The Partition Pairing Theorems I

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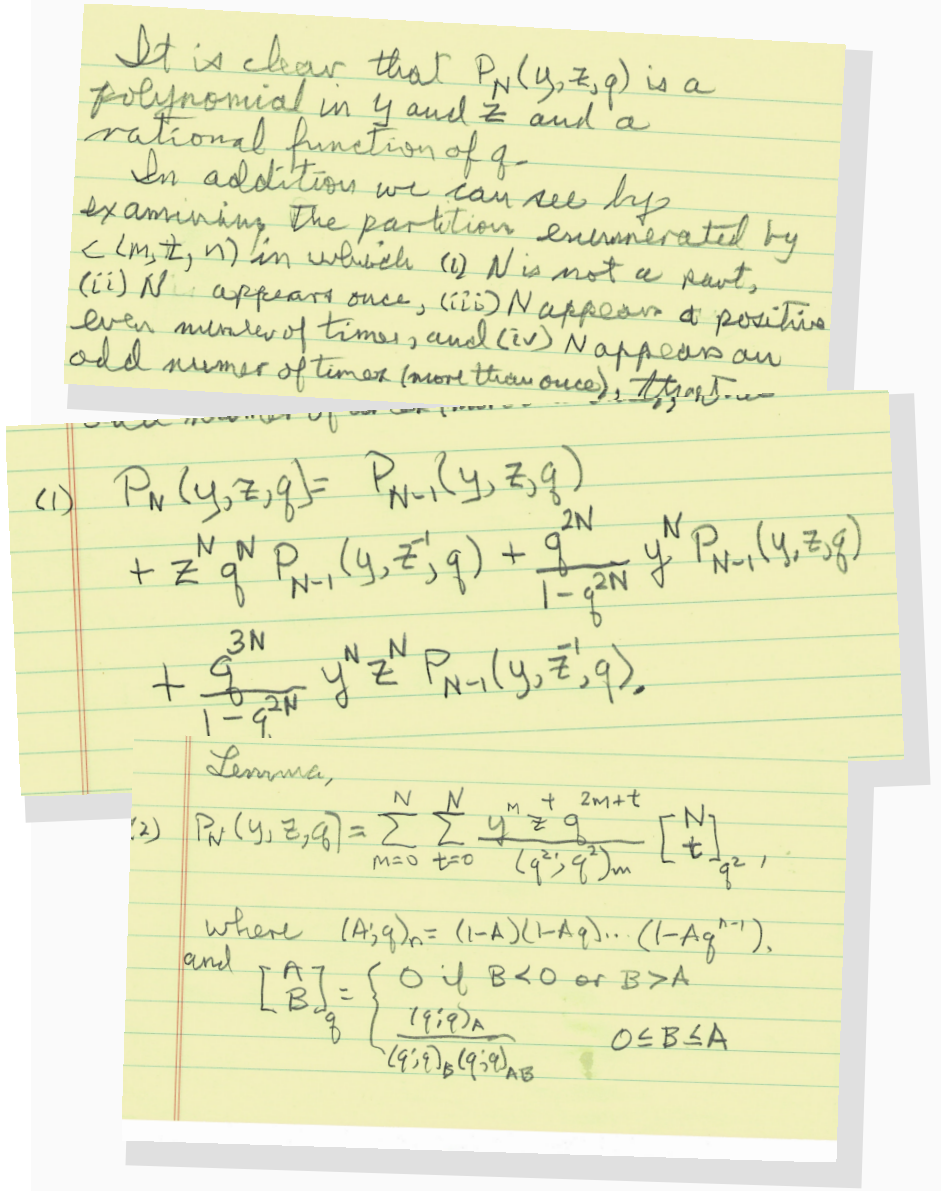


Motivation: Parity Bias

For a partition of n, compare the number of odd parts with the number of even parts.

Let

$$p_o(n) = \#\{\lambda \vdash n : \lambda \text{ has more odd parts than even parts}\},$$

and

$$p_e(n) = \#\{\lambda \vdash n : \lambda \text{ has more even parts than odd parts}\}.$$

Kim–Kim–Lovejoy

For every positive integer $n \neq 2$,

$$\boxed{p_o(n) > p_e(n).}$$

Thus partitions exhibit a genuine parity bias toward odd parts.

Parity bias for distinct partitions

Kim, Kim and Lovejoy asked whether the same phenomenon persists for partitions into distinct parts.

Let

$$d_o(n) = \#\{\lambda \vdash n : \lambda \text{ distinct, with more odd parts than even parts}\},$$
$$d_e(n) = \#\{\lambda \vdash n : \lambda \text{ distinct, with more even parts than odd parts}\},$$

They conjectured that

$$d_o(n) > d_e(n) \qquad (n > 19).$$

We proved that

For every $n > 19$,

$$\boxed{d_o(n) > d_e(n).}$$

A parity bias in multiplicities

Instead of looking at the parity of the parts themselves, we can look at the parity of their multiplicities.

Let

$$q_o(n) = \#\{\lambda \vdash n : \text{more distinct part sizes have odd multiplicity}\},$$

and let

$$q_e(n) = \#\{\lambda \vdash n : \text{more distinct part sizes have even multiplicity}\}.$$

Theorem

For every positive integer $n \neq 2, 4$,

$$\boxed{q_o(n) > q_e(n).}$$

So parity bias persists even when we move from the parts to their multiplicities.

What does multiplicity parity forget?

Suppose a part j occurs m_j times.

$$m_j = 2p_j + \varepsilon_j, \qquad \varepsilon_j \in \{0, 1\}.$$

Looking only at whether m_j is odd or even records

$$\varepsilon_j,$$

but completely forgets

$$p_j.$$

For example,

$$(6, 6, 6, 6, 3) \qquad \text{and} \qquad (6, 6, 3)$$

have exactly the same multiplicity parities.

But the first contains two pairs of 6's, while the second contains only one.

Pairing

$$\lambda = (5, 5, 4, 3, 3, 3, 1).$$

Largest part participation

$$r(\lambda) = 5.$$

The largest part participating in a pair.

Alternating sum of unpaired parts

$$s(\lambda) = 4 - 3 + 1 = 2.$$

Write the unpaired parts as $u_1 > u_2 > \cdots$; then $s = u_1 - u_2 + u_3 - \cdots$.

Pairing index $T = r + s = 7$ Pairing rank $\rho = r - s = 3.$

The pairing index and length

Theorem 2.1 First Partition Pairing Theorem

For every $n, k \geq 0$, the number of partitions of n with pairing index k equals the number with exactly k parts.

$$\#\{\lambda \vdash n : T(\lambda) = k\} = \#\{\mu \vdash n : \ell(\mu) = k\}.$$

$$\sum_{\lambda} z^{T(\lambda)} q^{|\lambda|} = \frac{1}{(zq; q)_\infty}.$$

All seven partitions of 5

λ	r	s	T
(5)	0	5	5
$(4,1)$	0	3	3
$(3,2)$	0	1	1
$(3,1,1)$	1	3	4
$(2,2,1)$	2	1	3
$(2,1,1,1)$	1	1	2
$(1,1,1,1,1)$	1	1	2

The two distributions agree

k	$\#(T=k)$	$\#(\ell=k)$
1	1	1
2	2	2
3	2	2
4	1	1
5	1	1

For instance, $T=2$ occurs for

$$(2,1,1,1), \quad (1,1,1,1,1),$$

while length 2 occurs for $(4,1)$ and $(3,2)$.

A width built from the two portions

Pair span

Number of copies belonging to pairs.

$$(5, 5, 4, 3, 3, 3, 1) \qquad \longrightarrow \qquad 4.$$

Residual span

For nonempty unpaired remainder,

$$2(u_1 - s) + 1.$$

Here $2(4 - 2) + 1 = 5$.

Pairing width

$$W(\lambda) = \max\{\text{pair span}, \text{residual span}\}.$$

$$W(\lambda) = \max\{4, 5\} = 5.$$

The residual span is 0 when the remainder is empty.

The Triple Pairing Theorem

Theorem 2.3 (Triple Pairing Theorem)

For every $n, a, b, c \geq 0$, the number of partitions λ of n satisfying

$$\text{largest paired part } r(\lambda) = a, \qquad \text{alternate sum } s(\lambda) = b, \qquad W(\lambda) = c$$

equals the number of partitions of n with exactly a even parts, b odd parts, and largest part c.

Even and odd parts are counted with multiplicity.

Example: n = 14

The pairing triple $(r, s, W) = (3, 2, 4)$

λ after pairing	unpaired parts	$s(\lambda)$	$W(\lambda)$
$(3, 3, 3, 3, 2)$	(2)	2	$\max\{4, 1\} = 4$
$(3, 3, 3, 2, 2, 1)$	$(3, 1)$	$3 - 1 = 2$	$\max\{4, 3\} = 4$
$(3, 3, 3, 2, 1, 1, 1)$	$(3, 2, 1)$	$3 - 2 + 1 = 2$	$\max\{4, 3\} = 4$

Partitions of 14 with largest paired part 3, sum of alternate unpaired parts 2, pairing width 4.

Example: 14

The ordinary triple $(e, o, \mu_1) = (3, 2, 4)$

μ	even parts	odd parts	μ_1
$(4, 4, 4, 1, 1)$	$(4, 4, 4)$	$(1, 1)$	4
$(4, 4, 3, 2, 1)$	$(4, 4, 2)$	$(3, 1)$	4
$(4, 3, 3, 2, 2)$	$(4, 2, 2)$	$(3, 3)$	4

Partitions of 14 with exactly 3 even parts 3, exactly 2 odd parts, and largest part 4.

Classical identities inside the triple

Specialization	Result recovered
Sum over s and W	Largest repeated part $\sim$ number of even parts
$r = 0$; retain s	Distinct parts, alternating sum $b \sim$ exactly b odd parts
$r = 0$; sum over s	Euler: distinct parts $\sim$ odd parts
$r = 0$; mark odd-indexed sum	Schmidt: $\#\{u_1 > u_2 > \cdots : \ u_1 + u_3 + \cdots = n\} = p(n)$

Andrews–Deutsch (2016) Bessenrodt (1994)

Euler (1748) Schmidt; Andrews–Paule (2022)

Corollary I: Andrews–Deutsch

Forget $s(\lambda)$ and $W(\lambda)$.

The Triple Pairing Theorem says:

$$r(\lambda) = a \quad \longleftrightarrow \quad \#\text{ even parts} = a.$$

Therefore:

Andrews–Deutsch
The number of partitions of n whose largest repeated part is a equals the number of partitions of n having exactly a even parts.

Corollary II: Bessenrodt

Now set

$$r(\lambda) = 0.$$

This means: no part is repeated. So λ is a distinct partition.

The Triple Pairing Theorem becomes:

$$s(\lambda) = b \quad \longleftrightarrow \quad \#\text{ odd parts} = b.$$

Bessenrodt
The number of distinct partitions of n with alternating sum b equals the number of partitions of n into exactly b odd parts.

Corollary III: Euler

From Bessenrodt, now forget the value of b.

Left side:

$$\text{distinct partitions of } n.$$

Right side:

$$\text{partitions of } n \text{ into odd parts.}$$

Euler's odd–distinct theorem

The number of partitions of n into distinct parts equals the number of partitions of n into odd parts.

Corollary IV: the pairing index theorem

The Triple Pairing Theorem gives

$$r(\lambda) \longleftrightarrow \#\text{ even parts}, \qquad s(\lambda) \longleftrightarrow \#\text{ odd parts}.$$

Add the two sides:

$$r(\lambda) + s(\lambda) \quad \longleftrightarrow \quad \#\text{ even parts} + \#\text{ odd parts}.$$

But

$$T(\lambda) = r(\lambda) + s(\lambda), \qquad \ell(\mu) = \#\text{ even parts} + \#\text{ odd parts}.$$

First Partition Pairing Theorem

The number of partitions of n with pairing index k equals the number of partitions of n into exactly k parts.

Corollary V: Schmidt's theorem

In the distinct-part case,

$$r(\lambda) = 0.$$

The generating function becomes

$$\sum_{\lambda \text{ distinct}} z^{s(\lambda)} q^{|\lambda|} = \frac{1}{(zq; q^2)_\infty}.$$

Now set $z = q$ and then replace q^2 by q.

Schmidt's theorem

$p(n)$ is the number of distinct partitions for which the sum of the odd-indexed parts is n.

Kummer's theorem

Kummer, 1852

Let p be a prime and let $R, S \geq 0$. Then the highest power of p dividing $\binom{R+S}{R}$ = the number of carries produced when R and S are added in base p.

Example: Four carries in base 3 for $\binom{246}{119}$

Take $R = 119$ and $S = 127$

$$119 = 11102_3, \qquad 127 = 11201_3,$$

$$119 + 127 = 246 = 100010_3.$$

$$\begin{array}{r} 11102_3 \\ +\,11201_3 \\ \hline 100010_3 \end{array}$$

position	3^0	3^1	3^2	3^3	3^4
carry out	1	0	1	1	1

Example: Factorization

$$\binom{246}{119} = 2^7 \cdot 3^4 \cdot 5 \cdot 7 \cdot 11^2 \cdot 13 \cdot 41 \cdot 43 \cdot 47$$
$$\cdot 61 \cdot 67 \cdot 71 \cdot 73 \cdot 79 \cdot 131 \cdot 137 \cdot 139$$
$$\cdot 149 \cdot 151 \cdot 157 \cdot 163 \cdot 167 \cdot 173 \cdot 179 \cdot 181$$
$$\cdot 191 \cdot 193 \cdot 197 \cdot 199 \cdot 211 \cdot 223$$
$$\cdot 227 \cdot 229 \cdot 233 \cdot 239 \cdot 241.$$

The index–width theorem

Theorem 2.5

For every $n \geq 0$ and $a, b \geq 0$,

$$\#\{\lambda \vdash n : T(\lambda) = a,\ W(\lambda) = b\} = \#\{\mu \vdash n : \ell(\mu) = a,\ \mu_1 = b\}.$$

pairing index T	$\longleftrightarrow$	number of parts ℓ
pairing width W	$\longleftrightarrow$	largest part μ_1

$$N_{a,b}(n) := \#\{\lambda \vdash n : T(\lambda) = a,\ W(\lambda) = b\}.$$

Pairing form of Kummer's theorem

Let p be prime and let $a, b \geq 1$. The p-adic valuation of the total number, over all sizes, of partitions having pairing index a and pairing width b equals the number of carries that occur when $a-1$ and $b-1$ are added in base p.

Bounded width: a finite generating function

Define

$$F_B(x,q) := \sum_{\substack{\lambda \\ W(\lambda)\le B}} x^{T(\lambda)} q^{|\lambda|}.$$

The index–width theorem, summed over $b \le B$, gives

$$F_B(x,q) = \sum_{\substack{\mu \\ \mu_1 \le B}} x^{\ell(\mu)} q^{|\mu|}.$$

Therefore

$$\boxed{F_B(x,q) = \prod_{j=1}^{B} \frac{1}{1-xq^j} = \frac{1}{(xq;q)_B}}.$$

x : pairing index / number of parts,
q : size of the partition,
B : width / largest-part bound.

Exact width produces a Gaussian polynomial

Exact width is obtained by subtraction:

$$F_b(x,q) - F_{b-1}(x,q) = \frac{xq^b}{(xq;q)_b}.$$

Taking the coefficient of x^a,

$$\begin{aligned}\sum_{n\geq 0} N_{a,b}(n)q^n &= [x^a]\frac{xq^b}{(xq;q)_b} \\ &= q^{a+b-1}\begin{bmatrix} a+b-2 \\ a-1 \end{bmatrix}_q.\end{aligned}$$

The pairing statistics have become a finite Gaussian polynomial.

At $q = 1$, the binomial coefficient appears

From the Gaussian polynomial,

$$\sum_{n\geq 0} N_{a,b}(n) = q^{a+b-1} \begin{bmatrix} a+b-2 \\ a-1 \end{bmatrix}_q \Bigg|_{q=1}.$$

Hence

$$\boxed{\sum_{n\geq 0} N_{a,b}(n) = \binom{a+b-2}{a-1}.}$$

Put $r = a - 1$ and $s = b - 1$

Then

$$\sum_{n\geq 0} N_{r+1,s+1}(n) = \binom{r+s}{r}.$$

Kummer's theorem as a special case

Thus Kummer's theorem is a special case of the pairing index–width theorem by putting

$$a = r + 1, \qquad b = s + 1,$$

and then setting $q = 1$.

Indeed,

$$\sum_{n \geq 0} N_{r+1,s+1}(n) = \binom{r+s}{r}.$$

The cyclotomic factors of the Gaussian polynomial give the proof of Kummer's theorem.

$5n + 4$ yet again!

We are all familiar with Ramanujan's famous congruence

$$\boxed{p(5n + 4) \equiv 0 \pmod 5}.$$

Recently, George and I proved that the total sums of odd parts and even parts over the partitions of $5n + 4$ satisfy the same congruence:

$$\boxed{S_o(5n + 4) \equiv 0 \pmod 5.}$$

$$\boxed{S_e(5n + 4) \equiv 0 \pmod 5.}$$

But the surprises do not end there.

Record the entire distribution

For a partition λ, let

$$\delta(\lambda) = c_1(\lambda) - c_3(\lambda),$$

where $c_1(\lambda)$ and $c_3(\lambda)$ count unpaired parts congruent to 1 and 3 (mod 4).

Instead of recording only the sum of the values of δ, record the entire distribution:

$$A_j(N) := \#\{\lambda \vdash N : \delta(\lambda) = j\}.$$

Package these numbers into the Laurent polynomial

$$G_N(z) := \sum_{j \in \mathbb{Z}} A_j(N) z^j = \sum_{\lambda \vdash N} z^{\delta(\lambda)}.$$

The stronger $5n + 4$ theorem

The surprising congruence

For every $n \geq 0$,

$$\boxed{A_j(5n+4) \equiv 0 \pmod 5 \qquad \text{for every } j \in \mathbb{Z}.}$$

So Ramanujan's famous congruence becomes
a special case of our theorem looking at partitions over all j.

$$\boxed{p(5n+4) \equiv 0 \pmod 5}$$

A revealing example: $14 = 5 \cdot 2 + 4$

For the 135 partitions of 14,

j	-2	0	2
$A_j(14)$	5	110	20

Thus

$$A_{-2}(14) \equiv A_0(14) \equiv A_2(14) \equiv 0 \pmod 5.$$

The five partitions with $j = -2$

$$(11, 3), \quad (7, 4, 3), \quad (7, 3, 2^2),$$
$$(7, 3, 2, 1^2), \quad (7, 3, 1^4).$$

So δ is a semi-crank

We call a statistic c a *semi-crank modulo* ℓ for the progression $\ell n + r$ if

$$\#\{\lambda \vdash \ell n + r : c(\lambda) = j\} \equiv 0 \pmod{\ell}$$

for every j.

The classes need not have equal cardinality.

For

$$\delta(\lambda) = c_1(\lambda) - c_3(\lambda),$$

our theorem gives

$$A_j(5n+4) = \#\{\lambda \vdash 5n+4 : \delta(\lambda) = j\} \equiv 0 \pmod{5}.$$

Summing over j recovers

$$p(5n+4) \equiv 0 \pmod{5}.$$

From arithmetic pairing to diagonal pairing

The statistic δ records an arithmetic imbalance among the parts left unpaired.

Can pairing produce analogous semi-cranks for the other classical Ramanujan congruences?

ℓ	Ramanujan progression
5	$5n+4$
7	$7n+5$
11	$11n+6$

The answer is yes.
But for that we need the concept of diagonal pairing that we will introduce shortly!

Pairing rank and self-conjugate partitions

Recall the definition of pairing rank

$$\rho(\lambda) = r(\lambda) - s(\lambda).$$

Let $E(n)$ and $O(n)$ denote the numbers of partitions of n with even and odd pairing rank, respectively. Let $\mathrm{sc}(n)$ denote the number of self-conjugate partitions of n.

Theorem 3.1

$$\boxed{E(n) - O(n) = (-1)^n \,\mathrm{sc}(n) \qquad (n \geq 0).}$$

Example: $n = 14$

For the 135 partitions of 14,

pairing-rank	number of partitions
even	$E(14) = 69$
odd	$O(14) = 66$

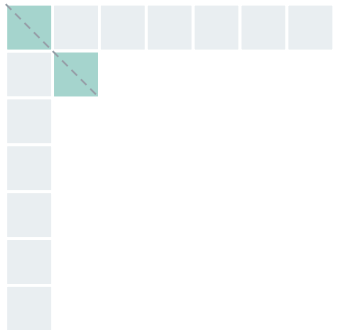
Thus

$$E(14) - O(14) = 69 - 66 = 3.$$

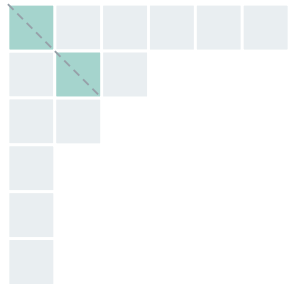
There are exactly three self-conjugate partitions of 14, so

$$(-1)^{14}\,\mathrm{sc}(14) = 3.$$

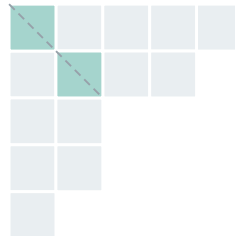
The three self-conjugate partitions of 14



$\lambda = (7, 2, 1^5)$



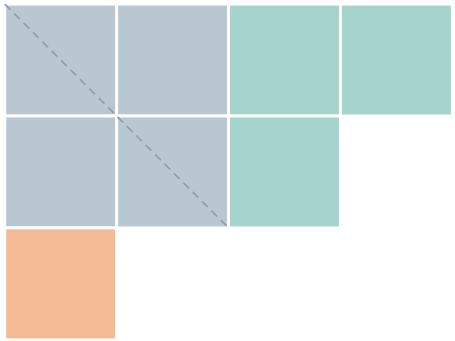
$\lambda = (6, 3, 2, 1^3)$



$\lambda = (5, 4, 2^2, 1)$

$$\mathrm{sc}(14) = 3, \qquad E(14) - O(14) = 3.$$

Diagonal pairing: fold the two wings



$\lambda = (4, 3, 1), \qquad d = 2$

Remove the $d \times d$ Durfee square and write

$$\alpha_i = \lambda_i - d, \qquad \beta_i = \lambda'_i - d.$$

For $\lambda = (4, 3, 1)$,

$$\alpha = (2, 1), \qquad \beta = (1, 0).$$

The lower wing is reflected across the main diagonal and superimposed on the right wing.

Cells that coincide are paired. The remaining cells form the unpaired part of the diagram.

Their row-by-row differences are the successive ranks:

$$\sigma_i(\lambda) = \alpha_i - \beta_i = \lambda_i - \lambda'_i.$$

The folded-wing span

For a partition λ with Durfee size d, define

$$R(\lambda) = \max_{1 \le i \le d} |\sigma_i(\lambda)|, \qquad \sigma_i(\lambda) = \lambda_i - \lambda'_i.$$

Thus $R(\lambda)$ is the largest row-by-row difference between the two folded wings.

For $\ell \in \{5, 7, 11\}$, group these spans in blocks of length ℓ:

$$\boxed{\kappa_\ell(\lambda) = \left\lfloor \frac{R(\lambda) + 1}{\ell} \right\rfloor}$$

The statistic κ_ℓ is the diagonal semi-crank for the prime ℓ.

Three diagonal semi-cranks

ℓ	partition size	semi-crank
5	$5n+4$	κ_5
7	$7n+5$	κ_7
11	$11n+6$	κ_{11}

For every k, the exact-value classes satisfy

$$\boxed{\#\{\lambda \vdash \ell n + r_\ell : \kappa_\ell(\lambda) = k\} \equiv 0 \pmod{\ell}}$$

Summing over k recovers the corresponding Ramanujan congruence.

The first informative 7-case: $12 = 7 \cdot 1 + 5$

There are

$$p(12) = 77$$

partitions of 12.

For this example, the diagonal semi-crank is

$$\kappa_7(\lambda) = \left\lfloor \frac{R(\lambda) + 1}{7} \right\rfloor, \qquad R(\lambda) = \max_i |\sigma_i(\lambda)|.$$

Thus the values are grouped as follows:

$\kappa_7(\lambda)$	$R(\lambda)$
0	$0 \le R(\lambda) \le 5$
1	$6 \le R(\lambda) \le 12$

We now examine one partition in each class.

A partition in the class $\kappa_7 = 0$

Consider

$$\lambda = (4, 3, 2, 2, 1) \vdash 12.$$

Its conjugate is

$$\lambda' = (5, 4, 2, 1), \qquad d = 2.$$

The folded wings are

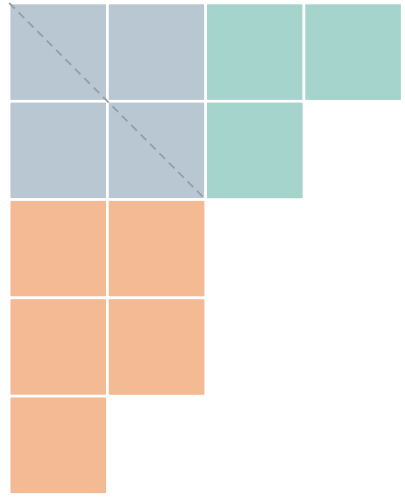
$$\alpha = (2, 1), \qquad \beta = (3, 2).$$

Hence

$$\sigma(\lambda) = \alpha - \beta = (-1, -1),$$

so

$$R(\lambda) = 1, \qquad \kappa_7(\lambda) = \left\lfloor \frac{1+1}{7} \right\rfloor = 0.$$



right wing: $\alpha = (2, 1)$

lower wing: $\beta = (3, 2)$

A partition in the class $\kappa_7 = 1$

Consider

$$\lambda = (10, 2) \vdash 12.$$

Its Durfee size is $d = 2$, and

$$\lambda' = (2, 2, 1^8).$$

The folded wings are

$$\alpha = (8, 0), \qquad \beta = (0, 0).$$

Therefore

$$\sigma(\lambda) = \alpha - \beta = (8, 0),$$

so

$$R(\lambda) = 8, \qquad \kappa_7(\lambda) = \left\lfloor \frac{8+1}{7} \right\rfloor = 1.$$

right wing: $\alpha = (8, 0)$

reflected lower wing: $\beta = (0, 0)$

The two semi-crank classes of 12

For the 77 partitions of 12, the distribution of κ_7 is

k	range of $R(\lambda)$	$\#\{\lambda \vdash 12 : \kappa_7(\lambda) = k\}$
0	$0 \le R(\lambda) \le 5$	$63 = 7 \cdot 9$
1	$6 \le R(\lambda) \le 12$	$14 = 7 \cdot 2$

Thus every exact semi-crank class is divisible by 7:

$$\boxed{63 \equiv 14 \equiv 0 \pmod 7}$$

Adding the two classes gives

$$63 + 14 = 77 = p(12),$$

and hence

$$p(7n + 5) \equiv 0 \pmod 7$$

Questions raised by semi-cranks

We have seen that the same prime can admit different, explicit semi-cranks.

- How many genuinely different semi-cranks can a prime have?
- Would explicit semi-cranks help us find
  new proofs of congruences for larger primes?
- Can pairing theory predict the right statistic before the congruence is known?

Diagonal pairing and principal hooks

Why this comes from pairing

Diagonal pairing separates a partition into paired structure and parity residue. In Frobenius language, that same separation is detected by the principal hooks.

Principal-hook overpartition theorem

For every $m \geq 0$,

$$\#\{\lambda \vdash 2m : \text{ every principal hook of } \lambda \text{ is even}\} = \overline{p}(m).$$

Here $\overline{p}(m)$ denotes the number of overpartitions of m.

$$\sum_{m \geq 0} \#\{\lambda \vdash 2m : \text{ all principal hooks even}\} q^m = \frac{(-q;q)_\infty}{(q;q)_\infty}.$$

What is a principal hook?

For a partition with Frobenius symbol

$$\lambda = \begin{pmatrix} a_1 & a_2 & \cdots & a_d \\ b_1 & b_2 & \cdots & b_d \end{pmatrix},$$

the hook based at the i-th diagonal cell is the principal hook

$$h_i = a_i + b_i + 1.$$

Example

Take

$$\lambda = (4, 3, 1).$$

Its Frobenius symbol is

$$\begin{pmatrix} 3 & 1 \\ 2 & 0 \end{pmatrix}.$$

What is a principal hook?

For

$$\lambda = (4, 3, 1), \qquad \begin{pmatrix} 3 & 1 \\ 2 & 0 \end{pmatrix},$$

the two principal hook lengths are

$$h_1 = 3 + 2 + 1 = 6, \qquad h_2 = 1 + 0 + 1 = 2.$$

Thus every principal hook of λ is even.

In general,

$$\boxed{\text{all principal hooks even}} \iff a_i \not\equiv b_i \pmod 2 \quad \text{for every } i.$$

Example: $m = 4$

$$\#\{\lambda \vdash 8 : \text{ all principal hooks even}\} = \overline{p}(4) = 14.$$

Even principal hooks in partitions of 8

λ	principal hooks
(8)	8
$(7,1)$	8
$(6,1,1)$	8
$(5,3)$	$6,2$
$(5,1,1,1)$	8
$(4,3,1)$	$6,2$
$(4,2,2)$	$6,2$
$(4,1,1,1,1)$	8
$(3,3,1,1)$	$6,2$
$(3,2,2,1)$	$6,2$
$(3,1,1,1,1,1)$	8
$(2,2,2,1,1)$	$6,2$
$(2,1,1,1,1,1,1)$	8
(1^8)	8

Overpartitions of 4

$$4$$
$$\overline{4}$$
$$3+1$$
$$\overline{3}+1$$
$$3+\overline{1}$$
$$\overline{3}+\overline{1}$$
$$2+2$$
$$\overline{2}+2$$
$$2+1+1$$
$$\overline{2}+1+1$$
$$2+\overline{1}+1$$
$$\overline{2}+\overline{1}+1$$
$$1+1+1+1$$
$$\overline{1}+1+1+1$$

A refined principal-hook theorem

Theorem

Fix $d \geq 1$.

The number of partitions of $2n$ whose

- principal hooks are all even, and
- Durfee square has side length d,

is equal to the number of overpartitions of n whose Frobenius representation has exactly d columns.

Simply paired partitions

We now return to pairing equal parts, with one restriction:

At most one pair of each size.

We call these partitions *simply paired*.

Their enumeration links to interesting objects such as odd divisors and overpartitions into odd parts, and to the Kontsevich–Zagier strange function.

Allow each part at most three times

Simply paired partitions

Each part size has multiplicity $0, 1, 2$, or 3.

Let A be the set of paired sizes and U the set of unpaired sizes.

$$m_j = 2\mathbf{1}_{j \in A} + \mathbf{1}_{j \in U}.$$

Write $b(\lambda) = |A|$.

A negative-rank example

$$\lambda = (6, 3, 3, 3, 2, 1) \vdash 18.$$

$$A = \{3\}, \qquad U = \{6, 3, 2, 1\}.$$

$$r = 3, \qquad s = 6 - 3 + 2 - 1 = 4.$$

$$\rho = -1, \qquad b = 1.$$

The size 3 belongs to both sets.

A signed count becomes an odd-divisor count

Theorem 6.2 Odd-divisor theorem

Among simply paired partitions with $\rho < 0$, let $E_{\mathrm{sp}}, O_{\mathrm{sp}}$ count even and odd b.

$$E_{\mathrm{sp}}(n) - O_{\mathrm{sp}}(n) = d_{\mathrm{odd}}(n), \qquad n \geq 1.$$

Here $d_{\mathrm{odd}}(n)$ counts the odd positive divisors of n.
At $n = 6$,

$$4 - 2 = 2 = \#\{1, 3\}.$$

$\lambda \vdash 6$	ρ	b	Sign
(6)	-6	0	$+$
$(5,1)$	-4	0	$+$
$(4,2)$	-2	0	$+$
$(4,1,1)$	-3	1	$-$
$(3,2,1)$	-2	0	$+$
$(3,1,1,1)$	-1	1	$-$

The unsigned count gives odd overpartitions

Theorem 6.3 Odd-overpartition theorem

Let $M(n)$ count simply paired partitions of n with negative pairing rank. Then

$$M(n) = \frac{1}{2}\,\overline{p}_{\mathrm{odd}}(n), \qquad n \geq 1.$$

n	4	5	6	7	8	9	10
$M(n)$	3	4	6	8	11	15	20
$\overline{p}_{\mathrm{odd}}(n)$	6	8	12	16	22	30	40

Here $\overline{p}_{\mathrm{odd}}(n)$ counts overpartitions of n into odd parts.

At rank minus two, squares control parity

Let $A_2(n)$ count simply paired partitions of n with $\rho = -2$. Define $\chi(m) = 1$ if the odd part of $m > 0$ is a square, and 0 otherwise; $\chi(0) = 0$.

Theorem 7.3 Square-switch theorem

$$A_2(n+8) + A_2(n) \equiv n + 1 + \chi(n+3) + \chi(n+7) \pmod 2, \qquad n \geq 0.$$

For even n, the two counts have the same parity exactly when one of $n+3$ and $n+7$ is a perfect square.

The parity switch is visible in small values

Even n

n	$A_2(n)$	$A_2(n+8)$	Square among $n+3, n+7$
0	0	3	none
2	1	5	9
4	1	6	none
6	2	10	9
8	3	14	none

At $n = 2$ and $n = 6$, the parity stays the same.

Corollary 7.4

For every $n \geq 0$,

$$\begin{aligned} A_2(16n) &\equiv 0, \\ A_2(16n+4) &\equiv 1, \\ A_2(16n+8) &\equiv 1, \\ A_2(16n+12) &\equiv 0 \end{aligned} \pmod 2.$$

The Kontsevich–Zagier strange identity

Define

$$F(q) := \sum_{n\geq 0} (q;q)_n = 1 + (1-q) + (1-q)(1-q^2) + \cdots .$$

Zagier proved the remarkable "strange identity"

$$\boxed{F(q) \text{ “}=\text{”} -\frac{1}{2}\sum_{n\geq 1} n\left(\frac{12}{n}\right) q^{(n^2-1)/24}}$$

where $\left(\frac{12}{n}\right)$ is the Kronecker symbol.

The two sides never converge simultaneously.

The equality means that their asymptotic expansions agree as q approaches roots of unity.

The Kontsevich–Zagier strange function

Consider the series

$$F(Q) = 1 + (1 - Q) + (1 - Q)(1 - Q^2) + \cdots$$

It does not converge for $|Q| < 1$.

At a root of unity Q of order N, every term containing $1 - Q^N$ vanishes. Thus

$$F(Q) = \sum_{j=0}^{N-1} (Q; Q)_j.$$

A finite family of simply paired partitions

Let $\mathcal{S}_N$ contain the simply paired partitions with parts in $\{1, \ldots, N\}$, including the empty partition.

Each multiplicity is $0, 1, 2$, or 3, so

$$|\mathcal{S}_N| = 4^N.$$

We now allow every pairing rank.

The sign

$b(\lambda)$ counts the part sizes that participate in a pair.

$$(-1)^{b(\lambda)}$$

The pairing rank

$$\rho(\lambda) = r(\lambda) - s(\lambda).$$

Here s is the alternating sum of the unpaired parts.

Two signed cancellations

Let $N \geq 1$ and let ζ be a primitive $2N$-th root of unity.

The signed total

$$\sum_{\lambda \in \mathcal{S}_N} (-1)^{b(\lambda)} \zeta^{|\lambda|} = 0.$$

The first pairing-rank moment

$$\sum_{\lambda \in \mathcal{S}_N} (-1)^{b(\lambda)} \rho(\lambda) \zeta^{|\lambda|} = 0.$$

Pairing rank and the strange function

Let $N \geq 1$ and let ζ be a primitive $2N$-th root of unity.

Theorem

The signed second moment of the pairing rank gives the Kontsevich–Zagier strange function:

$$\sum_{\lambda \in \mathcal{S}_N} (-1)^{b(\lambda)} \rho(\lambda)^2 \zeta^{|\lambda|} = -2N\, F(\zeta^2).$$

Since ζ^2 has order N, the right side is also a finite sum.

Example: $N = 2$, even b

$|\mathcal{S}_2| = 16, \quad \zeta = i, \qquad w(\lambda) = (-1)^{b(\lambda)}\rho(\lambda)^2 i^{|\lambda|}.$

The eight partitions with even b

λ	b	ρ	$w(\lambda)$
$\varnothing$	0	0	0
(1)	0	-1	i
(2)	0	-2	-4
$(2,1)$	0	-1	$-i$
$(2^2,1^2)$	2	2	-4
$(2^2,1^3)$	2	1	$-i$
$(2^3,1^2)$	2	0	0
$(2^3,1^3)$	2	1	i

One entry

For $\lambda = (2,2,1,1)$,

$$b = 2, \qquad \rho = 2, \qquad |\lambda| = 6.$$

Thus

$$w(\lambda) = 2^2 i^6 = -4.$$

Subtotal

$$i - 4 - i - 4 - i + i = -8.$$

Example: $N = 2$, odd b

$|\mathcal{S}_2| = 16, \quad \zeta = i, \qquad w(\lambda) = (-1)^{b(\lambda)}\rho(\lambda)^2 i^{|\lambda|}.$

The eight partitions with odd b

λ	b	ρ	$w(\lambda)$
(1^2)	1	1	1
(1^3)	1	0	0
$(2,1^2)$	1	-1	-1
$(2,1^3)$	1	0	0
(2^2)	1	2	-4
$(2^2,1)$	1	1	$-i$
(2^3)	1	0	0
$(2^3,1)$	1	1	i

All sixteen partitions

$$\underbrace{-8}_{b \text{ even}} + \underbrace{(-4)}_{b \text{ odd}} = -12.$$

The strange function

$$F(-1) = 1 + 2 = 3.$$
$$-2N\,F(\zeta^2) = -4 \cdot 3 = -12.$$

Collect the entire pairing-rank distribution

For each integer m, put

$$R_m(q) = \sum_{\rho(\lambda)=m} q^{|\lambda|}.$$

$$\sum_{m\in\mathbb{Z}} R_m(q) z^m = \frac{1}{(zq^2;q^2)_\infty (z^{-1}q;q^2)_\infty}.$$

Form the Toeplitz determinant

$$D_k(q) = \det\bigl(R_{i-j}(q)\bigr)_{1\le i,j\le k}, \qquad D_3(q) = \begin{vmatrix} R_0 & R_{-1} & R_{-2} \\ R_1 & R_0 & R_{-1} \\ R_2 & R_1 & R_0 \end{vmatrix}.$$

The finite determinant is a Schur sum

Theorem 8.1 Finite determinant identity

For every $k \geq 1$,

$$D_k(q) = \sum_{\ell(\nu) \leq k} s_\nu(q^2, q^4, q^6, \ldots)\, s_\nu(q, q^3, q^5, \ldots).$$

Here

$$s_\nu(x) = \det\big(h_{\nu_i - i + j}(x)\big), \qquad \sum_{r \geq 0} h_r(x) z^r = \prod_{j \geq 1} \frac{1}{1 - x_j z},$$

is the usual Schur function.

$$D_2(q) = R_0(q)^2 - R_{-1}(q) R_1(q).$$

Every $D_k(q)$ has nonnegative coefficients, and the coefficients increase weakly with k.

The limit has a MacMahon-type product

Theorem 8.2 Limiting determinant identity

Coefficientwise as a formal power series,

$$\lim_{k\to\infty} D_k(q) = \prod_{r\geq 1} \frac{1}{(1-q^{2r+1})^r}.$$

The identity also holds analytically for $|q| < 1$.

MacMahon (1916): plane partitions	The pairing-rank determinant
$\prod_{r\geq 1}(1-q^r)^{-r}$	$\prod_{r\geq 1}(1-q^{2r+1})^{-r}$

The multiplicity is r; its exponent is $2r+1$ in the pairing product.

What pairing reveals

Pairing construction	Structure it reveals
Equal parts	Length, even/odd part counts, and a rectangle refinement
Cells across the diagonal	Boolean classes and nonnegative successive ranks
Signed and unsigned counts	Self-conjugacy, divisors, and overpartitions
Signed pairing-rank moments	The Kontsevich–Zagier strange function
The complete rank distribution	Schur functions and a MacMahon-type product

Thank you!

Questions?

Appendix

Reference: higher pairing-rank moments

Let $N, k \geq 1$, let ζ have order $2N$, and bound the largest part by kN.

All moments through degree k vanish

$$\sum_{\lambda \in \mathcal{S}_{kN}} (-1)^{b(\lambda)} \rho(\lambda)^j \zeta^{|\lambda|} = 0, \qquad 0 \leq j \leq k.$$

The next moment

$$\sum_{\lambda \in \mathcal{S}_{kN}} (-1)^{b(\lambda)} \rho(\lambda)^{k+1} \zeta^{|\lambda|} = -(k+1)!\, N^k F(\zeta^2).$$

For $j = 0$, the first sum means the signed total.
The case $k = 1$ is the second-moment theorem in the talk.

Reference: conventions

$$(a;q)_N = \prod_{j=0}^{N-1}(1-aq^j), \qquad \begin{bmatrix} A \\ B \end{bmatrix}_q = \frac{(q;q)_A}{(q;q)_B(q;q)_{A-B}}.$$

The empty partition has $r = s = T = W = 0$, Durfee size 0, and no diagonal blocks.

If $P_N(y,z,q)$ restricts the largest part of the original partition to at most N, Lemma 2.2 states

$$P_N(y,z,q) = \sum_{m=0}^{N}\sum_{t=0}^{N} \frac{y^m z^t q^{2m+t}}{(q^2;q^2)_m} \begin{bmatrix} N \\ t \end{bmatrix}_{q^2}.$$

This is a bound on λ_1. The triple theorem instead bounds $W(\lambda)$.

Reference: width at the boundary

λ	r	s	Pair span	Residual span	T	W
$\varnothing$	0	0	0	0	0	0
$(4,4,2,2)$	4	0	4	0	4	4
$(5,3,1)$	0	3	0	5	3	5
$(1,1,1)$	1	1	2	1	2	2
$(5,5,4,3,3,3,1)$	5	2	4	5	7	5

For a nonempty remainder, the residual span is always odd.

The pair span is always even.

Reference: all nonnegative representatives of 8

k	Partitions in $\mathcal{P}^+$	Count
0	$(4,2,1,1),\ (3,3,2)$	2
1	$(8),\ (7,1),\ (6,2),\ (6,1,1)$	
	$(5,3),\ (5,2,1),\ (5,1,1,1),\ (4,4)$	8
2	$(4,3,1)$	1

$$\sum_{\substack{\lambda \vdash 8 \\ \lambda \in \mathcal{P}^+}} t^{\kappa(\lambda)} = 2 + 8t + t^2.$$

$$\sum_{\lambda \vdash 8} z^{\kappa(\lambda)} = 2 + 16z + 4z^2.$$

Reference: the Frobenius statistic for overpartitions

An overpartition Frobenius representation with d columns has:

A top row of d distinct nonnegative integers.

A bottom row that is an overpartition into d nonnegative parts.

Weight d plus the sum of all entries.

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \qquad \begin{pmatrix} 1 & 0 \\ \overline{0} & 0 \end{pmatrix}$$

are the two representations of weight 3 with $d = 2$.

The refinement uses this column statistic on overpartitions.

Reference: determinant coefficients

Power n	$[q^n]D_1$	$[q^n]D_2$	$[q^n]D_3$	Product limit
0	1	1	1	1
3	1	1	1	1
5	2	2	2	2
7	3	3	3	3
9	5	5	5	5
10	5	6	6	6
12	9	11	11	11
14	16	21	21	21
21	70	117	118	118

The product also counts colored partitions in which size $2r + 1$ is available in r colors, for $r \geq 1$.

Reference: proof of the strange-function theorem I

Let

$$G_B(y,z;q) = \sum_{\lambda \in \mathcal{S}_B} (-1)^{b(\lambda)} y^{r(\lambda)} z^{s(\lambda)} q^{|\lambda|}.$$

A simply paired partition splits into its paired part-sizes A and its unpaired part-sizes U. Hence

$$G_B(y,z;q) = A_B(y;q^2)\, D_B(z;q),$$

with

$$A_B(y;Q) = 1 - \sum_{r=1}^{B} y^r Q^r (Q;Q)_{r-1},$$

and

$$D_B(z;q) = \sum_{t=0}^{B} z^t q^t \begin{bmatrix} B \\ t \end{bmatrix}_{q^2}.$$

Now let $B = N$, let ζ be a primitive $2N$-th root of unity, and put $Q = \zeta^2$.
Since $(Q;Q)_N = 0$,

$$A_N(y;Q) = (1-y) \sum_{j=0}^{N-1} y^j (Q;Q)_j.$$

Also,

$$D_N(z;\zeta) = 1 - z^N.$$

Therefore

$$\boxed{G_N(y,z;\zeta) = (1-y)(1-z^N) \sum_{j=0}^{N-1} y^j (Q;Q)_j.}$$

Reference: proof of the strange-function theorem II

Set

$$y = e^{v}, \qquad z = e^{-v}.$$

Then

$$y^{r(\lambda)} z^{s(\lambda)} = e^{v(r(\lambda)-s(\lambda))} = e^{v\rho(\lambda)}.$$

Hence

$$G_N(e^{v}, e^{-v}; \zeta) = \sum_{\lambda \in \mathcal{S}_N} (-1)^{b(\lambda)} e^{v\rho(\lambda)} \zeta^{|\lambda|}.$$

On the other hand,

$$G_N(e^{v}, e^{-v}; \zeta) = (1 - e^{v})(1 - e^{-Nv}) \sum_{j=0}^{N-1} e^{jv} (Q; Q)_j.$$

Since

$$(1 - e^{v})(1 - e^{-Nv}) = -Nv^2 + O(v^3),$$

and

$$\sum_{j=0}^{N-1} (Q; Q)_j = F(Q),$$

we obtain

$$G_N(e^{v}, e^{-v}; \zeta) = -NF(Q)v^2 + O(v^3).$$

Taking the second derivative at $v = 0$,

$$\boxed{\sum_{\lambda \in \mathcal{S}_N} (-1)^{b(\lambda)} \rho(\lambda)^2 \zeta^{|\lambda|} = -2N\, F(\zeta^2).}$$

Appendix: Schur functions

For an alphabet $x = (x_1, x_2, \ldots)$, define $h_r(x)$ by

$$\sum_{r \geq 0} h_r(x) z^r = \prod_{i \geq 1} \frac{1}{1 - x_i z},$$

with $h_0(x) = 1$ and $h_r(x) = 0$ for $r < 0$.

For a partition λ, the Schur function is

$$\boxed{s_\lambda(x) = \det\big(h_{\lambda_i - i + j}(x)\big).}$$

In our theorem:

$$x = (q^2, q^4, q^6, \ldots), \qquad y = (q, q^3, q^5, \ldots).$$

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