MACMAHON'S PARTITION ANALYSIS, SCHMIDT-TYPE PARTITIONS, AND MODULAR FORMS

George E. Andrews
(joint work with Peter Paule)

September 23, 2021

IN 1999, FRANK SCHMIDT PROPOSED THE FOLLOWING PROBLEM IN THE AMERICAN MATH. MONTHLY:
LET $p(n)$ DENOTE THE NUMBER OF PARTITIONS OF n, AND LET $f(n)$ DENOTE THE NUBER OF PARTITIONS $\lambda_1 + \lambda_2 + \lambda_3 + \cdots$ SATISFYING $\lambda_1 > \lambda_2 > \lambda_3 > \cdots$ AND $n = \lambda_1 + \lambda_3 + \lambda_5 + \cdots$. THEN FOR $n \geq 1$

$$p(n) = f(n).$$

FOR EXAMPLE, $p(6) = 11$:

$$6, 5+1, 4+2, 4+1+1, 3+3, 3+2+1,$$
$$3+1+1+1, 2+2+2, 2+2+1+1,$$
$$2+1+1+1+1, 1+1+1+1+1+1,$$

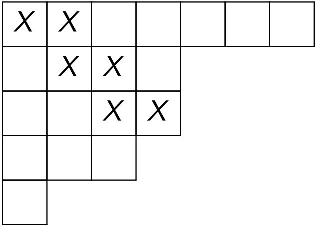
$f(6) = 11$

$$6, 6+5, 6+4, 6+3, 6+2, 6+1$$
$$5+4+1, 5+3+1, 5+2+1, 4+3+2,$$
$$4+3+2+1$$

THE SOLUTION PUBLISHED WAS BY PETER MORK. THE MONTHLY ALSO NOTED THAT EIGHT OTHERS SOLVED THE PROBLEM: D. BECKWITH, R.J. CHAPMAN, G. LORD, C.C ROUSSEAU, P, SIMEONOV, J. H. STEELMAN, AND F. SCHMIDT (THE PROPOSER).

MORK'S SOLUTION IS QUITE CLEVER. IT IS A BIJECTION BETWEEN THE TWO SETS. HERE IS THE IDEA:

$$7 + 4 + 4 + 3 + 1$$



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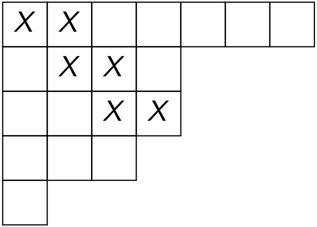

NOW COMPUTE THE HOOKS AT THE SQUARES MARKED X:

$$11 + 9 + 5 + 4 + 3 + 1$$

BY CONSTRUCTION, WE KNOW A PRIORI THAT

$$7 + 4 + 4 + 3 + 1 = 11 + 5 + 3.$$

WE REVERSE ENGINEER TO GET THE INVERSE MAP. START WITH THE SMALLEST PART OF $1 + 3 + 4 + 5 + 9 + 11$.
1 IS A PART OF EVEN INDEX; SO ITS SQUARE IS OFF THE DIAGONAL.

$$\boxed{X}$$

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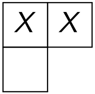

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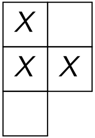

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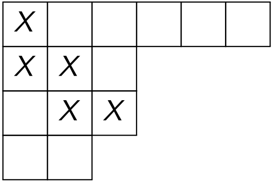

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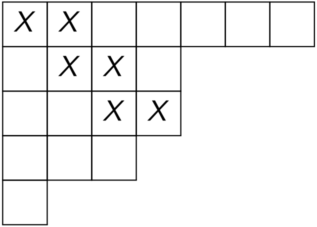

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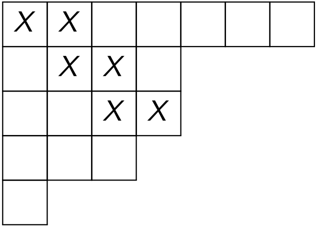

NOW READING ROWS WE RETRIEVE $7 + 4 + 4 + 3 + 1$.

MORK'S SOLUTION WAS QUITE INGENIOUS. HOWEVER, ITS VERY CLEVERNESS MAKES IT HARD TO SEE THIS PROBLEM AS ONE OF A GENERAL CLASS OF RESULTS.

UPON SEEING THIS PROBLEM, ONE ALMOST IMMEDIATELY ASKS WHAT WOULD HAPPEN IF WE RELAXED THE INEQUALITIES IN THE DEFINITION OF $f(n)$? LET $f_1(n)$ DENOTE THE NUMBER OF PARTITIONS $\lambda_1 + \lambda_2 + \lambda_3 + \cdots$ SATISFYING $\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \lambda_4 \geq \cdots$ AND $n = \lambda_1 + \lambda_3 + \lambda_5 + \cdots$

$$\sum_{n\geq 0} f_1(n)q^n = 1 + 2q + 5q^2 + 10q^3 + 20q^4 + 36q^5 \cdots$$
$$+ 65q^6 + 110q^7 + 185q^8 + 300q^9 + 481q^{10} + \cdots$$

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IN OEIS, THIS IS A032442

OF PARTITIONS INTO PARTS OF TWO KINDS

$$f_1(2) = 5 : 2_1, 2_2, 1_2 + 1_2, 1_2 + 1_1, 1_1 + 1_1$$

THE FACT THAT MANY COEFFICIENTS ARE DIVISIBLE BY 5 FOLLOWS FROM:

$$\begin{aligned}\prod_{n=1}^{\infty} \frac{1}{(1-q^n)^2} &= \prod_{n=1}^{\infty} \frac{(1-q^n)^3}{(1-q^n)^5} \\ &= \prod_{n=1}^{\infty} \frac{1}{1-q^{5n}} \sum_{n=0}^{\infty} (-1)^n (2n+1) q^{\binom{n+1}{2}} \pmod 5\end{aligned}$$

(JACOBI), AND TRIANGULAR NUMBERS ARE ALL $n \equiv 0, 1, \text{ or } 3 \pmod 5$. NOTE ALSO THAT $\binom{n+1}{2} \equiv 3 \pmod 5$ IMPLIES $n \equiv 2 \pmod 5$, SO $2n+1 \equiv 0 \pmod 5$.

QUESTION:
IS THERE A METHOD THAT WILL EASILY EXPLAIN THESE RESULTS AND BE APPLICABLE TO MANY SUCH PROBLEMS?

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ANSWER:
P. A. MACMAHON'S PARTITION ANALYSIS

Figure: P. A. MacMahon [1]

[1] https://commons.wikimedia.org/w/index.php?curid=6707227

THE MACMAHON OPERATOR, $\Omega_{\geq}$, IS GIVEN BY

$$\Omega_{\geq} \sum_{s_1=-\infty}^{\infty} \cdots \sum_{s_r=-\infty}^{\infty} A_{s_1,\ldots,s_r} \lambda_1^{s_1} \lambda_2^{s_2} \cdots \lambda_r^{s_r} = \sum_{s_1 \geq 0} \cdots \sum_{s_r \geq 0} A_{s_1,\ldots,s_r}$$

THE A'S ARE USUALLY RATIONAL FUNCTIONS OF SEVERAL COMPLEX VARIABLES OVER $\mathbb{C}$, AND THE λ_i LIE IN A NEIGHBORHOOD OF $|\lambda_i| = 1$.

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HERE IS THE SIMPLEST EXAMPLE: CONSIDER
$n = a_1 + a_2 + a_3 + a_4$ WITH

$$\begin{array}{ccc} a_4 & \overset{\geq}{\text{———}} & a_3 \\ \rotatebox{90}{$\leq$} & & \rotatebox{90}{$\leq$} \\ a_2 & \overset{\geq}{\text{———}} & a_1 \end{array}$$

USING MACMAHON'S NOTATION THE GENERATING FUNCTION IS

$$\underset{\geq}{\Omega} \sum_{a_i \geq 0} q^{a_1+a+2+a_3+a_4} \lambda_1^{a_1-a_2} \lambda_2^{a_1-a_3} \lambda_3^{a_2-a_4} \lambda_4^{a_3-a_4}$$

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$$= \underset{\geq}{\Omega} \frac{1}{(1-\lambda_1\lambda_2 q)\left(1-q\frac{\lambda_3}{\lambda_1}\right)\left(1-q\frac{\lambda_4}{\lambda_2}\right)\left(1-\frac{q}{\lambda_3\lambda_4}\right)}$$

AND TO ELIMINATE ALL THE λ'S, WE NEED ONLY TWO OF MACMAHON'S FORMULAS:

$$\underset{\geq}{\Omega} \frac{1}{(1-x\lambda)(1-y\lambda)\left(1-\frac{z}{\lambda}\right)} = \frac{1-xyz}{(1-x)(1-y)(1-xz)(1-yz)}$$

$$\underset{\geq}{\Omega} \frac{1}{(1-x\lambda)(1-\frac{y}{\lambda})\left(1-\frac{z}{\lambda}\right)} = \frac{1}{(1-x)(1-xy)(1-xz)}$$

AND A COROLLARY

$$\underset{\geq}{\Omega} \frac{1}{(1-x\lambda)(1-\frac{y}{\lambda})} = \frac{1}{(1-x)(1-xy)}.$$

HENCE

$$\underset{\geq}{\Omega} \frac{1}{(1-\lambda_1\lambda_2 q)\left(1-q\frac{\lambda_3}{\lambda_1}\right)\left(1-q\frac{\lambda_4}{\lambda_2}\right)\left(1-q\frac{q}{\lambda_3\lambda_4}\right)}$$

$$= \underset{\geq}{\Omega} \frac{1}{(1-\lambda_1\lambda_2 q)\left(1-q\frac{\lambda_3}{\lambda_1}\right)\left(1-\frac{q}{\lambda_2}\right)\left(1-\frac{q^2}{\lambda_3}\right)} \text{ (by eliminating } \lambda_4\text{)}$$

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$$=\underset{\geq}{\Omega}\frac{1}{(1-\lambda_1 q)\left(1-\frac{q}{\lambda_1}\right)(1-\lambda_1 q^2)(1-q^4)} \text{ (by eliminating } \lambda_2\text{)}$$

HENCE

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$$= \frac{(1-q^4)}{(1-q)(1-q^2)(1-q^2)(1-q^3)(1-q^4)}$$

HENCE

$$
\begin{aligned}
&\underset{\geq}{\Omega}\frac{1}{(1-\lambda_1\lambda_2 q)\left(1-q\frac{\lambda_3}{\lambda_1}\right)\left(1-q\frac{\lambda_4}{\lambda_2}\right)\left(1-q\frac{q}{\lambda_3\lambda_4}\right)} \\
&= \underset{\geq}{\Omega}\frac{1}{(1-\lambda_1\lambda_2 q)\left(1-q\frac{\lambda_3}{\lambda_1}\right)\left(1-\frac{q}{\lambda_2}\right)\left(1-\frac{q^2}{\lambda_3}\right)} \text{ (by eliminating } \lambda_4) \\
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&= \underset{\geq}{\Omega}\frac{1}{(1-\lambda_1 q)\left(1-\frac{q}{\lambda_1}\right)(1-\lambda_1 q^2)(1-q^4)} \text{ (by eliminating } \lambda_2) \\
&= \frac{(1-q^4)}{(1-q)(1-q^2)(1-q^2)(1-q^3)(1-q^4)} \\
&= \frac{1}{(1-q)(1-q^2)^2(1-q^3)} \text{ (by eliminating } \lambda_1)
\end{aligned}
$$

MACMAHON ADMITTED THAT HE FAILED TO TREAT THE FULL PLANE PARTITION PROBLEM USING PARTITION ANALYSIS.
HOWEVER, PARTITION ANALYSIS IS THE IDEAL TOOL TO SOLVE SCHMIDT'S PROBLEM AND THE NATURAL GENERALIZATION WE PROPOSED.

TO PROVE THESE 2 THEOREMS WE NEED ONLY

$$\Omega_{\geq} \frac{\lambda^{-s}}{(1 - A\lambda)(1 - B\lambda^{-r})} = \frac{A^s}{(1 - A)(1 - A^r B)}$$

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PROOF:

$$\begin{aligned}
\Omega_{\geq} \frac{\lambda^{-s}}{(1 - A\lambda)(1 - B\lambda^{-t})} &= \Omega_{\geq} \sum_{n \geq 0} \sum_{m \geq 0} A^n B^m \lambda^{n - mr - s} \\
&= \sum_{m \geq 0} \sum_{n \geq mr + s} A^n B^m \\
&= \sum_{m,n \geq 0} A^{n + mr + s} B^m \\
&= \frac{A^s}{(1 - A)(1 - A^r B)}.
\end{aligned}$$

$\square$

THE FOLLOWING IS THE GENERATING FUNCTION FOR PARTITIONS WITH n DISTINCT PARTS WITH x_i ACCOUNTING FOR THE $i^{\underline{\text{th}}}$ PART

$$\sum_{\substack{j_1,\dots,j_n \geq 0 \\ j_1-j_2\geq 1, j_2-j_3\geq 1,\dots,j_{n-1}-j_n\geq 1}} x_1^{j_1} x_2^{j_2} \cdots x_n^{j_n}$$

$$= \Omega_{\geq} \sum_{j_1,\dots,j_n\geq 0} x_1^{j_1} x_2^{j_2} \cdots x_n^{j_n} \lambda_1^{j_1-j_2-1} \lambda_2^{j_2-j_3-1} \cdots \lambda_{n-1}^{j_{n-1}-j_n} \lambda_n^{j_n}$$

$$= \frac{x_1(x_1x_2)(x_1x_2x_3)\cdots(x_1x_2x_3\cdots x_r)}{(1-x_1)(1-x_1x_2)\cdots(1-x_1x_2\cdots x_r)}$$

NOW SET $n = 2k$

$$x_1 = q, x_2 = 1, x_3 = q, x_4 = 1, x_5 = q, \ldots, x_{2k-1} = q, x_{2k} = 1.$$

THIS WILL GENERATE THOSE PARTITIONS INTO DISTINCT PARTS WHERE ONLY THE $1^{\underline{\text{st}}}, 3^{\underline{\text{rd}}}, 5^{\underline{\text{th}}}, \ldots, (2k-1)^{\underline{\text{st}}}$ PARTS ARE ADDED:

$$\frac{q^{k^2}}{(1-q)^2(1-q^2)^2\cdots(1-q^k)^2}$$

SUMMING OVER ALL $k \geq 0$, WE FIND

$$\sum_{n\geq 0} f(n)q^n = \sum_{k\geq 0} \frac{q^{k^2}}{(1-q)^2(1-q^2)^2\cdots(1-q^n)^2}$$

$$= \prod_{n=1}^{\infty} \frac{1}{1-q^n} \text{ (JACOBI)}$$

$$= \sum_{n\geq 0} p(n)q^n \qquad \square$$

TREATING $f_1(n)$ IS EVEN EASIER:

$$\Omega_{\geq} \sum_{n_1,n_2,n_3,\ldots} x_1^{n_1} x_2^{n_2} x_3^{n_3} \cdots \lambda_1^{n_1-n_2} \lambda_2^{n_2-n_3} \lambda_3^{n_3-n_4} \cdots$$

$$= \Omega_{\geq} \frac{1}{(1-x_1\lambda_1)(1-x_2\frac{\lambda_2}{\lambda_1})(1-x_3\frac{\lambda_3}{\lambda_2})(1-x_4\frac{\lambda_4}{\lambda_3})\cdots}$$

$$= \frac{1}{(1-x_1)(1-x_1x_2)(1-x_1x_2x_3)(1-x_1x_2x_3x_4)\cdots}$$

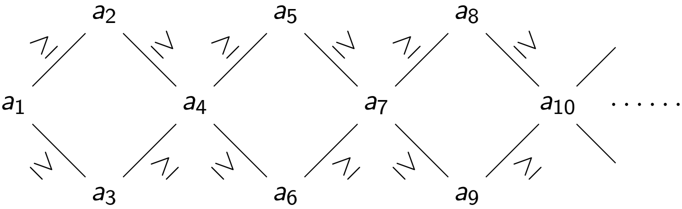
NOW SET $x_1 = q, x_2 = 1, x_3 = q, x_4 = 1, \ldots$. THE RESULT IS

$$\frac{1}{(1-q)^2(1-q^2)^2(1-q^3)^2(1-q^4)^2\cdots},$$

THE GENERATING FUNCTION FOR PARTITIONS INTO PARTS OF TWO KINDS.

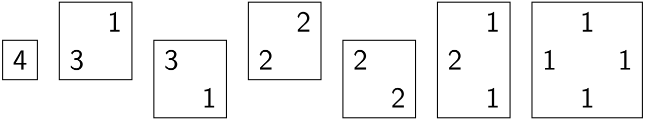
THE SCHMIDT IDEA CAN BE APPLIED TO NUMEROUS TYPES OF PARTITIONS. WE NEXT EXPLORE ITS APPLICATION TO PLANE PARTITION DIAMONDS.

PLANE PARTITION DIAMONDS:



$$n = a_1 + a_2 + a_3 + a_4 + a_5 + a_6 + \cdots$$

THERE ARE SEVEN SUCH PARTITIONS OF 4:



PARTITION ANALYSIS REVEALS THAT THE GENERATING FUNCTION FOR PLANE PARTITION DIAMONDS IS

$$\prod_{n\geq 1} \frac{1 - X_{3n-2}X_{3n}}{\left(1 - \frac{X_{3n}}{x_{3n-1}}\right)(1 - X_n)}$$

WHERE $X_k = x_1 x_2 \cdots x_k$.
SETTING ALL $x_k = q$ YIELDS

$$\prod_{n=1}^{\infty} \frac{1 + q^{3n-1}}{1 - q^n} = 1 + q + 2q^2 + 4q^3 + 7q^4 + \cdots$$

NOW APPLY THE SCHMIDT IDEA
JUST ADD UP THE SUMMANDS AT THE LINKS:

$$n = a_1 + a_4 + a_7 + a_{10} + \cdots$$

NOW THE GENERATING FUNCTION IS

$$\prod_{n=1}^{\infty} \frac{1+q^n}{(1-q^n)^3} = 1 + 4q + 13q^2 + 36q^3 + 7q^4 + \cdots$$

THE FOUR PARTITIONS FOR $n = 1$ ARE

$$\begin{array}{|c|}\hline 1 \\ \hline \end{array} \quad \begin{array}{|cc|}\hline & 1 \\ 1 & \\ \hline \end{array} \quad \begin{array}{|cc|}\hline 1 & \\ & 1 \\ \hline \end{array} \quad \begin{array}{|cc|}\hline & 1 \\ 1 & \\ & 1 \\ \hline \end{array}$$

NOTE THAT THE COMPLETE GENERATING FUNCTION FOR PLANE PARTITION DIAMONDS (PPD) IS NOT A MODULAR FORM,

BUT

THE GENERATING FUNCTION FOR ADDING JUST THE LINKS (PPDL): $a_1 + a_4 + a_7 + \cdots$ IS:

$$\frac{(-q;q)_\infty}{(q;q)_\infty^3} = \frac{(q^2;^2)_\infty}{(q;q)_\infty^4} = \frac{q^{-1/4}\eta(2\tau)}{\eta(\tau)4}, q = e^{2\pi i\tau}$$

THE MODULARITY LEADS TO EXTENSIVE ARITHMETIC INFORMATION. FOR EXAMPLE,

$$\begin{aligned}\prod_{n=1}^{\infty}\frac{1+q^n}{(1-q^n)^3} &= \prod_{n=1}^{\infty}\frac{1-q^{2n}}{(1-q^n)^4}\\ &= \prod_{n=1}^{\infty}\frac{1-q^{2n}}{1-4q^n+6q^{2n}-4q^{3n}+q^{4n}}\\ &\equiv \prod_{n=1}^{\infty}\frac{1-q^{2n}}{(1-2q^{2n})^2} \pmod 4\\ &= \prod_{n=1}^{\infty}\frac{1}{1-q^{2n}} = \sum_{n=0}^{\infty}p(n)q^{2n}.\end{aligned}$$

THUS THE NUMBER OF PPDLS OF $2n-1$ IS DIVISIBLE BY 4.

INDEED MUCH MORE IS TRUE.

$$\sum_{m=0}^{\infty} \text{ppdl}(2m+1)q^m = 4\frac{(q^2;q^2)_\infty^2 (q^4;q^4)_\infty^2}{(q;q)_\infty^9}$$

$$\sum_{m=0}^{\infty} \text{ppdl}(2m)q^m = 4\frac{(q^2;q^2)_\infty^4 (q^8;q^8)_\infty^4}{(q;q)_\infty^9 (q^4;q^4)_\infty^2} + \frac{(q^4;q^4)_\infty^{10}}{(q;q)_\infty^9 (q^8;q^8)_\infty^4},$$

WHERE $(A;q)_\infty = \prod_{n=0}^{\infty}(1-Aq^n)$.

IT IS POSSIBLE TO PROVE THE FIRST IDENTITY IN THE RAMANUJAN MANNER:

$$2\sum_{m\geq 0} \text{ppdl}(2m+1)q^{2m+1} = \frac{(-q;q)_\infty}{(q;q)_\infty^3} - \frac{(q;-q)_\infty}{(-q;-q)_\infty^3}$$

$$\vdots$$

$$= \frac{1}{(q^2;q^2)_\infty^7}(\psi(q^4) - \psi(-q^4))$$

WHERE $\psi(q) = \sum_{n=0}^{\infty} q^{\binom{n+1}{2}} = \frac{(q^2;q^2)_\infty}{(q;q^2)_\infty}.$

WE CAN FIND (SLIGHTLY DISGUISED) IN N. J. FINE'S BOOK ON q-HYPERGEOMETRIC SERIES:

$$\psi(q)^4 - \psi(-q)^4 = 8q\frac{(q^8;q^8)_\infty^4}{(q^2;q^4)_\infty^2}.$$

FILLING THIS INTO THE LAST EXPRESSION THEN YIELDS THE DESIRED RESULT AFTER SIMPLIFICATION.

THE OTHER IDENTITY (AND MANY MORE IN OUR FURTHER APPLICATIONS) REQUIRES SILVIU RADU'S "RAMANUJAN-KOLBERG" ALGORITHM WHICH HAS BEEN IMPLEMENTED IN MATHEMATICA BY NICHOLAS SMOOT. IT IS CALLED

RaduRK

AND IS AVAILABLE FROM RISC.

THANK YOU!