Construction of Evidently Positive Series and An Alternative Construction for a Family of Partition Generating Functions due to Kanade and Russell

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Outline

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1. Introduction
2. An Alternative Construction for a Family of Partition Generating Functions due to Kanade and Russell
3. Construction of Evidently Positive Series for a Family of Partitions due to Kanade and Russell

Definition

A **partition** of an integer n is a finite non-decreasing sequence of positive integers $\lambda_1, \ldots, \lambda_k$ such that $\lambda_1 + \ldots + \lambda_k = n$.
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Example. Let $n = 4$. The partitions of 4 are:

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Notation

We use the standard notations concerning q-Pochhammer symbols:

$$(a;q)_n = (1-a)(1-aq)(1-aq^2)\ldots(1-aq^n)$$

$$(a;q)_\infty = \lim_{n\to\infty}(a;q)_n = (1-a)(1-aq)(1-aq^2)(1-aq^3)\ldots.$$

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The **generating function** $f(q)$ for the sequence $a_0, a_1, a_2, a_3, \dots$ is the power series $f(q) = \sum_{n \geq 0} a_n q^n$.

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Example. The generating function $P(q) = \sum_{n \geq 0} p(n) q^n$ counts all partitions of all non-negative integers n, where $p(n)$ denotes the number of all partitions of n and the exponent of q is the number being partitioned.

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Euler's Identities

$$\frac{1}{(x;q)_\infty} = \sum_{n\geq 0} \frac{x^n}{(q;q)_n} \tag{1}$$

$$(-x;q)_\infty = \sum_{n\geq 0} \frac{x^n q^{n(n-1)/2}}{(q;q)_n} \tag{2}$$

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- (c) *For a contiguous sub-partition $\lambda_i + \lambda_{i+1} + \lambda_{i+2}$, we have $|\lambda_i - \lambda_{i+2}| \geq 4$ if λ_{i+1} is even and appears more than once.*

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For $n, m \in \mathbb{N}$, let $kr_1(n, m)$ denote the number of partitions of n into m parts such that the partitions satisfy the conditions (a), (b), (c) and (d).

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For $n, m \in \mathbb{N}$, let $kr_1(n,m)$ denote the number of partitions of n into m parts such that the partitions satisfy the conditions (a), (b), (c) and (d). Then,

$$\sum_{m,n\geq 0} kr_1(n,m) q^n t^m = \sum_{i,j,k\geq 0} (-1)^k \frac{t^{i+2j+3k} q^{(i+2j+3k)(i+2j+3k-1)+i+6j+3k^2+6k}}{(q;q)_i (q^4;q^4)_j (q^6;q^6)_k}$$

Example. Let $n = 12$.

$$\checkmark 12 \quad \checkmark 1+11 \quad \checkmark 2+10 \quad \checkmark 3+9 \quad \checkmark 4+8 \quad \checkmark 1+3+8$$
$$\checkmark 5+7 \quad \checkmark 1+4+7 \quad \checkmark 6+6 \quad \checkmark 2+4+6$$

$$\times 2+2+8 \quad \times 1+1+10 \quad \times 4+4+4$$

Proof of Theorem 1. We start with a definition:

Definition

Let $\lambda = \lambda_1 + \ldots + \lambda_m$ be a partition counted by $kr_1(n,m)$. If there exist repeating even parts $(2k) + (2k)$ in λ, then we rewrite those parts as consecutive odd parts $(2k-1) + (2k+1)$. We call the partition formed after this transformation a **seed partition**.

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Example.

$$\lambda = 3+5+8+\underbrace{12+12}+16+18+\underbrace{24+24}$$

$$\Big\downarrow \text{we write the repeating even parts as the consecutive odd parts}$$

$$\overline{\lambda} = 3+5+8+\underbrace{11+13}+16+18+\underbrace{23+25}$$

Let $\lambda = \lambda_1 + \ldots + \lambda_m$ be a partition counted by $kr_1(n, m)$ and $\overline{\lambda} = \overline{\lambda_1} + \ldots + \overline{\lambda_m}$ be the corresponding seed partition to λ.

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We define $\beta = 1+3+5+\ldots+2m-1$ as the **base partition** corresponding to the seed partition $\overline{\lambda}$. Observe that β is the partition with m consecutive odd parts.

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Let $\mu = \mu_1 + \ldots + \mu_m$, where $\mu_i = \overline{\lambda_i} - \beta_i$ for all $i = 1, \ldots, m$.

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Let $\mu = \mu_1 + \ldots + \mu_m$, where $\mu_i = \overline{\lambda_i} - \beta_i$ for all $i = 1, \ldots, m$.

Whenever the seed partition $\overline{\lambda}$ has a streak of consecutive odd parts, the number of parts in that streak must be even and that streak will give rise to a group of same repeating even parts in μ.

Example.

$$\lambda = \underbrace{4+4}+8+\underbrace{12+12}+19+21+\underbrace{24+24}$$

$$\downarrow$$

$$\overline{\lambda} = 3+5+8+11+13+19+21+23+25$$

$$\beta = 1+3+5+7+9+11+13+15+17$$

$$\mu = 2+2+3+4+4+8+8+8+8$$

Example.

$$\lambda = \underbrace{4 + 4} + 8 + \underbrace{12 + 12} + 19 + 21 + \underbrace{24 + 24}$$

$$\downarrow$$

$$\overline{\lambda} = 3 + 5 + 8 + 11 + 13 + 19 + 21 + 23 + 25$$

$$\beta = 1 + 3 + 5 + 7 + 9 + 11 + 13 + 15 + 17$$

$$\mu = 2 + 2 + 3 + 4 + 4 + 8 + 8 + 8 + 8$$

The number of non-zero even parts that appear an even number of times in μ determines the number of partitions that can be generated from the seed partition $\overline{\lambda}$. We have three non-zero even parts that appear an even number of times in μ, namely 2, 4 and 8.

So, from the seed partition $\overline{\lambda}$, we can generate $2^3 = 8$ partitions counted by $kr_1(n, m)$, where $n = 128$, $m = 9$:

$$3 + 5 + 8 + 11 + 13 + 19 + 21 + 23 + 25 = \overline{\lambda}$$

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Therefore, we need a generating function for ordinary partitions, where we keep track of the number of non-zero even parts that appear an even number of times.

Proposition 1.

The ordinary partitions in which 0 *may appear as a part is generated by*

$$A(t;q;a) = \prod_{n=1}^{\infty} \frac{(1 + tq^{2n} + (a-1)t^2q^{4n})}{(1 - tq^{2n-1})(1 - t^2q^{4n})} . \frac{1}{(1-t)},$$

where the exponent of t *keeps track of the number of parts and the exponent of* a *keeps track of the number of non-zero even parts that appear an even number of times.*

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partitions satisfying the conditions, together with the seed partition itself. Therefore, we plug $a = 2$ into $A(t; q; a)$ and we get the following infinite product:

$$A(t; q; 2) = \prod_{n=1}^{\infty} \frac{(1 + tq^{2n} + t^2 q^{4n})}{(1 - tq^{2n-1})(1 - t^2 q^{4n})} \cdot \frac{1}{(1-t)}.$$

$$
\begin{aligned}
A(t;q;2) &= \prod_{n=1}^{\infty} \frac{(1+tq^{2n}+t^2q^{4n})}{(1-tq^{2n-1})(1-t^2q^{4n})} \cdot \frac{1}{(1-t)} \\
&= \prod_{n=1}^{\infty} \frac{(1-t^3q^{6n})}{(1-tq^{2n-1})(1-tq^{2n})(1-t^2q^{4n})(1-t)} \\
&= \frac{(t^3q^6;q^6)_\infty}{(t;q)_\infty (t^2q^4;q^4)_\infty} \\
&= \left(\sum_{i\geq 0} \frac{t^i}{(q;q)_i} \right) \left(\sum_{j\geq 0} \frac{t^{2j}q^{4j}}{(q^4;q^4)_j} \right) \left(\sum_{k\geq 0} \frac{(-1)^k t^{3k} q^{6k+3k(k-1)}}{(q^6;q^6)_k} \right) \\
&= \sum_{i,j,k\geq 0} \frac{(-1)^k t^{i+2j+3k} q^{4j+3k^2+3k}}{(q;q)_i (q^4;q^4)_j (q^6;q^6)_k}. \qquad (3)
\end{aligned}
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$$\sum_{m,n\geq 0} kr_1(n,m)q^n t^m$$

$$= \sum_{i,j,k\geq 0} \frac{(-1)^k t^{i+2j+3k} q^{4j+3k^2+3k} q^{(i+2j+3k)^2}}{(q,q)_i (q^4,q^4)_j (q^6,q^6)_k}$$

$$= \sum_{i,j,k\geq 0} (-1)^k \frac{t^{i+2j+3k} q^{(i+2j+3k)(i+2j+3k-1)+i+6j+3k^2+6k}}{(q;q)_i (q^4;q^4)_j (q^6;q^6)_k}.$$

Theorem 2 (Kurşungöz, Ömrüuzun Seyrek 2021-Kanade, Russell 2019).

Consider the partitions satisfying the following conditions:

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(b) *Odd parts do not repeat.*

(c) *For a contiguous sub-partition* $\lambda_i + \lambda_{i+1} + \lambda_{i+2}$, *we have* $|\lambda_i - \lambda_{i+2}| \geq 4$ *if* λ_{i+1} *is even and appears more than once.*

(d') 1 *is not allowed to appear as a part.*

For $n, m \in \mathbb{N}$, *let* $kr_2(n,m)$ *denote the number of partitions of* n *into* m *parts such that the partitions satisfy the conditions* (a), (b), (c) *and* (d'). *Then,*

$$\sum_{m,n\geq 0} kr_2(n,m) q^n t^m$$

$$= \sum_{i,j,k\geq 0} (-1)^k \frac{t^{i+2j+3k} q^{(i+2j+3k)(i+2j+3k-1)+2i+2j+3k^2+6k}}{(q;q)_i (q^4;q^4)_j (q^6;q^6)_k}.$$

Theorem 3 (Kurşungöz, Ömrüuzun Seyrek 2021-Kanade, Russell 2019).

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(d'') $1, 2$ *and* 3 *are not allowed to appear as parts.*

For $n, m \in \mathbb{N}$, let $kr_3(n, m)$ denote the number of partitions of n into m parts such that the partitions satisfy the conditions (a), (b), (c) and (d''). Then,

$$\sum_{m,n \geq 0} kr_3(n, m) q^n t^m$$

$$= \sum_{i,j,k \geq 0} (-1)^k \frac{t^{i+2j+3k} q^{(i+2j+3k)(i+2j+3k-1)+4i+6j+3k^2+12k}}{(q;q)_i (q^4;q^4)_j (q^6;q^6)_k}.$$

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Theorem 4 (Kurşungöz, Ömrüuzun Seyrek 2021).

For $n, m \in \mathbb{N}$, let $h(n,m)$ denote the number of partitions of n into m parts such that each part appears at most twice. Then,

$$\sum_{m,n \geq 0} h(n,m) q^n t^m = \sum_{\substack{n_1, n_2 \geq 0 \\ n_1 = n_{11} + n_{12} \\ \beta}} \frac{q^{|\beta|} t^{2n_2 + n_1}}{(q;q)_{n_{12}} (q^3;q^3)_{n_2}}$$

where β is the base partition with n_2 pairs, n_{11} immobile singletons, n_{12} moveable singletons.

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$$\begin{array}{c} \lambda = 1, [\mathbf{4}, \mathbf{4}], [5, 6], 6, [9, 10], [11, 12], 12, 14 \\ \Big\downarrow \text{one backward move} \\ 1, [\mathbf{2}, \mathbf{3}], [5, 6], 6, [9, 10], [11, 12], 12, 14 \end{array}$$

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$$\Big\downarrow \text{regrouping the pairs}$$

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$$[1,2], [3,4], 4, [6,6], 7, [\mathbf{11},\mathbf{12}], 12, 14$$

We observe that the pair $[6,6]$ can not be moved further back and 4 is an immobile singleton.

$$\Big\downarrow \text{one backward move}$$

$$[1,2],[3,4],4,[6,6],7,[\mathbf{10},\mathbf{10}],12,14$$

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$$[1,2],[3,4],4,[6,6],[7,8],\mathbf{9},12,14$$

$$\Big\downarrow \text{one backward move}$$

$$[1,2],[3,4],4,[6,6],[7,8],8,\mathbf{12},14$$

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$$\Big\downarrow \text{two backward moves}$$

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$$[1,2],[3,4],4,[6,6],[7,8],8,10,\mathbf{14}$$

$$\Big\downarrow \text{two backward moves}$$

$$\beta = [1,2],[3,4],4,[6,6],[7,8],8,10,12$$

We have $\mu = 3 + 3 + 6 + 6$ and $\theta = 0 + 1 + 2 + 2$. We observe that

$$|\lambda| = 94 = |\beta| + |\mu| + |\theta| = 71 + 18 + 5.$$

Lemma 5.

For $m_1, m_2, m_3 \in \mathbb{N}$ and $m \in \mathbb{Z}$, let $P(m_1, m_2, m_3, m+1; q)$ be the generating function of base partitions β's defined in the proof of Theorem 4 with m_1 pairs of two repeating parts, m_2 pairs of two consecutive parts and m_3 blocks, where a block is a partition into five parts which have the form $[k-1, k], k, [k+2, k+2]$. Then,

$$\begin{aligned} P(m_1, m_2, m_3, m+1; q) =& P_0(m_1, m_2, m_3, m+1; q) \\ &+ P_1(m_1, m_2, m_3, m+1; q) \end{aligned}$$

where $P_0(m_1, m_2, m_3, m+1; q)$ is the generating function of the base partitions in which the largest pair is $[m, m]$ and $P_1(m_1, m_2, m_3, m+1; q)$ is the generating function of the base partitions in which the largest pair is $[m, m+1]$.

$P_0(m_1, m_2, m_3, m+1; q)$ and $P_1(m_1, m_2, m_3, m+1; q)$ satisfy the following functional equations:

$$\begin{aligned}P_0(m_1, m_2, m_3, m+1; q) =& q^{2m}\Big[P_0(m_1-1, m_2, m_3, m; q)\\ &+ P_1(m_1-1, m_2, m_3, m-1; q)\\ &+ P_0(m_1-1, m_2, m_3, m-1; q)\Big]\\ &+ q^{5m-7}\Big[P_1(m_1, m_2, m_3-1, m-3; q)\\ &+ P_0(m_1, m_2, m_3-1, m-3; q)\\ &+ P_1(m_1, m_2, m_3-1, m-4; q)\Big]\end{aligned}$$

$$\begin{aligned}P_1(m_1, m_2, m_3, m+1; q) =& q^{2m+1}\Big[P_1(m_1, m_2-1, m_3, m; q)\\ &+ P_0(m_1, m_2-1, m_3, m; q)\\ &+ P_1(m_1, m_2-1, m_3, m-1; q)\Big]\end{aligned}$$

$$
\begin{aligned}
P_{0/1}(m_1, m_2, m_3, m; q) &= 0 \qquad \text{if } m < 0 \\
P_{0/1}(m_1, m_2, m_3, 0; q) &= 1 \\
P_{0/1}(0, 0, 0, m; q) &= 0 \qquad \text{if } m \neq 1 \\
P_0(0, 0, 0, 1; q) &= 1 \\
P_1(0, 0, 0, 1; q) &= 0
\end{aligned}
$$

Moreover, $P(m_1, m_2, m_3, m+1; q)$'s are the polynomials of q with evidently positive coefficients.

Theorem 6 (Kurşungöz, Ömrüuzun Seyrek 2021).

For $n, m \in \mathbb{N}$, let $h(n,m)$ denote the number of partitions of n into m parts such that each part appears at most twice. Then,

$$\begin{aligned} H(t;q) &= \sum_{m,n \geq 0} h(n,m) q^n t^m \\ &= \sum_{\substack{m_1, m_2, m_3, \\ m, n_{12} \geq 0}} \frac{P(m_1, m_2, m_3, m+1; q) q^{m n_{12} + n_{12}^2} t^{2m_1 + 2m_2 + 5m_3 + n_{12}}}{(q;q)_{n_{12}} (q^3; q^3)_{m_1 + m_2 + 2m_3}} \end{aligned}$$

where $P(m_1, m_2, m_3, m+1; q)$'s are the polynomials of q with evidently positive coefficients constructed in Lemma 5.

Theorem 7 (Kurşungöz, Ömrüuzun Seyrek 2021).

Consider the partitions satisfying the following conditions:

(a) *No consecutive parts allowed.*

(b) *Odd parts do not repeat.*

(c) *For a contiguous sub-partition* $\lambda_i + \lambda_{i+1} + \lambda_{i+2}$, *we have* $|\lambda_i - \lambda_{i+2}| \geq 4$ *if* λ_{i+1} *is even and appears more than once.*

(d) $2 + 2$ *is not allowed as a sub-partition.*

For $n, m \in \mathbb{N}$, *let* $kr_1(n, m)$ *denote the number of partitions of* n *into* m *parts such that the partitions satisfy the conditions* (a), (b), (c) *and* (d).

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(d) *$2 + 2$ is not allowed as a sub-partition.*

For $n, m \in \mathbb{N}$, let $kr_1(n,m)$ denote the number of partitions of n into m parts such that the partitions satisfy the conditions (a), (b), (c) and (d). Then,

$$\sum_{m,n\geq 0} kr_1(n,m) q^n t^m$$

$$= \sum_{\substack{m_1,m_2,m_3,\\ m,n_{12},i,j,k\geq 0}} \frac{P(m_1,m_2,m_3,m+1;q^2)}{(q^2;q^2)_{n_{12}}(q^6;q^6)_{m_1+m_2+2m_3}(q^2;q^2)_i(q^4;q^4)_j} \tag{4}$$

$$\times\, q^{2mn_{12}+2n_{12}^2+i+4j+(2m_1+2m_2+5m_3+n_{12}+i+2j+k)^2} t^{2m_1+2m_2+5m_3+n_{12}+i+2j+k}$$

where $P(m_1,m_2,m_3,m+1;q)$'s are the polynomials of q with evidently positive coefficients constructed in Lemma 5. Moreover, the generating function (4) is an evidently positive series.

Theorem 8 (Kurşungöz, Ömrüuzun Seyrek 2021).

Consider the partitions satisfying the following conditions:

- (a) *No consecutive parts allowed.*
- (b) *Odd parts do not repeat.*
- (c) *For a contiguous sub-partition $\lambda_i + \lambda_{i+1} + \lambda_{i+2}$, we have $|\lambda_i - \lambda_{i+2}| \geq 4$ if λ_{i+1} is even and appears more than once.*
- (d') 1 *is not allowed to appear as a part.*

For $n, m \in \mathbb{N}$, let $kr_2(n,m)$ denote the number of partitions of n into m parts such that the partitions satisfy the conditions (a), (b), (c) and (d'). Then,

$$\begin{aligned} &\sum_{m,n\geq 0} kr_2(n,m) q^n t^m \\ &= \sum_{\substack{m_1,m_2,m_3,\\ m,n_{12},i,j\geq 0}} \frac{P(m_1,m_2,m_3,m+1;q^2)}{(q^2;q^2)_{n_{12}}(q^6;q^6)_{m_1+m_2+2m_3}(q^2;q^2)_i(q^4;q^4)_j} \\ &\times q^{2mn_{12}+2n_{12}^2+i+(2m_1+2m_2+5m_3+n_{12}+i+2j)^2} t^{2m_1+2m_2+5m_3+n_{12}+i+2j} \end{aligned} \tag{5}$$

where $P(m_1, m_2, m_3, m+1; q)$'s are the polynomials of q with evidently positive coefficients constructed in Lemma 5. Moreover, the generating function (5) is an evidently positive series.

Theorem 9 (Kurşungöz, Ömrüuzun Seyrek 2021).

Consider the partitions satisfying the following conditions:

- (a) *No consecutive parts allowed.*
- (b) *Odd parts do not repeat.*
- (c) *For a contiguous sub-partition $\lambda_i + \lambda_{i+1} + \lambda_{i+2}$, we have $|\lambda_i - \lambda_{i+2}| \geq 4$ if λ_{i+1} is even and appears more than once.*
- (d'') $1, 2$ *and* 3 *are not allowed to appear as parts.*

For $n, m \in \mathbb{N}$, let $kr_3(n, m)$ denote the number of partitions of n into m parts such that the partitions satisfy the conditions (a), (b), (c) and (d''). Then,

$$\begin{aligned} &\sum_{m,n \geq 0} kr_3(n,m) q^n t^m \\ &= \sum_{\substack{m_1, m_2, m_3, \\ m, n_{12}, i, j \geq 0}} \frac{P(m_1, m_2, m_3, m+1; q^2)}{(q^2;q^2)_{n_{12}} (q^6;q^6)_{m_1+m_2+2m_3} (q^2;q^2)_i (q^4;q^4)_j} \\ &\times q^{2mn_{12} + 2n_{12}^2 + 3i + 4j + 4m_1 + 4m_2 + 10m_3 + 2n_{12} + (2m_1 + 2m_2 + 5m_3 + n_{12} + i + 2j)^2} \\ &\times t^{2m_1 + 2m_2 + 5m_3 + n_{12} + i + 2j} \end{aligned} \tag{6}$$

where $P(m_1, m_2, m_3, m+1; q)$'s are the polynomials of q with evidently positive coefficients constructed in Lemma 5. Moreover, the generating function (6) is an evidently positive series.

THANK YOU.