\end{tikzcd}$$

This implies

$$\mathrm{Span}\big(H_0N_0,\ H_2N_0,\ H_1N_1,\ H_3N_1,\ H_5N_2,\ H_4N_2\big)$$

is a six dimensional space invariant under $SL_2(\mathbb{Z})$.

Considering the action on $H_0N_0 + H_2N_0$, we find

$$\begin{tikzcd} H_0N_0 + H_2N_0 \arrow[loop left, "S"] \arrow[r, leftrightarrow, "T"] & H_1N_1 + H_3N_1 \arrow[r, leftrightarrow, "S"] & H_4N_2 + H_5N_2 \arrow[loop right, "T"] \end{tikzcd}$$

In fact, defining

$$W_1 = \left[\, H_0N_0 + H_2N_0, \quad H_1N_1 + H_3N_1, \quad H_4N_2 + H_5N_2 \,\right]^T,$$

we have

$$W_1(\tau)\big|_{2,\sigma_1}\gamma = W_1(\tau) \qquad \text{for } \gamma \in Mp_2(\mathbb{Z}),$$

where

$$\sigma_1(T) = \begin{pmatrix} 0 & \zeta_{24} & 0 \\ \zeta_{24} & 0 & 0 \\ 0 & 0 & \zeta_{12}^5 \end{pmatrix}, \qquad \sigma_1(S) = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix}.$$

$$\operatorname{Span}\big(H_0N_0 + H_2N_0,\ H_1N_1 + H_3N_1,\ H_4N_2 + H_5N_2\big)$$

is a three dimensional space invariant under $SL_2(\mathbb{Z})$.

So we define

$$
\begin{aligned}
W_1 &= [H_0N_0 + H_2N_0, H_1N_1 + H_3N_1, H_4N_2 + H_5N_2]^T, \\
W_2 &= [H_0N_0 - H_2N_0, H_1N_1 - H_3N_1, H_4N_2 - H_5N_2]^T, \\
W_3 &= [H_0N_2 + H_4N_0, H_1N_2 + H_5N_1, H_2N_1 + H_3N_0]^T, \\
W_4 &= [H_0N_2 - H_4N_0, H_1N_2 - H_5N_1, H_2N_1 - H_3N_0]^T, \\
W_5 &= [H_0N_1 + H_1N_0, H_2N_2 + H_5N_0, H_3N_2 + H_4N_1]^T, \\
W_6 &= [H_0N_1 - H_1N_0, H_2N_2 - H_5N_0, H_3N_2 - H_4N_1]^T.
\end{aligned}
$$

Then we have

$$W_j(\tau)\big|_{2,\sigma_j}\gamma = W_j(\tau) \qquad \text{for } \gamma \in Mp_2(\mathbb{Z}),$$

for matrices $\sigma_j(T)$ and $\sigma_j(S)$.

PROPOSITION

$$\rho_{(3)} \otimes \rho \approx \sigma_1 \oplus \ldots \oplus \sigma_6$$

- For each odd j, we have $M_2(\sigma_j) = \mathrm{Span}(V_j)$, where

$$V_1(\tau) := \begin{pmatrix} \frac{\eta(\tau)^{16}}{\eta(\tau/2)^6\,\eta(2\tau)^6} \\ \frac{\eta(\tau/2)^6}{\eta(\tau)^2} \\ 8\,\frac{\eta(2\tau)^6}{\eta(\tau)^2} \end{pmatrix}, \quad V_3(\tau) := \cdots, V_5(\tau) := \cdots$$

- For each even j, we have $\dim_{\mathbf{C}}(M_2(\sigma_j)) = 0$ and $M_2(\sigma_j) = \{0\}$.

THEOREM

$$\pi_{\mathrm{hol}}(H_{(3)} \otimes N) = E,$$

where

$$\begin{aligned} E(\tau) :=& \frac{1}{4}(\mathfrak{e}_1 + \mathfrak{e}_7)\frac{\eta^{16}(\tau)}{\eta^6(\tau/2)\,\eta^6(2\tau)} + \frac{1}{6}(2\mathfrak{e}_2 + 2\mathfrak{e}_4 + \mathfrak{e}_8 + \mathfrak{e}_{10})\frac{\eta^7(\tau)}{\eta^3(2\tau)} \\ &+ \frac{\sqrt{2}}{3}(\mathfrak{e}_3 + 2\mathfrak{e}_9 + \mathfrak{e}_{13} + 2\mathfrak{e}_{16})\frac{\eta^7(\tau)}{\eta^3(\tau/2)} \\ &+ \frac{1}{4}(\mathfrak{e}_5 + \mathfrak{e}_{11})\frac{\eta^6(\tau/2)}{\eta^2(\tau)} \\ &+ \frac{\sqrt{2}}{3}(\mathfrak{e}_6 + 2\mathfrak{e}_{12} + 2\mathfrak{e}_{14} + \mathfrak{e}_{17})\frac{\eta^3(\tau/2)\,\eta^3(2\tau)}{\eta^2(\tau)} \\ &+ 2(\mathfrak{e}_{15} + \mathfrak{e}_{18})\frac{\eta^6(2\tau)}{\eta^2(\tau)}, \end{aligned}$$

THE IDENTITIES

$$H_{(3)} \otimes N = F_{(3)} \otimes N + G_{(3)} \otimes N.$$

$$\begin{aligned} E &= \pi_{\mathrm{hol}}(H_{(3)} \otimes N) = \pi_{\mathrm{hol}}(F_{(3)} \otimes N) + \pi_{\mathrm{hol}}(G_{(3)}) \otimes N) \\ &= F_{(3)} \otimes N + \pi_{\mathrm{hol}}(G_{(3)} \otimes N) \end{aligned}$$

$$F_{(3)} \otimes N = E - \pi_{\mathrm{hol}}(G_{(3)} \otimes N)$$

HOLOMORPHIC PROJECTION OF NONHOLOMORPHIC PARTS

$$\eta(\tau/2)^3 q^{-1/48}\psi(q^{1/2}) = \frac{1}{6}\frac{\eta(\tau)^7}{\eta(2\tau)^3} - \frac{1}{2}\pi_{\mathrm{hol}}\left(\eta(\tau/2)^3 G_2(\tau)\right).$$

where

$$G_{(3)} = \left(G_0, G_1, \ldots, G_5\right)^T.$$

$$G_2(\tau) = -\frac{i}{2\sqrt{6}}\sum_{n=1}^{\infty}\left(\frac{n}{6}\right) n \int_{-\bar{\tau}}^{i\infty} \frac{e^{\frac{i\pi n^2}{24}z}}{\sqrt{-i(z+\tau)}}\,\mathrm{d}z$$

$$\eta^3(\tau) = \sum_{n=1}^{\infty}\left(\frac{-4}{n}\right) n e^{\frac{i\pi n^2}{4}\tau}.$$

$$\eta^3\left(\tau/2\right)G_2(\tau)$$
$$= -\frac{i}{2\sqrt{6}}\left(\sum_{m=1}^{\infty}\left(\frac{-4}{m}\right) m e^{\frac{i\pi m^2}{8}\tau}\right)\sum_{n=1}^{\infty}\left(\frac{n}{6}\right) n \int_{-\bar{\tau}}^{i\infty} \frac{e^{\frac{i\pi n^2}{24}z}}{\sqrt{-i(z+\tau)}}\,\mathrm{d}z.$$

$$\frac{1}{2}\pi_{\mathrm{hol}}\left(\eta(\tau/2)^3 G_2(\tau)\right). = \frac{1}{6}\sum_{\substack{m=1 \\ 3m^2-n^2>0}}\sum_{n=1}\left(\frac{-4}{m}\right)\left(\frac{n}{6}\right)(3m-n\sqrt{3})q^{m^2/16-n^2/48}$$

$$= \frac{1}{2}\sum_{m=1}^{\infty}\sum_{n=1}^{\lfloor 3m/2\rfloor}\left(\frac{-4}{m}\right)\left(\frac{n}{6}\right)\left(m-\frac{2}{3}n\right)q^{m^2/16-n^2/48}$$

$$\eta(\tau/2)^3 q^{-1/48}\psi(q^{1/2})$$
$$= \frac{1}{6}\frac{\eta(\tau)^7}{\eta(2\tau)^3} - \frac{1}{2}\sum_{m=1}^{\infty}\sum_{n=1}^{\lfloor 3m/2 \rfloor}\left(\frac{-4}{m}\right)\left(\frac{n}{6}\right)\left(m - \frac{2}{3}n\right) q^{m^2/16 - n^2/48}$$

$$(q;q)_\infty^3\,\psi(q)$$
$$= \frac{1}{6}\frac{(q^2;q^2)_\infty^7}{(q^4;q^4)_\infty^3} - \frac{1}{2}\sum_{m=1}^{\infty}\sum_{n=1}^{\lfloor \frac{3}{2}m \rfloor}\left(\frac{-4}{m}\right)\left(\frac{n}{6}\right)\left(m - \frac{2}{3}n\right) q^{\frac{m^2}{8} - \frac{n^2}{24} - \frac{1}{12}},$$

MAPLE EXPERIMENT

$$> \quad \frac{1}{2}\pi_{\mathrm{hol}}\left(\eta(\tau/2)^3 G_2(\tau)\right). = \frac{1}{6}\sum_{\substack{m=1\\3m^2-n^2>0}}\sum_{n=1}\left(\frac{-4}{m}\right)\left(\frac{n}{6}\right)(3m-n\sqrt{3})q^{m^2/16-n^2/48}$$

```
> with(qseries):
> kron:=(a,b)->NumberTheory[KroneckerSymbol](a,b):
>
> BIGS:=select(L->if 3*L[1]^2-L[2]^2>0 then true else false fi, [seq(seq([m,
  n],m=1..1000),n=1..1000)]):
> nops(BIGS);
```

$$711536$$

```
> KS:=add(kron(-4,L[1])*kron(L[2],6)*(3*L[1]-L[2]*sqrt(3))*q^(L[1]^2/8-L[2]
  ^2/24-1/12),L in BIGS):
> series(KS,q,50);
```

$$-1350+780\sqrt{3}+\left(-465+265\sqrt{3}\right)q+\left(-480+280\sqrt{3}\right)q^2+\left(471-265\sqrt{3}\right)q^3+\left(672-392\sqrt{3}\right)q^4+\left(576-336\sqrt{3}\right)q^5+\left(51-41\sqrt{3}\right)q^6+\left(-1920+1120\sqrt{3}\right)q^8+\left(627-355\sqrt{3}\right)q^9+\left(1011-575\sqrt{3}\right)q^{10}+\left(-873+497\sqrt{3}\right)q^{13}+\left(1575-933\sqrt{3}\right)q^{14}+\left(-153+71\sqrt{3}\right)q^{15}+\left(-753+451\sqrt{3}\right)q^{16}+\left(879-497\sqrt{3}\right)q^{19}+\left(-321+205\sqrt{3}\right)q^{20}+\left(9+19\sqrt{3}\right)q^{21}+\left(-1365+781\sqrt{3}\right)q^{23}+\left(-459+255\sqrt{3}\right)q^{24}+\left(459-287\sqrt{3}\right)q^{26}+\left(-645+355\sqrt{3}\right)q^{27}+\left(27-41\sqrt{3}\right)q^{28}+\left(300-180\sqrt{3}\right)q^{29}+\left(513-285\sqrt{3}\right)q^{30}+\left(-450+270\sqrt{3}\right)q^{31}+\left(-150+90\sqrt{3}\right)q^{33}+\left(-453+287\sqrt{3}\right)q^{34}+\left(651-355\sqrt{3}\right)q^{35}+\left(-21\right.$$

```
> KS2:=evalf(%);
```

$$KS2 := 0.999630 - 6.0065359\,q + 4.9742262\,q^{2} + 12.0065359\,q^{3} - 6.9639167\,q^{4} - 5.9690715\,q^{5} - 20.01408313\,q^{6} + 19.896905\,q^{8} + 12.1219632\,q^{9} + 15.0707854\,q^{10} - 12.1707484\,q^{13} - 41.003404\,q^{14} - 30.0243926\,q^{15} + 28.1549144\,q^{16} + 18.1707484\,q^{19} + 34.0704156\,q^{20} + 41.90896535\,q^{21} - 12.268319\,q - 17.3270440\,q^{24} - 38.0985819\,q^{26} - 30.1219632\,q^{27} - 44.01408313\,q^{28} - 11.7691454\,q^{29} + 19.3655197\,q^{30} + 17.6537182\,q^{31} + 5.8845727\,q^{33} + 44.0985819\,q^{34} + 36.1219632\,q^{35} + 50.01408313\,q + 44.846467\,q^{38} - 91.731409\,q^{40} - 47.076582\,q^{43} - 28.6279444\,q^{44} - 54.0243926\,q^{45} - 50.961894\,q^{48} + \mathrm{O}(q^{50})$$

```
> KS3:=add(round(coeff(KS2,q,j))*q^j,j=0..48);
```

$$KS3 := -51\,q^{48} - 54\,q^{45} - 29\,q^{44} - 47\,q^{43} - 92\,q^{40} + 45\,q^{38} + 50\,q^{36} + 36\,q^{35} + 44\,q^{34} + 6\,q^{33} + 18\,q^{31} + 19\,q^{30} - 12\,q^{29} - 44\,q^{28} - 30\,q^{27} - 38\,q^{26} - 17\,q^{24} - 12\,q^{23} + 42\,q^{21} + 34\,q^{20} + 18\,q^{19} + 28\,q^{16} - 30\,q - 41\,q^{14} - 12\,q^{13} + 15\,q^{10} + 12\,q^{9} + 20\,q^{8} - 20\,q^{6} - 6\,q^{5} - 7\,q^{4} + 12\,q^{3} + 5\,q^{2} - 6\,q + 1$$

```
> with(ramamocktheta): psi3q:=getmockqs(psi,3):
> _MF:=aqprod(q,q,infinity)^3*_psi(q);
```

$$\_MF := (q,q)_\infty^3\,\_\psi(q)$$

```
> MF:=series(etaq(q,1,100)^3*psi3q,q,100):
> findprod([MF,KS3],6,10,30);
```

$$[[0,[-6,-1]],[1,[-1,0]],[1,[1,0]],[0,[6,1]]]$$

```
> newprodmake(6*MF+KS3,q,50,list);
```

$$[1,[0,7,0,4,0,7,0,4,0,7,0,4,0,7,0,4,0,7,0,4,0,7,0,4,0,7,0,4,0,7,0,4,0,7,0,4,0,6,0,-1,0,-11,-1,-51,-7,-140,-32,-352,-119]]$$

```
> etamake(6*MF+KS3,q,30);
```

$$\frac{\eta(2\tau)^7}{q^{1/12}\,\eta(4\tau)^3}$$

```
> _MF=1/6*% - _PI[holo];
```

$$(q,q)_\infty^3\,\_\psi(q) = \frac{\eta(2\tau)^7}{6\,q^{1/12}\,\eta(4\tau)^3} - \_\Pi_{holo}$$