Copartitions

Parity and Positivity

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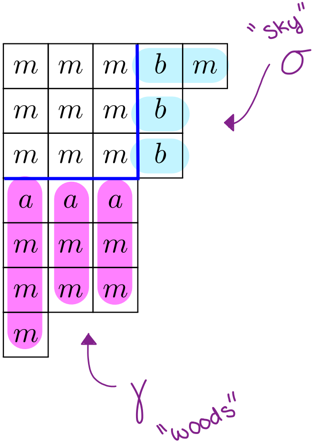
Definition



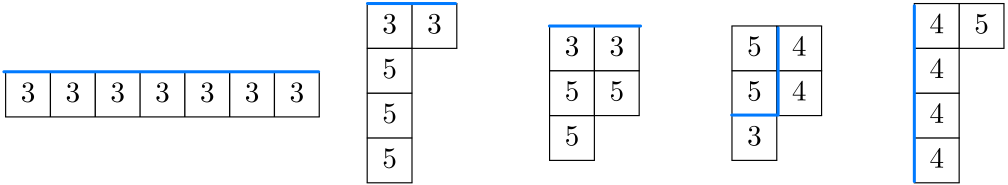
Definition

An (a, b, m)**-copartition** is a vector partition $\lambda = (\gamma, \rho, \sigma)$ where

- γ has parts of size a modulo m
- ρ is a $\#(\sigma) \times \#(\gamma)$ rectangle of m's.
- σ has parts of size b modulo m
- $|\lambda| = |\gamma| + |\rho| + |\sigma|$

Example

All the $(3, 4, 5)$-copartitions of size 21:



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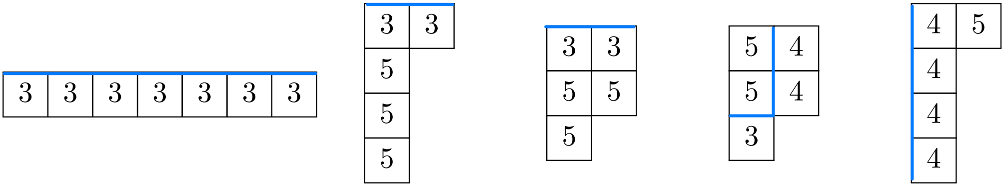

Let $\mathrm{cp}_{a,b,m}(n)$ be the function that counts the number of (a, b, m)-copartitions of size n. Then,

$$\mathrm{cp}_{3,4,5}(21) = 5.$$

Generating function

$\mathrm{cp}_{a,b,m}(n) :=$ the number of (a,b,m)-copartitions of size n.

Theorem (B., Eichhorn)

$$\begin{aligned}\mathbf{cp}_{a,b,m}(q) &:= \sum_{n=0}^{\infty} \mathrm{cp}_{a,b,m}(n) q^n \\ &= \sum_{w=0}^{\infty}\sum_{s=0}^{\infty} \frac{q^{mws+aw+bs}}{(q^m;q^m)_w (q^m;q^m)_s} \\ &= \frac{(q^{a+b};q^m)_\infty}{(q^a;q^m)_\infty (q^b;q^m)_\infty}\end{aligned}$$

Conjugation

m	m	m	b	m	m	m
m	m	m	b	m	m	m
m	m	m	b	m	m	
m	m	m	b			
a	a	a				
m	m	m				
m						
m						

$\longrightarrow$

m	m	m	m	a	m	m	m
m	m	m	m	a	m		
m	m	m	m	a	m		
b	b	b	b				
m	m	m					
m	m	m					
m	m						

Some facts about $\mathrm{cp}_{a,b,m}(n)$

- $\mathrm{cp}_{1,1,2}(n) = \mathcal{EO}^*(2n)$, where $\mathcal{EO}^*(n)$ counts the number of partitions of n such that all even parts are smaller than all odd parts and only the largest even part appears an odd number of times.

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Some facts about $\mathrm{cp}_{a,b,m}(n)$

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- $\mathrm{cp}_{a,b,m}(n) = \mathrm{cp}_{b,a,m}(n)$
- $\mathrm{cp}_{ka,kb,km}(kn) = \mathrm{cp}_{a,b,m}(n)$, so we will mostly assume that $\gcd(a, b, m) = 1$.

Plan

1. Background on Copartitions
2. Parity
3. Adding weight

A family of congruences

Theorem (B.-Eichhorn)

For even m,

$$\mathrm{cp}_{a,a,m}(2n+1) \equiv 0 \pmod{2}.$$

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proof

All (a,a,m)-copartitions of odd size can be paired by conjugation.

$$\begin{array}{|c|c|c|c|c|}
\hline
m & m & a & m & m \\ \hline
m & m & a & m & \\ \cline{1-4}
m & m & a & & \\ \cline{1-3}
a & a & & & \\ \cline{1-2}
m & m & & & \\ \cline{1-2}
m & m & & & \\ \cline{1-2}
m & & & & \\ \cline{1-1}
\end{array}
\quad \longleftrightarrow \quad
\begin{array}{|c|c|c|c|c|c|c|}
\hline
m & m & m & a & m & m & m \\ \hline
m & m & m & a & m & m & \\ \cline{1-6}
a & a & a & & & & \\ \cline{1-3}
m & m & & & & & \\ \cline{1-2}
m & & & & & & \\ \cline{1-1}
\end{array}$$

What is the deeper story?

Question

For different values of a, b, and m, how often is the (a, b, m)-copartition function even? How often is it odd?

Parity: ordinary partitions

Theorem (Kolberg)

The partition function $p(n)$ takes both even and odd values infinitely often.

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Open Problem

Show that $p(n)$ is even (or odd) with positive density, that is, show that

$$\lim_{n\to\infty} \frac{\#\{1 \le k \le n \mid p(k) \text{ is even}\}}{n} \ge c \qquad \text{for some } c > 0.$$

Copartitions: A first result

Recall: For even m,

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A closer look at self-conjugate copartitions

Theorem (B.-Eichhorn)

$\mathrm{scp}_{a,m}(n) :=$ # self-conjugate (a, a, m)-copartitions of size n.
Then, for a odd and m even,

$$\sum_{n=0}^{\infty} \mathrm{scp}_{a,m}(n) q^n = (-q^{m+2a}; q^{2m})_\infty.$$

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m	m	m	a	m	m	m
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a	a	a				
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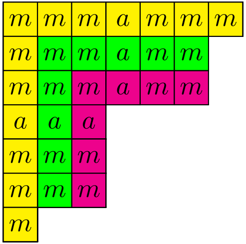

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Corollary

$$\sum_{n=0}^{\infty} \mathrm{cp}_{a,a,m}(n) q^n \equiv (-q^{m+2a}; q^{2m})_\infty \pmod 2$$

Applications to congruences modulo 2

Corollary

$$\sum_{n=0}^{\infty} \mathrm{cp}_{a,a,m}(n) q^n \equiv (-q^{m+2a}; q^{2m})_\infty \pmod{2}$$

Corollary

For $m \equiv 2 \pmod{4}$ and odd a,

$$\mathrm{cp}_{a,a,m}(4n+1) \equiv 0 \pmod{2}$$
$$\mathrm{cp}_{a,a,m}(4n+2) \equiv 0 \pmod{2}$$
$$\mathrm{cp}_{a,a,m}(4n+3) \equiv 0 \pmod{2}$$

Back to parity

Corollary

For $m \equiv 2 \pmod 4$ and odd a,

$$\begin{aligned} \mathrm{cp}_{a,a,m}(4n+1) &\equiv 0 \pmod 2 \\ \mathrm{cp}_{a,a,m}(4n+2) &\equiv 0 \pmod 2 \\ \mathrm{cp}_{a,a,m}(4n+3) &\equiv 0 \pmod 2 \end{aligned}$$

Theorem (B.-Eichhorn)

When $m \equiv 2 \pmod 4$ and a is odd,

$$\liminf_{n\to\infty} \frac{\#\{1 \le k \le n \mid \mathrm{cp}_{a,a,m}(k) \text{ is even}\}}{n} \ge \frac{3}{4}.$$

Some parity conjectures

Conjecture

When $m \equiv 0 \pmod 4$ and $a \equiv 1 \pmod 2$,
$\mathrm{cp}_{a,a,m}(n)$ is even with arithmetic density $\frac{1}{2} + \mathbf{\frac{1}{4}} = \frac{3}{4}$.

When $m \equiv 2 \pmod 4$, $a \equiv 1 \pmod 2$, and $m \neq 2a$,
$\mathrm{cp}_{a,a,m}(n)$ is even with arithmetic density $\frac{3}{4} + \mathbf{\frac{1}{8}} = \frac{7}{8}$.

When $m \equiv 1 \pmod 2$ and $\gcd(a, m) = 1$, $\mathrm{cp}_{a,a,m}(n)$ is even (odd) with arithmetic density $\mathbf{\frac{1}{2}}$.

Data

n	$d_{3,3,4}(n)$	$d_{1,1,6}(n)$	$d_{3,3,7}(n)$
1000	0.765	0.871	0.705
3000	0.752	0.875	0.575
5000	0.753	0.874	0.543
7000	0.749	0.875	0.534
9000	0.748	0.873	0.524
11000	0.749	0.874	0.519
13000	0.750	0.875	0.518
15000	0.749	0.875	0.516

$$d_{a,b,m}(n) = \frac{\#\{1 \leq k \leq n : \mathrm{cp}_{a,b,m}(k) \text{ is even}\}}{n}.$$

A very special case

Theorem (B.-Eichhorn, Barman-Ray)

$\mathrm{cp}_{a,a,2a}(n)$ is even with arithmetic density one.

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Remark

Barman and Ray used the theory of modular forms and proved that $\mathrm{cp}_{a,a,2a}(n)$ is almost always divisible by 2^k for any $k \in \mathbb{N}$.

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Barman and Ray used the theory of modular forms and proved that $\mathrm{cp}_{a,a,2a}(n)$ is almost always divisible by 2^k for any $k \in \mathbb{N}$.

Remark

We have an elementary proof using Euler's pentagonal number theorem.

An open problem

Question

For which $a \neq b$, is $\mathrm{cp}_{a,b,m}(n)$ even with arithmetic density $\frac{1}{2}$?

Question

When there is a parity bias, how often is $\mathrm{cp}_{a,b,m}(n)$ even?

What we know, part 1

Theorem (B.-Eichhorn)

For all a, m, $\mathrm{cp}_{a,m-a,m}(n)$ is even (odd) infinitely often.

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If $a = m/2$,

$$\mathbf{cp}_{m/2,m/2,m}(q) = \frac{(q^m;q^m)_\infty}{(q^{m/2};q^m)_\infty^2} \equiv \frac{(q^m;q^m)_\infty}{(q^m;q^{2m})_\infty} \equiv (q^{2m};q^{2m})_\infty \pmod 2.$$

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Apply Euler's pentagonal number theorem to the RHS.

What we know, part 1, odd case

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If $0 < a < m/2$,

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$$\mathbf{cp}_{a,m-a,m}(q) \sum_{n=-\infty}^{\infty} (-1)^n q^{an+mn(n-1)/2} \equiv \sum_{k=-\infty}^{\infty} q^{mk(3k-1)} \pmod 2$$

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If $\mathrm{cp}_{a,m-a,m}(n)$ has finitely many odd values, then there is a point at which the LHS has two close odd values, but the RHS does not.

What we know, part 1, even case

Theorem (B.-Eichhorn)

For all a, m, $\mathrm{cp}_{a,m-a,m}(n)$ is even infinitely often.

If $0 < a < m/2$, $\mathbf{cp}_{a,m-a,m}(q) \equiv \dfrac{(q^{2m};q^{2m})_\infty}{(q^{m-a};q^m)_\infty (q^a;q^m)_\infty (q^m;q^m)_\infty} \pmod 2$.

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Let $E_{a,m} := \{n \in \mathbb{N}_0 : \mathrm{cp}_{a,m-a,m}(n) \text{ is even}\}$ and $G_{a,m}(q) = \sum_{n \in E_{a,m}} q^n$.

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$$\frac{1}{1-q} \equiv \mathbf{cp}_{a,m-a,m}(q) + G_{a,m}(q) \pmod 2$$

What we know, part 1, even case

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$$\frac{\sum_{n=-\infty}^{\infty}(-1)^n q^{an+mn(n-1)/2}}{1-q} \equiv (q^{2m};q^{2m})_\infty + G_{a,m}(q) \sum_{n=-\infty}^{\infty}(-1)^n q^{an+mn(n-1)/2} \pmod 2$$

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positive density

$$\frac{1}{1-q} \equiv \mathbf{cp}_{a,m-a,m}(q) + G_{a,m}(q) \pmod 2$$

zero density

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$$\boxed{\frac{\sum_{n=-\infty}^{\infty}(-1)^n q^{an+mn(n-1)/2}}{1-q}} \equiv (q^{2m};q^{2m})_\infty + G_{a,m}(q) \boxed{\sum_{n=-\infty}^{\infty}(-1)^n q^{an+mn(n-1)/2}} \pmod 2$$

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zero density

∞ly many non-zero terms!

zero density

$$\boxed{\frac{\sum_{n=-\infty}^{\infty}(-1)^n q^{an+mn(n-1)/2}}{1-q}} \equiv (q^{2m};q^{2m})_\infty + G_{a,m}(q) \sum_{n=-\infty}^{\infty} (-1)^n q^{an+mn(n-1)/2} \pmod 2$$

What we know, part 2

Theorem (B.-Eichhorn)

$\mathrm{cp}_{3,1,4}(n)$ is even with arithmetic density one. Specifically, $\mathrm{cp}_{3,1,4}(n)$ is even if the prime factorization of $24n + 5$ includes a prime $\equiv 3 \pmod 4$ occuring with an odd exponent.

Theorem (B.-Eichhorn)

$\mathrm{cp}_{5,1,6}(n)$ is even with arithmetic density one. Specifically, $\mathrm{cp}_{5,1,6}(n)$ is even if the prime factorization of $6n + 1$ includes a prime $\equiv 2 \pmod 3$ occuring with an odd exponent.

Proof sketch

$$\sum_{n=0}^{\infty} \mathrm{cp}_{3,1,4}(n) q^n = \frac{(q^4;q^4)_\infty}{(q^3;q^4)_\infty (q;q^4)_\infty}$$

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$$\sum_{n=0}^{\infty} \mathrm{cp}_{3,1,4}(n)q^n = \frac{(q^4;q^4)_\infty}{(q^3;q^4)_\infty (q;q^4)_\infty} = \frac{(q^4;q^4)_\infty}{(q;q^2)_\infty}$$

Proof sketch

$$\begin{aligned}
\sum_{n=0}^{\infty} \mathrm{cp}_{3,1,4}(n) q^n &= \frac{(q^4;q^4)_\infty}{(q^3;q^4)_\infty (q;q^4)_\infty} = \frac{(q^4;q^4)_\infty}{(q;q^2)_\infty} \\
&= (-q;q)_\infty (q^4;q^4)_\infty \\
&\equiv (q;q)_\infty (q^4;q^4)_\infty \pmod 2
\end{aligned}$$

Proof sketch

$$\sum_{n=0}^{\infty} \mathrm{cp}_{3,1,4}(n)q^n = \frac{(q^4;q^4)_\infty}{(q^3;q^4)_\infty (q;q^4)_\infty} = \frac{(q^4;q^4)_\infty}{(q;q^2)_\infty}$$
$$= (-q;q)_\infty (q^4;q^4)_\infty$$
$$\equiv (q;q)_\infty (q^4;q^4)_\infty \pmod 2$$

Applying Euler's pentagonal number theorem:

$$\sum_{n=0}^{\infty} \mathrm{cp}_{3,1,4}(n)q^n \equiv \left(\sum_{j=-\infty}^{\infty} q^{j(3j+1)/2}\right)\left(\sum_{k=-\infty}^{\infty} q^{2k(3k+1)}\right) \pmod 2$$

Proof sketch, continued

$$\sum_{n=0}^{\infty} \mathrm{cp}_{3,1,4}(n) q^n \equiv \left(\sum_{j=-\infty}^{\infty} q^{j(3j+1)/2} \right) \left(\sum_{k=-\infty}^{\infty} q^{2k(3k+1)} \right) \pmod 2$$

$\mathrm{cp}_{3,1,4}(n)$ can be odd only if $n = j(3j+1)/2 + 2k(3k+1)$ for some integers j and k.

Proof sketch, continued

$$\sum_{n=0}^{\infty} \mathrm{cp}_{3,1,4}(n)q^n \equiv \left(\sum_{j=-\infty}^{\infty} q^{j(3j+1)/2}\right)\left(\sum_{k=-\infty}^{\infty} q^{2k(3k+1)}\right) \pmod 2$$

$\mathrm{cp}_{3,1,4}(n)$ can be odd only if $n = j(3j+1)/2 + 2k(3k+1)$ for some integers j and k.

Equivalently, $24n + 5 = (6j+1)^2 + 4(6k+1)^2 = A^2 + B^2$.

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Equivalently, $24n + 5 = (6j+1)^2 + 4(6k+1)^2 = A^2 + B^2$.

Note that $24n + 5$ is representable by $A^2 + B^2$ precisely when the prime factorization of $24n + 5$ has all primes $\equiv 3 \pmod{4}$ occur with an even exponent.

Some implied congruences, part 1

Corollary

For any prime $p > 3$, $p \equiv 3 \pmod 4$, let $24\delta \equiv 1 \pmod{p^2}$. Then,

$$\mathrm{cp}_{3,1,4}(p^2k + pt - 5\delta) \equiv 0 \pmod 2$$

for $t = 1, 2, \ldots, p-1$ and all $k \in \mathbb{N}$.

Example

For $r = 3, 17, 24, 31, 38, 45$ and any $k \in \mathbb{N}$,

$$\mathrm{cp}_{3,1,4}(49k + r) \equiv 0 \pmod 2$$

Some implied congruences, part 2

Corollary

For any prime $p > 2$, $p \equiv 2 \pmod{3}$, let $6\delta \equiv 1 \pmod{p^2}$. Then,

$$\mathrm{cp}_{5,1,6}(p^2k + pt - \delta) \equiv 0 \pmod{2}$$

for $t = 1, 2, \ldots, p-1$ and all $k \in \mathbb{N}$.

Example

For $r = 9, 14, 19, 24$ and any $k \in \mathbb{N}$,

$$\mathrm{cp}_{5,1,6}(25k + r) \equiv 0 \pmod{2}$$

More data

Questions

For which $a \neq b$, is $\mathrm{cp}_{a,b,m}(n)$ even with arithmetic density $\frac{1}{2}$?
When there is a parity bias, how often is $\mathrm{cp}_{a,b,m}(n)$ even?

n	$d_{1,5,6}(n)$	$d_{1,4,5}(n)$	$d_{3,5,8}(n)$
1000	0.581	0.503	0.628
2000	0.599	0.511	0.654
4000	0.623	0.509	0.681
8000	0.641	0.509	0.703
16000	0.653	0.508	0.720
32000	0.671	0.501	0.735

A bold conjecture

Conjecture

When $\gcd(a, b, m) = 1$, $a \neq b$, and $a + b \neq m$, $\mathrm{cp}_{a,b,m}(n)$ is even (odd) with arithmetic density $\frac{1}{2}$.

Table of Contents

1 Background on Copartitions

2 Parity

3 Adding weight

Motivation

Theorem (Chern 2021)

Define $\mathrm{eo}_0^*(n)$ (resp. $\mathrm{eo}_2^*(n)$) to be the number of partitions counted by $\mathcal{EO}^*(n)$ with largest even part congruent to 0 (resp. 2) modulo 4. Then,

$$\sum_{n\geq 0}(\mathrm{eo}_0^*(n) - \mathrm{eo}_2^*(n))q^n = \frac{(-q^4;q^4)_\infty}{(q^4;q^8)_\infty}$$

Corollary

$$\mathrm{eo}_0^*(n) \begin{cases} = \mathrm{eo}_2^*(n) & \text{if } 4 \nmid n \\ > \mathrm{eo}_2^*(n) & \text{if } 4 \mid n \end{cases}$$

Overpartition analogue

Define $\overline{\mathrm{eo}}_0^*(n)$ (resp. $\overline{\mathrm{eo}}_2^*(n)$) to be the number of overpartitions with all even parts smaller than all odd parts, only the largest even part appearing an odd number of times, and largest even part $\equiv 0$ (resp. 2) $\pmod 4$.

Theorem (Chern 2021)

$$\overline{\mathrm{eo}}_0^*(n) \begin{cases} = \overline{\mathrm{eo}}_2^*(n) & \text{if } 4 \nmid n \\ > \overline{\mathrm{eo}}_2^*(n) & \text{if } 4 \mid n \end{cases}$$

Copartitions version

Let $\mathrm{cp}^{e}_{1,1,2}(n)$ (resp. $\mathrm{cp}^{o}_{1,1,2}(n)$) be the number of $(1,1,2)$-copartitions with an even (resp. odd) number of woods parts.

Theorem

$$\mathrm{cp}^{e}_{1,1,2}(n) \begin{cases} = \mathrm{cp}^{o}_{1,1,2}(n) & \text{if } n \text{ is odd} \\ > \mathrm{cp}^{o}_{1,1,2}(n) & \text{if } n \text{ is even} \end{cases}$$

Combinatorial proof idea

2	2	2	2	2	1	2	2	2	2
2	2	2	2	2	1	2	2	2	
2	2	2	2	2	1	2	2	2	
1	1	1	1	1					
2	2	2	2	2					
2	2	2	2						
2	2								

Combinatorial proof idea

2	2	2	2	2	1	2	2	2	2
2	2	2	2	2	1	2	2	2	
2	2	2	2	2	1	2	2	2	
1	1	1	1	1					
2	2	2	2	2					
2	2	2	2						
2	2								

$\longrightarrow$

2	2	2	2	2	1	2	2	2	2
2	2	2	2	2	1	2	2	2	
2	2	2	2	2	1	2	2	2	
2	2	2	2	1					
2	2	2	2	2					
1	1	1	1						
2	2								

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2	2	2	2	2	1	2	2	2	2
2	2	2	2	2	1	2	2	2	
2	2	2	2	2	1	2	2	2	
1	1	1	1	1					
2	2	2	2	2					
2	2	2	2						
2	2								

$\longrightarrow$

2	2	2	2	2	1	2	2	2	2
2	2	2	2	2	1	2	2	2	
2	2	2	2	2	1	2	2	2	
2	2	2	2	1					
2	2	2	2	2					
1	1	1	1						
2	2								

$\longrightarrow$

2	2	2	2	2	1	2	2	2	2
2	2	2	2	2	1	2	2	2	
2	2	2	2	2	1	2	2	2	
2	2	2	2	1					
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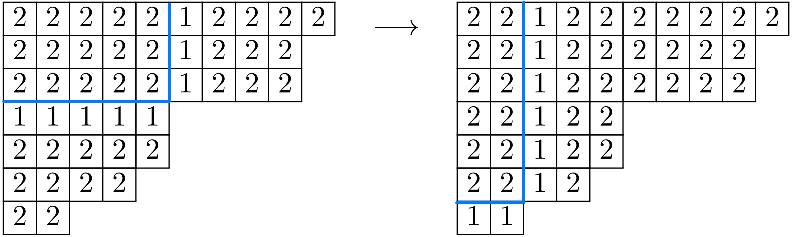
2	2								

$\longrightarrow$

2	2	2	2	2	1	2	2	2	2
2	2	2	2	2	1	2	2	2	
2	2	2	2	2	1	2	2	2	
2	2	2	2	1					
2	2	2	2	1					
2	2	2	1						
1	1								

Combinatorial proof idea



Overpartition version

Remark

Our injection preserves the sum of the diversities (number of distinct part sizes) of the woods and the sky.

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$$\overline{\mathrm{cp}}^{e}_{1,1,2}(n) \begin{cases} = \overline{\mathrm{cp}}^{o}_{1,1,2}(n) & \text{if } n \text{ is odd} \\ > \overline{\mathrm{cp}}^{o}_{1,1,2}(n) & \text{if } n \text{ is even} \end{cases}$$

General version: a conjecture

Let $\mathrm{cp}^{e}_{a,b,m}(n)$ (resp. $\mathrm{cp}^{o}_{a,b,m}(n)$) be the number of (a,b,m)-copartitions with an even (resp. odd) number of woods parts.

Conjecture

If $b|a$, then

$$\mathrm{cp}^{e}_{a,b,m}(n) \geq \mathrm{cp}^{o}_{a,b,m}(n)$$

for all $n \in \mathbb{N}$.

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Let $\mathrm{cp}^e_{a,b,m}(n)$ (resp. $\mathrm{cp}^o_{a,b,m}(n)$) be the number of (a, b, m)-copartitions with an even (resp. odd) number of woods parts.

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Conjecture, reframed

When expanded as a q-series,

$$\frac{(-q^{a+b}; q^m)_\infty}{(-q^a; q^m)_\infty (q^b; q^m)_\infty}$$

has non-negative coefficients when $b|a$.

Related conjectures and progress

Conjecture (Seo-Yee)

For $m \geq 4$,

$$\frac{1}{(q;q^m)_\infty(-q^{m-1};q^m)_\infty}$$

has non-negative coefficients (when expanded as a q-series).

Remark

Will Craig proved the case $m = 4$ of the above conjecture.

Open problems

What other divisibility properties does $\mathrm{cp}_{a,b,m}(n)$ have?

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By changing the restrictions on the ground and sky, there are many options for generalizations. Which generalizations have interesting properties?

Is there a logical way to unite three (or more) partitions?

To learn more

H. E. Burson and D. Eichhorn. Copartitions. arXiv:2111.04171

H. E. Burson and D. Eichhorn. On the parity of the number of (a, b, m)-copartitions of n. arXiv:2201.04247

H. E. Burson and D.Eichhorn. On the positivity of a family of infinite products. In preparation.