EXTENSIONS OF MACMAHON'S SUMS OF DIVISORS

GEORGE ANDREWS

JOINT

WORK

WITH

TEWODROS AMDEBERHAN

AND

ROBERTO TAURASO

+

EARLIER WORK WITH

SIMON ROSE

Percy Alexander MacMahon (his presidential portrait for the Royal Astronomical Society)

MACMAHON'S STARTING POINT IS THAT THE NUMBER OF DIVISORS OF n IS ALSO THE NUMBER OF PARTITIONS OF n INTO PARTS OF ONLY ONE SIZE.

EX. $n=6$

DIVISORS

1	1+1+1+1+1+1
2	2+2+2
3	3+3
6	6

LET $d(n)$ DENOTE THE NUMBER OF DIVISORS OF n, AND $\sigma_1(n)$ THE SUM OF THE DIVISORS. THEN

$$\sum_{n\geq 1} d(n) q^n = \sum_{n\geq 1} \frac{q^n}{(1-q^n)}$$

$$\sum_{n\geq 1} \sigma_1(n) q^n = \sum_{n\geq 1} \frac{q^n}{(1-q^n)^2}$$

WE DENOTE THE LAST FUNCTION AS $A_1(q)$.

MACMAHON CONSIDERS

$$\sum_{1 \le n_1 < n_2 < \cdots < n_k} \frac{q^{n_1+n_2+\cdots+n_k}}{(1-q^{n_1})(1-q^{n_2})\cdots(1-q^{n_k})}$$

AND

$A_k(q) :=$

$$\sum_{1 \le n_1 < n_2 < \cdots < n_k} \frac{q^{n_1+n_2+\cdots+n_k}}{(1-q^{n_1})^2(1-q^{n_2})^2\cdots(1-q^{n_k})^2}$$

MACMAHON CONJECTURED THAT $A_k(q)$ IS IN THE RING OF QUASI-MODULAR FORMS, AND HE PROVED THIS FOR $k = 2, 3, 4$.

THE CONJECTURE WAS PROVED BY SIMON ROSE AND G.A. IN 2013

THE CRUX OF THE PROOF RELIES ON THE FOLLOWING:

$T_n(x) =$ the n^{th} Chebychev polynomial

$$T_n(\cos\theta) = \cos(n\theta).$$

DEFINE

$$F(x,q) = 2\sum_{n=0}^{\infty} T_{2n+1}\left(\frac{x}{2}\right) q^{n^2+n},$$

THEN

$$F(x,q) = (q^2;q^2)_\infty^3 \sum_{h=0}^{\infty} A_h(q^2)\, x^{2h+1}$$

where

$$(A;q)_N = (1-A)(1-Aq)\cdots(1-Aq^{n-1})$$

PROOF OF CRUCIAL IDENTITY

LET $x = 2\cos\theta$

$$F(x) = 2\sum_{n\geq 0} T_{2n+1}(\cos\theta)\, q^{n^2+n}$$

$$= 2\sum_{n\geq 0} \cos((2n+1)\theta)\, q^{n^2+n}$$

$$= \sum_{n\geq 0} \left(e^{i(2n+1)\theta} + e^{-i(2n+1)\theta}\right) q^{n^2+n}$$

$$= e^{i\theta} \sum_{n=-\infty}^{\infty} e^{(2i\theta)n}\, q^{n^2+n}$$

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$$= e^{i\theta} \sum_{n=-\infty}^{\infty} e^{(2i\theta)n}\, q^{n^2+n}$$

$$= e^{i\theta}(-e^{-2i\theta}; q^2)_\infty (-q^2 e^{2i\theta}; q^2)_\infty$$

$$\times (q^2; q^2)_\infty$$

$$= (e^{i\theta} + e^{-i\theta})(q^2; q^2)_\infty$$

$$\times \prod_{n=1}^{\infty} \left(1 + 2\cos(2\theta) q^{2n} + q^{4n}\right)$$

$$= (e^{i\theta} + e^{-i\theta})(q^2;q^2)_\infty$$

$$\times \prod_{n=1}^{\infty} \left(1 + 2\cos(2\theta) q^{2n} + q^{4n}\right)$$

$$= x (q^2;q^2)_\infty \prod_{n=1}^{\infty} \left(1 + (x^2-2) q^{2n} + q^{4n}\right)$$

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$$= x (q^2;q^2)_\infty^3 \prod_{n=1}^{\infty} \left(1 + \frac{x^2 q^{2n}}{(1-q^{2n})^2}\right)$$

$$= (q^2;q^2)_\infty^3 \sum_{k \geq 0} A_k(q^2)\, x^{2k+1}$$

RECENTLY, AMDEBERHAN TAURASO AND I EXPLORED CONGRUENCES

$$A_k(q) = \sum_{n \geq 0} MO(k,n) q^n$$

SAMPLE OF CONGRUENCES

$$MO(2,n) \equiv 0 \pmod{3} \text{ IF } 3 \nmid n$$

$$MO(2,5n+1) \equiv 0 \pmod{5}$$

$$MO(3,7n+3) \equiv 0 \pmod{7}$$

WE NOW COME TO THE MOST RECENT (UNPUBLISHED) RESULTS.

LET US CONSIDER

$$U_t(a;q) = \sum_{1 \le n_1 < n_2 < \cdots < n_t} \frac{q^{n_1+n_2+\cdots+n_t}}{(1+aq^{n_1}+q^{2n_1})\cdots(1+aq^{n_t}+q^{2n_t})}$$

THE PREVIOUS WORK WAS FOR

$$a = -2$$

HOWEVER, ALL THE CASES WHERE

$$1 + ax + x^2 = 0$$

HAS ROOTS OF UNITY AS ROOTS ARE OF INTEREST. I.E.

$$a = 0, \pm 1, \pm 2$$

LET US START WITH

$a=1,\quad t=2$

$$U_2(1;q)=\sum_{1\le n_1<n_2}\frac{q^{n_1+n_2}}{(1+q^{n_1}+q^{2n_1})(1+q^{n_2}+q^{2n_2})}$$

$$= q^3 +$$

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$$= q^3 + 3q^6 +$$

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$$= q^3+3q^6+4q^9+$$

LET US START WITH

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$$= q^3 + 3q^6 + 4q^9 + 7q^{12} + 6q^{15} + 12q^{18} + 8q^{21} + \cdots\cdots$$

OFF TO OEIS. FIND

$$\underbrace{1, 3, 4, 7, 6, 12, 8, \ldots}$$

UP COMES

A000203

$\sigma_1(n)$

THE SUM OF THE DIVISORS OF n.

SO IMMEDIATELY WE HAVE A WONDERFUL MYSTERY.

DOES

$$\sum_{1\le n_1<n_2} \frac{q^{n_1+n_2}}{(1+q^{n_1}+q^{2n_1})(1+q^{n_2}+q^{2n_2})}$$

$$= \sum_{n\ge 1} \frac{q^{3n}}{(1-q^{3n})^2} \ ?$$

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$$= \sum_{n \ge 1} \frac{q^{3n}}{(1-q^{3n})^2} \;?$$

YES! WE WILL PROVIDE A COMBINATORIAL PROOF.

TO PROVE

$$\sum_{1 \le n_1 < n_2} \frac{q^{n_1+n_2}}{(1+q^{n_1}+q^{2n_1})(1+q^{n_2}+q^{2n_2})}$$

$$= \sum_{n \ge 1} \frac{q^{3n}}{(1-q^{3n})^2}$$

COMBINATORIALLY, WE REQUIRE A FEW PRELIMINARIES.

$$\sum_{1\le n_1<n_2} \frac{q^{n_1+n_2}}{(1+q^{n_1}+q^{2n_1})(1+q^{n_2}+q^{2n_2})}$$

$$= \sum_{1\le n_1<n_2} \frac{q^{n_1+n_2}(1-q^{n_1})(1-q^{n_2})}{(1-q^{3n_1})(1-q^{3n_2})}$$

$$= \sum_{1\le n_1<n_2} \frac{(q^{n_1}-q^{2n_1})(q^{n_2}-q^{2n_2})}{(1-q^{3n_1})(1-q^{3n_2})}$$

$$= \sum_{n\ge 1} \left(P_0(n)-P_1(n)\right) q^n$$

$$= \sum_{1 \leq n_1 < n_2} \left(\sum_{f \geq 0} q^{(3f+1)n_1} - \sum_{f \geq 0} q^{(3f+2)n_1} \right)$$

$$\times \left(\sum_{f \geq 0} q^{(3f+1)n_2} - \sum_{f \geq 0} q^{(3f+2)n_2} \right)$$

$P_O(n)$ DENOTES THE NUMBER OF PARTITIONS OF n INTO TWO DIFFERENT PARTS (EACH MAY BE REPEATED) n_1 (APPEARING f_1 TIMES) AND n_2 (APPEARING f_2 TIMES)

$$3 \nmid f_1$$

$$3 \nmid f_2$$

$$f_1 \equiv f_2 \pmod 3$$

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$$3 \nmid f_1$$

$$3 \nmid f_2$$

$$f_1 \not\equiv f_2 \pmod 3$$

WE WANT TO PROVE

$$P_0(n) - P_1(n) = \begin{cases} 0 & \text{IF } 3 \nmid n \\ \sigma_1\left(\frac{n}{3}\right) & \text{IF } 3 \mid n \end{cases}$$

IN THE FOLLOWING

$$n_1^{f_1} \; n_2^{f_2}$$

DENOTES THE PARTITION OF n IN WHICH THE PART n_1 APPEARS f_1 TIMES AND n_2 APPEARS f_2 TIMES.

THE FOLLOWING IS AN INJECTION OF $\mathcal{P}_1$ PARTITIONS INTO $\mathcal{P}_0$ PARTITIONS $(f_1 \not\equiv f_2 \pmod 3)$

$$n_1^{f_1} n_2^{f_2} \rightarrow (n_1+n_2)^{f_1} n_2^{f_2-f_1}$$

(WHERE W.L.O.G. $f_2 > f_1$)

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(WHERE W.L.O.G. $f_2 > f_1$)

HERE IS THE INVERSE MAP

$$n_1^{f_1} n_2^{f_2} \longrightarrow (n_2-n_1)^{f_1} n_2^{f_1+f_2}$$

(WHERE W.L.O.G. $n_2 > n_1$)

WHOOPS! IT APPEARS WE HAVE PRODUCED A BIJECTION WHICH WOULD IMPLY

$$P_0(n) = P_1(n)$$

ALWAYS (CONTRADICTING $P_0(3) = 1$ $\quad P_1(3) = 0$).

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SO WHEN AND HOW CAN THE BIJECTION FAIL?

THE ONLY CASE OF FAILURE IS WHEN

$n_2 = d \qquad n_1 = 2d$

BECAUSE THEN THE INVERSE MAP DOES NOT GIVE A PARTITION WITH TWO DIFFERENT PARTS.

SO LET US EXAMINE

$$(2d)^{f_1} d^{f_2}$$

WITH $f_1 \equiv f_2 \pmod{3}$

IN THIS CASE

$$\begin{aligned} n &= f_1(2d) + f_2 d \\ &= d(2f_1 + f_2) \\ &\equiv d\,3f_1 \\ &\equiv 0 \pmod{3}. \end{aligned}$$

THUS THERE IS A BIJECTION IF $3 \nmid n$. SO,

$$P_0(n) - P_1(n) = 0 \quad \text{IF } 3 \nmid n$$

WHAT ABOUT THE CASE

$n = 3\nu$?

HOW MANY WAYS CAN WE SOLVE

$$f_1 (2d) + f_2 d = 3\nu$$

OR

$$2f_1 + f_2 = 3\nu/d$$

THIS IS UNIQUELY SOLVABLE FOR f_1 PROVIDED

$$3\nu/d - f_2$$

IS EVEN.

THUS f_2 MAY BE CHOSEN FROM

1 OR 2, 4 OR 5, 7 OR 8,

$\ldots 3\frac{\nu}{d}-2$ OR $3\frac{\nu}{d}-1$.

I.E. THERE ARE $\frac{\nu}{d}$ CHOICES FOR f_2

HENCE FOR EACH DIVISOR d OF ν (REMEMBER $n=3\nu$) THERE ARE ν/d P_0 PARTITIONS THAT HAVE NO PRE-IMAGE IN P_1

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HENCE

$$P_0(3\nu) - P_1(3\nu) = \sum_{d\mid\nu} \frac{\nu}{d}$$

$$= \sum_{d\mid\nu} d = \sigma_1(\nu) = \sigma_1\left(\frac{n}{3}\right).$$

Ex. $n=7$

P_1		P_0
511	$\rightarrow$	61
331	$\rightarrow$	43
322	$\rightarrow$	52
211111	$\rightarrow$	31111

Ex. $n = 9$

$711 \rightarrow 81$

$522 \rightarrow 72$

$441 \rightarrow 54$

$411111 \rightarrow 51111$

63

22221

2211111

21111111

ONE OF THE VERY INTERESTING ASPECTS OF THIS PROOF IS THAT WE SEE CONGRUENCE CONDITIONS ON THE FREQUENCIES OF THE PARTS RATHER THAN ON THE PARTS THEMSELVES.

IT IS UNLIKELY THAT THIS IS THE ONLY THEOREM OF THIS NATURE IN THE THEORY OF PARTITIONS,

IT IS UNLIKELY THAT THIS IS THE ONLY THEOREM OF THIS NATURE IN THE THEORY OF PARTITIONS ESPECIALLY WHEN WE WRITE PARTITIONS AS

$$n = f_1 n_1 + f_2 n_2 + \cdots + f_j n_j$$

REVEALING THE SYMMETRIC ROLE PLAYED BY FREQUENCIES f_i AND PARTS n_i

THERE IS MUCH MORE TO THE STORY.

THE $U_t(a;q)$ ARE ALL QUASI-MODULAR FORMS FOR

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THE $U_t(a;q)$ ARE ALL QUASI-MODULAR FORMS FOR

$$a = 0, \pm 1, \pm 2.$$

THERE ARE OTHER CONGRUENCES. FOR EXAMPLE

$$MO(1;t;3n+2) \equiv 0 \pmod 3$$

WHERE

$$U_t(a;q) = \sum_{n \geq 0} MO(a;t;n)\, q^n.$$

THERE ARE RELATED PROJECTS.

K. ONO ET AL. HAVE RELATED

$M0(-2; t, n)$

TO THE WORK OF MATIYASEVICH ON HILBERT'S 10^{th} PROBLEM.

THEY HAVE SHOWN THAT THERE ARE LINEAR COMBINATIONS OF THE $MO(-2;t,n)$ WHOSE POSITIVE VALUES ARE ALWAYS PRIME.

THANK
YOU