# Combinatorial constructions of generating functions of cylindric partitions with small profiles into unrestricted or distinct parts

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Cylindric partitions were introduced by Gessel and Krattenthaler, 1997.

## Definition 1.

Let $k$ and $\ell$ be positive integers. Let $c = (c_1, c_2, \ldots, c_k)$ be a composition, where $c_1 + c_2 + \cdots + c_k = \ell$. A *cylindric partition with profile* $c$ is a vector partition $\lambda = (\lambda^{(1)}, \lambda^{(2)}, \ldots, \lambda^{(k)})$, where each $\lambda^{(i)} = \lambda_1^{(i)} + \lambda_2^{(i)} + \cdots + \lambda_{s_i}^{(i)}$ is a partition, such that for all $i$ and $j$,

$$\lambda_j^{(i)} \geq \lambda_{j+c_{i+1}}^{(i+1)} \quad \text{and} \quad \lambda_j^{(k)} \geq \lambda_{j+c_1}^{(1)}.$$

## Example 2.

The sequence $\lambda = ((6,5,4,4),(8,8,5,3),(7,6,4,2))$ is a cylindric partition with profile $(1,2,0)$:

$$\begin{array}{ccccccc} & & & 6 & 5 & 4 & 4 \\ & 8 & 8 & 5 & 3 & & \\ & 7 & 6 & 4 & 2 & & \\ \mathbf{6} & \mathbf{5} & \mathbf{4} & \mathbf{4} & & & \end{array}$$

**Definition 3.**

The size $|\lambda|$ of a cylindric partition $\lambda = (\lambda^{(1)}, \lambda^{(2)}, \ldots, \lambda^{(k)})$ is defined to be the sum of all the parts in the partitions $\lambda^{(1)}, \lambda^{(2)}, \ldots, \lambda^{(k)}$.

The largest part of a cylindric partition $\lambda$ is defined to be the maximum part among all the partitions in $\lambda$, and it is denoted by $\max(\lambda)$.

The generating function for cylindric partitions, where $\mathcal{P}_c$ denotes the set of all cylindric partitions with profile $c$ is

$$F_c(z,q) := \sum_{\lambda \in \mathcal{P}_c} z^{\max(\lambda)} q^{|\lambda|}$$

**Theorem 4 (Borodin, 2007).**

*Let $k$ and $\ell$ be positive integers, and let $c = (c_1, c_2, \ldots, c_k)$ be a composition of $\ell$. Define $t := k + \ell$ and $s(i,j) := c_i + c_{i+1} + \cdots + c_j$. Then,*

$$F_c(1,q) = \frac{1}{(q^t;q^t)_\infty} \prod_{i=1}^{k} \prod_{j=i}^{k} \prod_{m=1}^{c_i} \frac{1}{(q^{m+j-i+s(i+1,j)};q^t)_\infty}$$
$$\times \prod_{i=2}^{k} \prod_{j=2}^{i} \prod_{m=1}^{c_i} \frac{1}{(q^{t-m+j-i-s(j,i-1)};q^t)_\infty}.$$

**Theorem 5 (Borodin, 2007-Kurşungöz, Ö.S., 2022).**
*Let* $c = (1,1)$*. Then, the generating function of cylindric partitions with profile* $c$ *is given by*

$$F_c(1,q) = \frac{(-q;q^2)_\infty}{(q;q)_\infty}.$$

**Sketch of the proof:** We show that each cylindric partition $\lambda$ with profile $c = (1, 1)$ corresponds to a unique pair of partitions $(\mu, \beta)$, where $\mu$ is an ordinary partition and $\beta$ is a partition with distinct odd parts.

Conversely, we show that each pair of partitions $(\mu, \beta)$ will correspond to a unique cylindric partition with profile $c = (1, 1)$, where $\mu$ is an ordinary partition and $\beta$ is a partition into distinct odd parts.

In this way, we get the desired generating function for cylindric partitions with profile $c = (1, 1)$.

$$F_c(1, q) = \sum_{\lambda} q^{|\lambda|} = \sum_{(\mu,\beta)} q^{|\mu|+|\beta|} = \left(\sum_{\mu} q^{|\mu|}\right) \left(\sum_{\beta} q^{|\beta|}\right)$$
$$= \frac{1}{(q; q)_\infty} (-q; q^2)_\infty,$$

where $\lambda$, $\mu$, and $\beta$ are as described above.

$\square$

How do we decompose a given cylindric partition $\lambda$ with profile $c = (1,1)$ into to a unique pair of partitions $(\mu, \beta)$, where $\mu$ is an ordinary partition and $\beta$ is a partition with distinct odd parts?

$\lambda$:a cylindric partition with profile $c = (1,1)$

$\mu$: an ordinary partition

$\beta$: a partition with distinct odd parts

We will make this decomposition performing a series of combinatorial moves.

Let $\lambda$ be the following cylindric partition with profile $c = (1,1)$:

$$\begin{array}{ccccc} & 7 & 4 & 4 & 3 \\ 6 & 5 & 4 & & \end{array}$$

We read the parts of $\lambda$ as pairs: $[6,7], [5,4], [4,4]$ and $[0,3]$.

$$\begin{array}{ccccc} & 7 & 4 & 4 & 3 \\ 6 & 5 & 4 & 0 & \end{array}$$

We now start to perform the moves. We first change the places of $0$ and $3$ in the rightmost pair and we get the following intermediate partition:

$$\begin{array}{ccccc} & 7 & 4 & 4 & 0 \\ 6 & 5 & 4 & 3 & \end{array}$$

$\Big\downarrow$ subtract $1$ from circled 3 and the parts take place above and on the left of it

$$\begin{array}{ccccc} & 6 & 3 & 3 & 0 \\ 5 & 4 & 3 & 2 & \end{array}$$

We changed the total weight by $7$, so we have $\beta_1 = 7$. We do not touch to the pairs $[3,3]$ and $[4,3]$ since $3 \geq 3$ and $4 \geq 3$. We now correct the places of $6$ and $5$, then we perform the last possible move:

$$\begin{array}{ccccc} & 5 & 3 & 3 & 0 \\ 6 & 4 & 3 & 2 & \end{array}$$

$$\Big\downarrow \text{subtract } 1 \text{ from circled } 6$$

$$\begin{array}{ccccc} & 5 & 3 & 3 & 0 \\ 5 & 4 & 3 & 2 & \end{array}$$

We changed the total weight by $1$, so we have $\beta_2 = 1$. Therefore, we decomposed $\lambda$ into the pair of partitions $(\mu, \beta)$, where $\beta = 7 + 1$ and $\mu = 5 + 5 + 4 + 3 + 3 + 3 + 2$.

**Theorem 6 (Borodin, 2007-Kurşungöz, Ö.S., 2022).**
*Let $c = (2, 0)$. Then, the generating function of cylindric partitions with profile $c$ is given by*

$$F_c(1, q) = \frac{(-q^2; q^2)_\infty}{(q; q)_\infty}.$$

**Lemma 7 (Kurşungöz, Ö.S., 2022).**

*Let $d_{(1,1)}(m,n)$ denote the number of cylindric partitions of $n$ into distinct parts with profile $c=(1,1)$ and the largest part equal to $m$. Then,*

$$D_{(1,1)}(t,q)=\sum_{n,m\geq 0} d_{(1,1)}(m,n)t^m q^n=\sum_{n\geq 0}\frac{q^{\binom{2n}{2}}t^{2n-1}2^n}{(tq;q)_{2n}}.$$

If all parts in a cylindric partition with profile $c = (1, 1)$ are distinct, then the inequalities between parts are strict. Given such a partition, if we label the parts in the top row as $a_1, a_2, \ldots$ and the parts in the bottom row as $b_1, b_2, \ldots$

$$\begin{array}{ccccccccccccc}
 & & & a_1 & & a_2 & & \cdots & & a_{r-1} & a_r & \cdots & a_n \\
 & & b_1 & b_2 & \cdots & & b_{r-1} & & b_r & \cdots & b_n & & \\
a_1 & a_2 & \cdots & a_{r-1} & & a_r & & \cdots & a_n & & & &
\end{array} \tag{1}$$

We have the inequalities

$$a_r > a_{r+1}, \qquad b_r > b_{r+1}, \qquad a_r > b_{r+1}, \qquad \text{and} \qquad b_r > a_{r+1}$$

for $r = 1, 2, \ldots, n-1$. In particular,

$$\min\{a_r, b_r\} > \max\{a_{r+1}, b_{r+1}\} \tag{2}$$

for $r = 1, 2, \ldots, n-1$.

We lose no generality by assuming that the top row and the bottom row have equal number of parts. We achieve this by allowing one of $a_n$ or $b_n$ to be zero. The inequality (2) ensures that we can switch the places of $a_r$ and $b_r$ without violating the condition for cylindric partition with profile $c = (1,1)$ for $r = 1, 2, \ldots, n$. There are $2^n$ ways to do this.

Therefore, given a cylindric partition into $2n$ distinct parts with profile $c = (1, 1)$, we can switch places of $a_r$ and $b_r$ to make $b_r > a_r$ for $r = 1, 2, \ldots, n$ so that

$$b_1 > a_1 > b_2 > a_2 > \cdots > b_n > a_n, \tag{3}$$

where $a_n$ is possibly zero. In other words, we obtain a partition into $2n$ distinct parts in which the smallest part is allowed to be zero.

Conversely, if we start with a partition into $2n$ distinct parts in which the smallest part can be zero, we can label the parts as in (3), then place them as in (1), and allow switching places of $a_r$ and $b_r$ for $r = 1, 2, \ldots, n$; then we will have generated a cylindric partition into $2n$ distinct parts with profile $c = (1, 1)$, where one of the parts is allowed to be zero.

It is clear that any such cylindric partition corresponds to a unique partition into an even number of distinct parts, and any partition into $2n$ distinct parts gives rise to $2^n$ cylindric partitions.

**Proof of Lemma 7:** A partition into $2n$ distinct parts is generated by

$$\frac{q^{\binom{2n}{2}}}{(q;q)_{2n}},$$

where the smallest part is allowed to be zero. When we want to keep track of the largest part, we start by the minimal partition into $2n$ distinct parts

$$(2n-1),(2n-2),\ldots,1,0,$$

hence the $t^{2n-1}$ in the numerator in the rightmost sum in the Lemma. Then, for $j=1,2,\ldots,2n$, the factor $(1-tq^j)$ in the denominator contributes to the $j$ largest parts in the partition into distinct parts.

**Lemma 8 (Kurşungöz, Ö.S., 2022).**

*Let $d_{(2,0)}(m,n)$ denote the number of cylindric partitions of $n$ into distinct parts with profile $c=(2,0)$ and the largest part equal to $m$. Then,*

$$D_{(2,0)}(t,q) = \sum_{n,m\geq 0} d_{(2,0)}(m,n) t^m q^n = \sum_{n\geq 0} \frac{q^{\binom{2n+1}{2}} t^{2n} 2^n}{(tq;q)_{2n+1}}.$$

## Theorem 9 (Kurşungöz, Ö.S., 2022).

*Let* $d_{(1,1)}(m,n)$ *and* $d_{(2,0)}(m,n)$ *be the number of cylindric partitions of* $n$ *into distinct parts where the largest part is* $m$ *with profiles* $c=(1,1)$ *and* $c=(2,0)$*, respectively. Let*

$$D_{(1,1)}(t,q) = \sum_{m,n\geq 0} d_{(1,1)}(m,n)t^m q^n,$$
$$D_{(2,0)}(t,q) = \sum_{m,n\geq 0} d_{(2,0)}(m,n)t^m q^n$$

*be the respective generating functions. Then,*

$$D_{(1,1)}(1,q) = \frac{(-\sqrt{2};q)_\infty + (\sqrt{2};q)_\infty}{2},$$
$$D_{(2,0)}(1,q) = \frac{(-\sqrt{2};q)_\infty - (\sqrt{2};q)_\infty}{2\sqrt{2}},$$
$$D_{(1,1)}(1,q) + \sqrt{2}\, D_{(2,0)}(1,q) = (-\sqrt{2};q)_\infty.$$

**Proof:** We dissect one of Euler's $q$-series identities

$$\sum_{n\geq 0} \frac{q^{\binom{n}{2}} a^n}{(q;q)_n} = (-a;q)_\infty$$

to separate odd and even powers of $a$.

$$\begin{aligned} \sum_{n\geq 0} \frac{q^{\binom{2n}{2}} (a^2)^n}{(q;q)_{2n}} &= \frac{(-a;q)_\infty + (a;q)_\infty}{2} \\ a \sum_{n\geq 0} \frac{q^{\binom{2n+1}{2}} (a^2)^n}{(q;q)_{2n+1}} &= \frac{(-a;q)_\infty - (a;q)_\infty}{2}. \end{aligned}$$

We plug in $t=1$ in the previous two Lemmas and $a=\sqrt{2}$ in the above formulas.

$\square$

*Thank you for your attention.*